

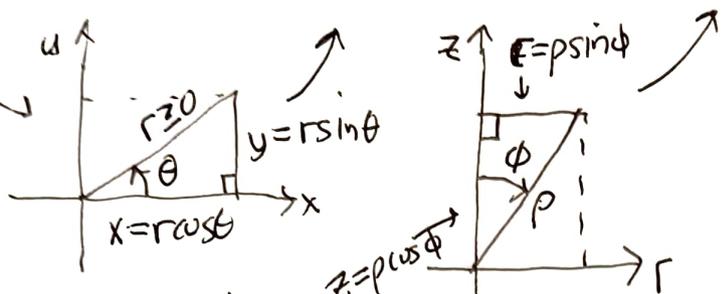
15.8 spherical coordinates

cartesian:                      cylindrical:

$$\begin{cases} x & = & r \cos \theta \\ y & = & r \sin \theta \\ z & = & z \end{cases}$$

spherical:                      ①

$$\begin{aligned} &= (\rho \sin \phi) \cos \theta \\ &= (\rho \sin \phi) \sin \theta \\ &= (\rho \cos \phi) \end{aligned}$$



theta up from positive horizontal axis in counterclockwise direction

introduce polar coords in r-z half plane

new angular coord:  
phi phi  
clockwise angle down from vertical  
phi = 0...pi  
up down

new radial coord  
rho = sqrt(r^2 + z^2) >= 0  
= sqrt(x^2 + y^2 + z^2)

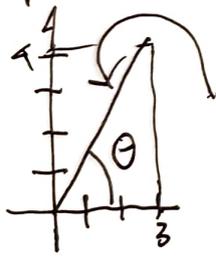
"radial" distance from origin

$$\cos \phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\phi = \arccos \frac{z}{\rho} \in [0, \pi] \quad \checkmark \text{ correct interval}$$

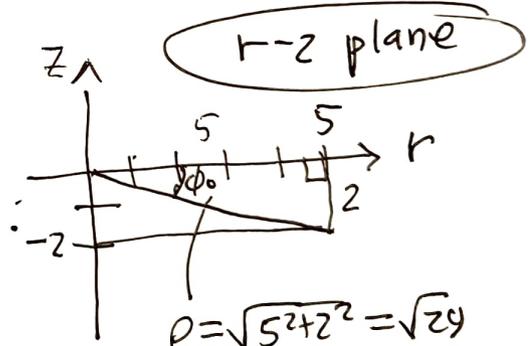
Calculate spherical coords in 2 steps using 2-d diagrams

Example (x, y, z) = (3, 4, -2)



$$\begin{aligned} r &= \sqrt{3^2 + 4^2} = 5 \\ \tan \theta &= \frac{4}{3} \\ \theta &= \arctan \frac{4}{3} \end{aligned}$$

x-y plane



$$\rho = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\cos \phi_0 = \frac{z}{\rho} = \frac{-2}{\sqrt{29}} \quad \phi_0 = \arccos \frac{-2}{\sqrt{29}}$$

$$\text{or } \tan \phi_0 = \frac{z}{r} = \frac{-2}{5} \quad \phi_0 = \arctan \frac{-2}{5}$$

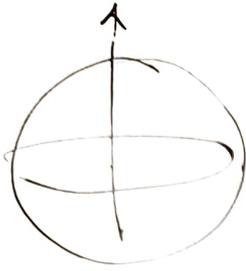
$$\phi = \frac{\pi}{2} + \arccos \frac{2}{\sqrt{29}}$$

15.8

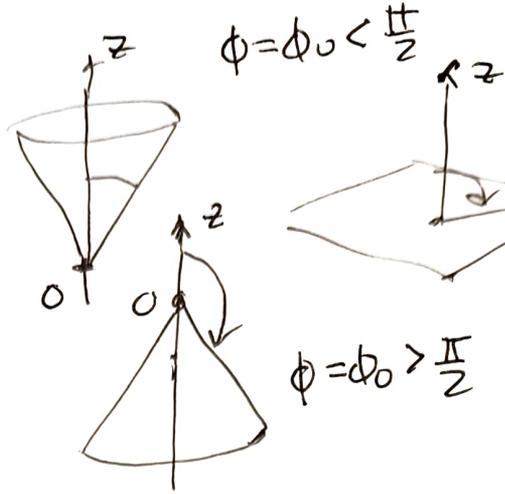
spherical coordinates

(2)

coordinate surfaces:

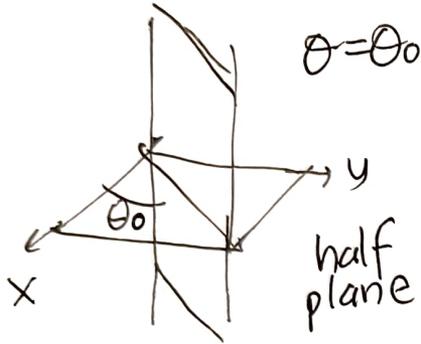


$\rho = \rho_0$   
sphere



$\phi = \phi_0 < \frac{\pi}{2}$

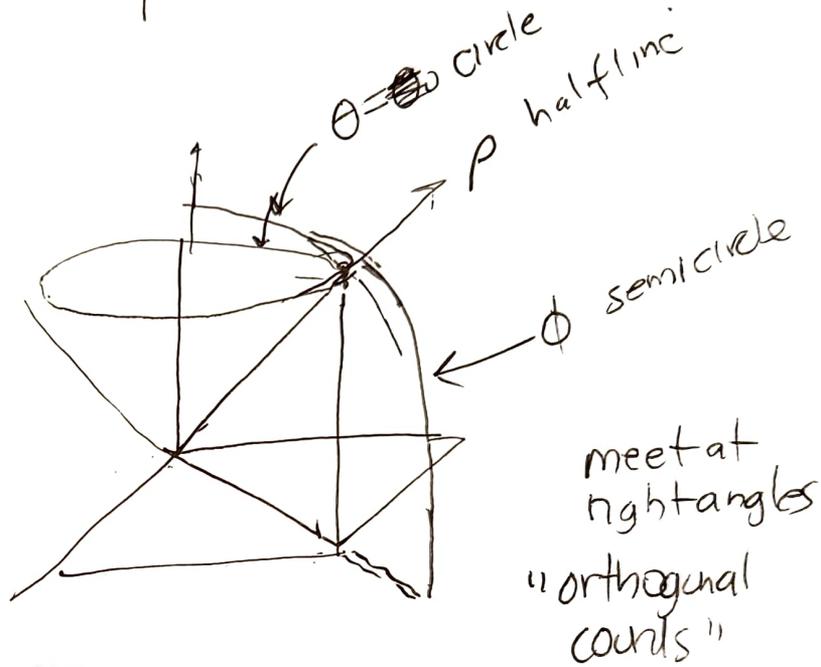
$\phi = \phi_0 > \frac{\pi}{2}$



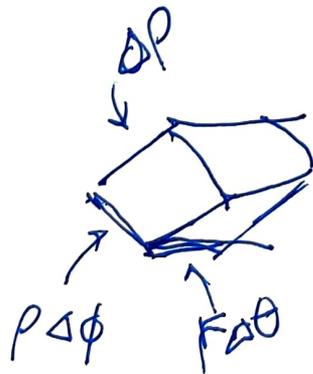
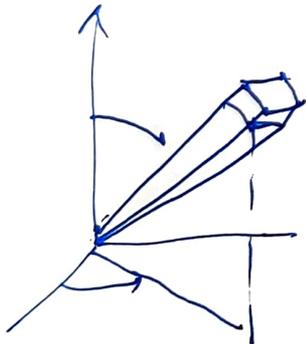
$\theta = \theta_0$

half plane

coordinate "lines"

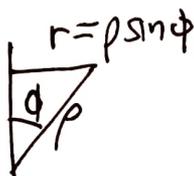


geometrical explanation of differential volume correction factor



$$\Delta V \approx (\Delta \rho)(\rho \Delta \phi)(r \Delta \theta)$$

$$= \underbrace{\rho \cdot r}_{\rho(\rho \sin \phi)} \Delta \rho \Delta \phi \Delta \theta$$



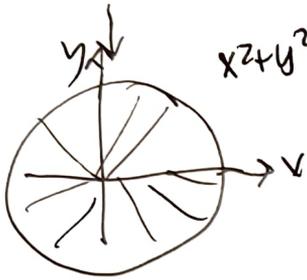
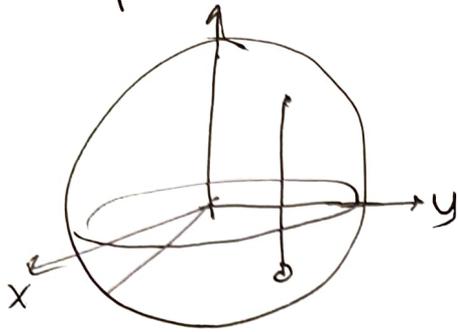
$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

usual order for most apps!

15.8 Spherical coordinates

(3)

sphere:



$$x^2 + y^2 + z^2 = a^2$$

$$V = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1 \, dz \, dy \, dx$$

$$= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} 1 \, dz \, (r \, dr \, d\theta)$$

$$1 \, dz \, dy \, dx$$

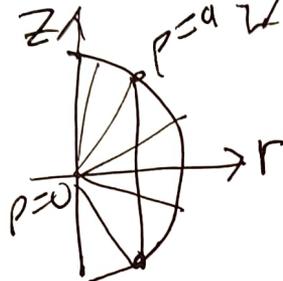
$$\downarrow$$

$$dz \, (r \, dr \, d\theta)$$

$$\downarrow$$

$$dp \, (p \, d\phi)$$

$$dV = \underbrace{p \cdot r}_{p^2 \sin \theta} dp \, d\phi \, d\theta$$



$$\int_0^{2\pi} \int_0^{\pi} \dots dp \, d\phi$$

$\uparrow$   $\uparrow$   
 $\phi$   $p$

inner double integral

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^a 1 \cdot p^2 \sin \phi \, dp \, d\phi \, d\theta$$

constant limits
factors
usual order

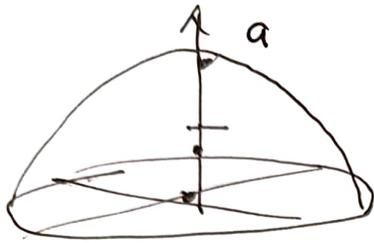
$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin \phi \, d\phi \int_0^a p^2 \, dp = (2\pi)(2)\left(\frac{a^3}{3}\right) = \frac{4\pi a^3}{3}$$

$\frac{p^3}{3} \Big|_0^a = \frac{a^3}{3}$   
 $-(-1) + (1) = 2$

15.8

spherical coordinates

④



hemisphere:  
 $\phi = 0 \dots \pi/2$   
 $\rho = 0 \dots a$   
 $\theta = 0 \dots 2\pi$

$$V = \frac{2\pi a^3}{3} \text{ (half)}$$

$$\iiint_H f \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a f \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

centroid: integrate  $x, y, z = \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi$

$$\langle M_{yz}, M_{xz}, M_{xy} \rangle = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi \rangle \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\langle \bar{x}, \bar{y}, \bar{z} \rangle = \frac{\langle M_{yz}, M_{xz}, M_{xy} \rangle}{V} = \langle \underbrace{0, 0}_{\text{symmetry}}, \bar{z} \rangle$$

$$\iiint z \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

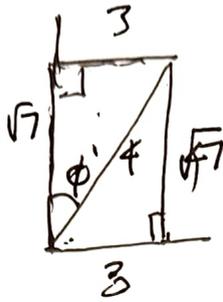
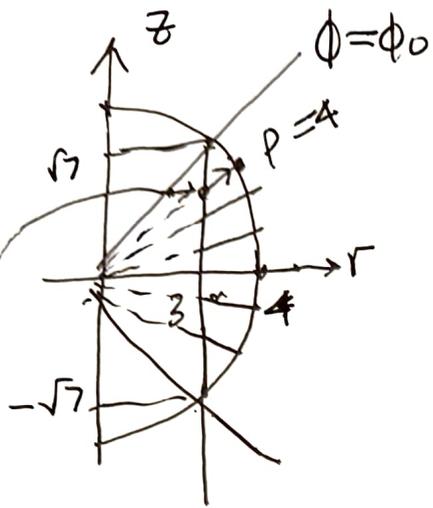
$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{\pi/2} \frac{\sin\phi \cos\phi \, d\phi}{\frac{1}{2}}}_{\frac{\sin^2\phi}{2} \Big|_0^{\pi/2} = \frac{1}{2}} \underbrace{\int_0^a \rho^3 \, d\rho}_{\frac{\rho^4}{4} \Big|_0^a = \frac{a^4}{4}} = \frac{\pi a^4}{4}$$

$$\bar{z} = \frac{\pi a^4 / 4}{2\pi a^3 / 3} = \frac{3a}{8} = .375a \text{ below } z = a/2$$

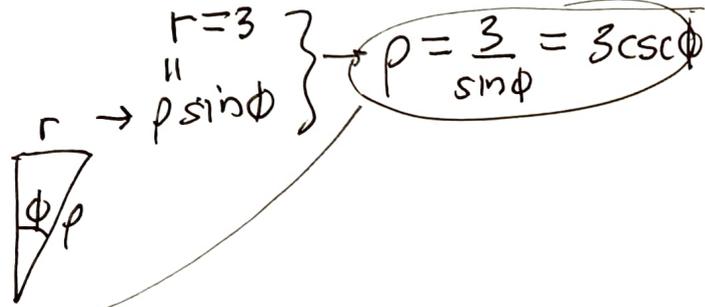
15.8 spherical coordinates

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wedding ring



$$\begin{aligned} \phi_0 &= \arccos \sqrt{7}/4 \\ &= \arctan 3/\sqrt{7} \\ &= \arcsin 3/4 \\ &\text{all equivalent} \\ &\text{(we have a choice)} \end{aligned}$$



$\rho = 3 \csc \phi$ ,  $\phi$  while  $\phi = \phi_0 \dots \pi - \phi_0$

$$V = \int_0^{2\pi} \int_{\phi_0}^{\pi - \phi_0} \int_{3 \csc \phi}^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \dots$$

$$\begin{aligned} &\left. \frac{\rho^3 \sin \phi}{3} \right|_{\rho=3 \csc \phi}^{\rho=4} \\ &\frac{1}{3} (16 - 9 \csc^2 \phi) \sin \phi \\ &= \left( \int_0^{2\pi} d\theta \right) \cdot \int_{\phi_0}^{\pi - \phi_0} \frac{1}{3} (16 \sin \phi - 9 \csc \phi) \, d\phi \\ &\quad \downarrow \text{easy} \quad \downarrow \text{easy (look up antiderivative)} \end{aligned}$$