

16.2a

line integrals: scalar & vector

①

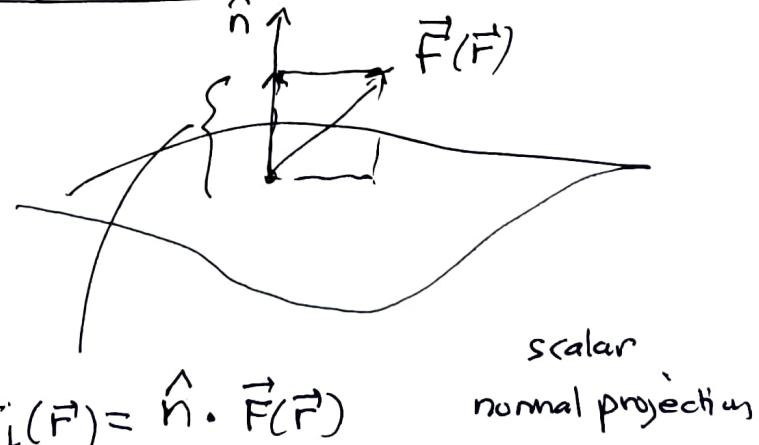
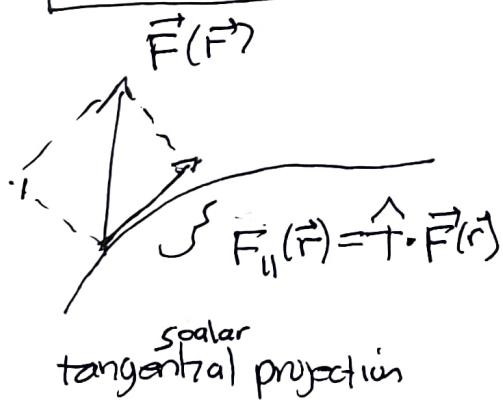
In Bob's review of the semester chapters only chapter 15 did not involve vectors: we could only integrate scalars over open subsets of \mathbb{R}^n — regions of the plane or space.

But we did integrate vectors in one application process: weighted averaging of the position vector over an open region of the plane or space. For a mass distribution this gave us the center of mass and for a homogeneous distribution we got the geometric center of the region — the centroid. For a probability distribution we got the expected value of the vector of random variables. The textbook drew NONE of these connections BUT that is one of the beautiful aspects of mathematics — it generalizes seemingly different ideas into one universal idea.

Now we want to extend integration of vectors to curves and surfaces within space, and to curves in the plane, namely lower-dimensional continuous subsets of \mathbb{R}^n ($n=2, 3, \dots$)

But we really can only integrate scalars so we need to produce scalars from vectors relevant to these curves & surfaces.

We can integrate vector fields over curves and surfaces by projecting them to scalars along these objects



"line integral"

"surface (flux) integral"

Scalar line integrals (1)

16.2a.1

scalar integration so far :

$$1d : \int_a^b f(x) dx$$

dx differential of arclength

$$dA = dx dy = r dr d\theta$$

$$2d : \iint_R f(x,y) dA$$

dA differential of area

$$3d : \iiint_R f(x,y,z) dV$$

dV differential of volume

$$dV = dx dy dz = r dz dr d\theta \\ = \rho^2 \sin\phi \, d\rho d\phi d\theta$$

2d, 3d parametrized curves

$$L = \int_C ds = \int_a^b \underbrace{|\vec{r}'(t)|}_{ds} dt$$

ds differential of arclength

$$ds = \sqrt{dx^2 + dy^2}, \\ \sqrt{dx^2 + dy^2 + dz^2}$$

"region of integration" is a subset of the whole space of lower dimension
in contrast with previous examples, but we are integrating a scalar only defined along the parametrized curve.

Functions ("scalar") defined over the whole space can be integrated over subspaces, like curves in the plane, or curves and surfaces in space. Just parametrize the subspace and evaluate those functions on the parametrized position vector

$$\text{parametrized curves: } \vec{r} = \vec{r}(t) : f(\vec{r}) \rightarrow f(\vec{r}(t)) \quad \text{evaluate on curve}$$

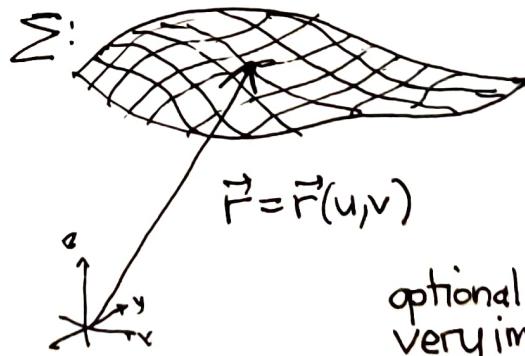
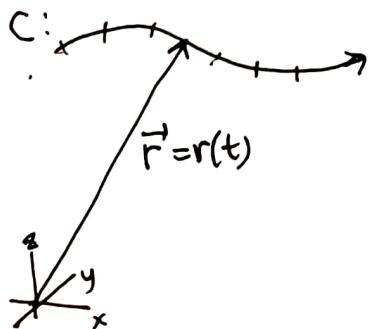
$$\text{parametrized surfaces: } \vec{r} = \vec{r}(u,v) : f(\vec{r}) \rightarrow f(\vec{r}(u,v)) \quad \text{evaluate on surface}$$

then integrate over curve/surface with respect to:

$$ds = |\vec{r}'(t)| dt \quad \text{or}$$

$$dS = (\text{?}) du dv$$

geometric correction factor to be discussed later



$$\vec{r} = \vec{r}(u,v)$$

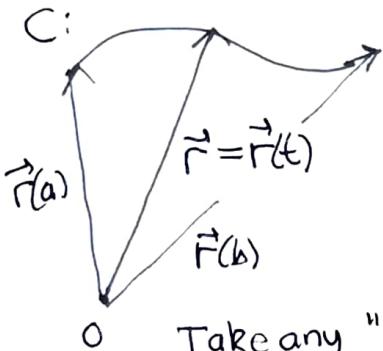
optional but very important!

Scalar line integrals (2)

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curves can be defined by a pair of equations in 3-d (intersection) of two surfaces or by a graph in 2-d

BUT we need a parametrization of the curve to evaluate a scalar line integral: $\vec{r} = \vec{r}(t), a \leq t \leq b$ curve segment



Take any "scalar field" $f(\vec{r})$
 \downarrow
 $f(\vec{r}(t))$

note arclength function:

$$s = \int_{t_0}^t |\vec{r}'(u)| du \Leftrightarrow \frac{ds}{dt} = |\vec{r}'(u)|$$

"dummy variable"

$$ds = |\vec{r}'(u)| du$$

geometric correction factor

$$\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

just plug in parametrization and integrate against differential of arclength

does not depend on parametrization!

$$\int_a^b f(\vec{r}(t)) \left| \frac{d\vec{r}(t)}{dt} \right| dt = \int_{t=a}^{t=b} f(\vec{r}(t(u))) \left| \frac{d}{dt} \vec{r}(t(u)) \right| dt(u)$$

\uparrow
change variable
 $\frac{d\vec{r}(t(u))}{dt} \frac{du}{dt}$
 $\frac{d\vec{r}(t(u))}{du}$
chain rule

$$= \int_{u_a}^{u_b} f(\vec{r}(t(u))) \left| \frac{d\vec{r}(t(u))}{du} \right| du$$

where $t = t(u) \Leftrightarrow u = u(t)$
 $u_a = u(a), u_b = u(b)$

same formula for new parametrization: $\vec{r} = \vec{r}(t(u))$

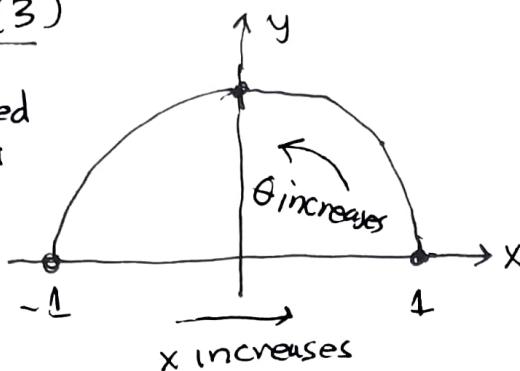
(notice absolute value sign here which we ignored in cancelling)

→ BUT $u_b \geq u_a$ only if $u = u(t)$ is an increasing function.
 HOWEVER, taking abs val sign into account reorders u_a and u_b to be in the right order!
 (sign of f determines sign of integral)

Scalar line integrals (3)

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Example: unparametrized curve in 2-d



unit semicircle:
 $x^2 + y^2 = 1$,
 $y \geq 0$

Cartesian coord parametrization

$$x^2 + y^2 = 1 \rightarrow y = \sqrt{1-x^2} \text{ for } y \geq 0$$

$$x = -1 \dots 1$$

so "y = $\sqrt{1-x^2}$ while x = -1 .. 1"

$$\vec{r} = \langle x, y \rangle = \langle x, \sqrt{1-x^2} \rangle$$

$$= \langle t, \sqrt{1-t^2} \rangle, t = -1..1$$

(choose $t=x$ to be the parameter)

note increasing x moves in clockwise direction

oppositely directed
parametrized
curves

$$\vec{r}'(t) = \left\langle 1, \frac{1}{2} \left(\frac{-2t}{\sqrt{1-t^2}} \right) \right\rangle = \left\langle 1, -\frac{t}{\sqrt{1-t^2}} \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{1 + \frac{t^2}{1-t^2}} = \sqrt{\frac{1-t^2+t^2}{1-t^2}} = \sqrt{\frac{1}{1-t^2}}$$

$$= \frac{1}{\sqrt{1-t^2}}$$

$$ds = \frac{dt}{\sqrt{1-t^2}}$$

$$\int_{C_{CW}} y \, ds = \int_{-1}^1 \left(\sqrt{1-t^2} \right) \frac{dt}{\sqrt{1-t^2}}$$

$$= \int_{-1}^1 1 \, dt = t \Big|_{-1}^1 = \boxed{2}$$

simple function
to integrate.

polar coord parametrization

$$1 = x^2 + y^2 = r^2 \rightarrow r = 1$$

so "r = 1 while $\theta = 0.. \pi$ "

$$x = r \cos \theta = \cos \theta$$

$$y = r \sin \theta = \sin \theta$$

(keep θ as parameter name, why not?)

$$\vec{r} = \langle x, y \rangle = \langle \cos \theta, \sin \theta \rangle$$

$$\theta = 0.. \pi$$

note increasing θ moves in counterclockwise direction

BUT doesn't matter for scalar line integrals!

$$\vec{r}'(\theta) = \langle -\sin \theta, \cos \theta \rangle$$

$$|\vec{r}'(\theta)| = \sqrt{\sin^2 \theta + \cos^2 \theta} = 1$$

(indeed θ is arclength on unit circle!)

$$ds = d\theta$$

$$\int_C y \, ds = \int_0^\pi \sin \theta \, d\theta$$

$$= -\cos \theta \Big|_0^\pi = \boxed{2}$$

Scalar line integrals (4)

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application: center of mass / centroid. (inhomogeneous wire)

given a linear mass density distribution on a curve:

$$dm = \rho(t) ds(t) \text{ differential of mass on } \vec{r} = \vec{r}(t)$$

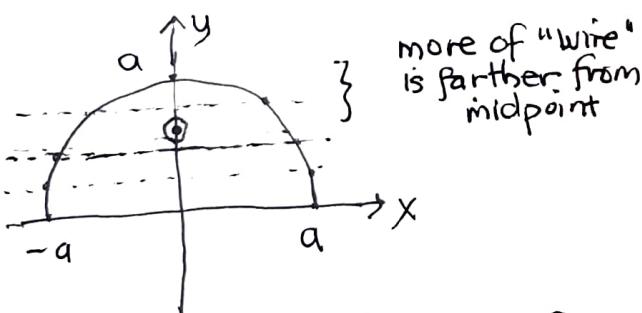
$$m = \int_C dm = \int_a^b \rho(t) ds(t) \quad \leftarrow ds(t) = |\vec{r}'(t)| dt$$

$$\langle M_y, M_x \rangle = \int_C \langle x, y \rangle dm = \int_a^b \langle x(t), y(t) \rangle \rho(t) ds(t)$$

$$\langle \bar{x}, \bar{y} \rangle = \frac{\langle M_y, M_x \rangle}{m} = \frac{\int_a^b \langle x(t), y(t) \rangle \rho(t) ds(t)}{\int_a^b \rho(t) ds(t)} = \int_a^b \langle x(t), y(t) \rangle \frac{\rho(t)}{m} ds(t)$$

If $\rho(b) = \rho_0$ (homogeneous wire), then ρ cancels out!
get centroid.

Example semicircular curve centroid



$$\vec{r}(\theta) = \langle a \cos \theta, a \sin \theta \rangle, \theta = 0.. \pi$$

$$|\vec{r}'(\theta)| = a, ds = a d\theta$$

$$L = \int_C ds = \pi a$$

$$\int_C \langle x, y \rangle ds = \langle 0, 2a^2 \rangle$$

$$\langle \bar{x}, \bar{y} \rangle = \frac{\langle 0, 2a^2 \rangle}{\pi a} = \langle 0, \frac{2a^2}{\pi} \rangle \approx \langle 0, 0.637 \rangle$$

a bit above the midpoint on the symmetry axis. ✓ makes sense!

variation:

suppose $\rho \propto y$ (heavier farther from x-axis)
 $= ky, (k > 0, y \geq 0)$

$$= k(a \sin \theta)$$

$$\underbrace{\rho}_{ds}$$

$$\langle M, M_y, M_x \rangle = \int_0^\pi \langle 1, a \cos \theta, a \sin \theta \rangle (k a \sin \theta) (a d\theta)$$

$$= k a \langle 2, 0, \frac{\pi a}{2} \rangle \rightarrow \bar{y} = \frac{\pi a / 2}{2} = \frac{\pi a}{4} \approx 0.785 a \text{ (higher as it should be)}$$

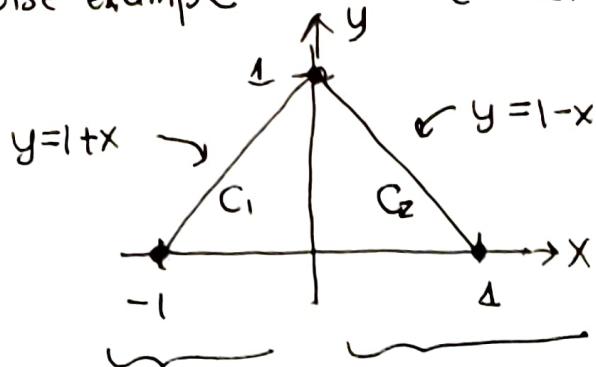
} fractional mass distribution

Scalar line integrals (5)

16.29.5

piecewise example

$$C = C_1 \cup C_2 \quad (\text{union!})$$



$$x = t = -1..0$$

$$y = 1+t$$

$$\vec{r} = \langle t, 1+t \rangle$$

$$\vec{r}' = \langle 1, 1 \rangle$$

$$|\vec{r}'| = \sqrt{1+1} = \sqrt{2}$$

$$x = t = 0..1$$

$$y = 1-t$$

$$\vec{r} = \langle t, 1-t \rangle$$

$$\vec{r}' = \langle 1, -1 \rangle$$

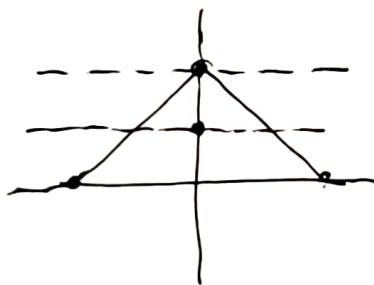
$$|\vec{r}'| = \sqrt{1+1} = \sqrt{2} \rightarrow ds = \sqrt{2} dt$$

$$\int_C \langle 1, y \rangle ds = \int_{C_1} \langle 1, y \rangle ds + \int_{C_2} \langle 1, y \rangle ds$$

$$= \int_{-1}^0 \langle 1, 1+t \rangle (\sqrt{2} dt) + \int_0^1 \langle 1, 1-t \rangle (\sqrt{2} dt)$$

$$= \dots = \langle \sqrt{2}, \sqrt{2}(1-\frac{1}{2}) \rangle + \langle \sqrt{2}, \sqrt{2}(1-\frac{1}{2}) \rangle$$

$$\stackrel{\text{easy by hand}}{=} \langle 2\sqrt{2}, \sqrt{2} \rangle \rightarrow \bar{y} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \quad \text{midpoint!}$$



each ΔS has same moment about middle line!