

15.7-8b

## cylindrical and spherical coordinate integration

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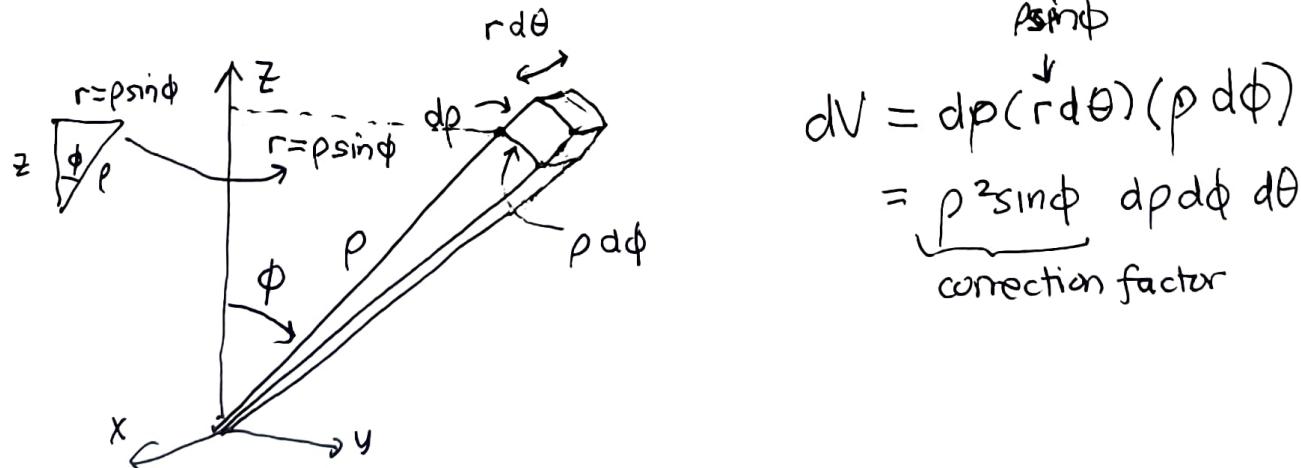
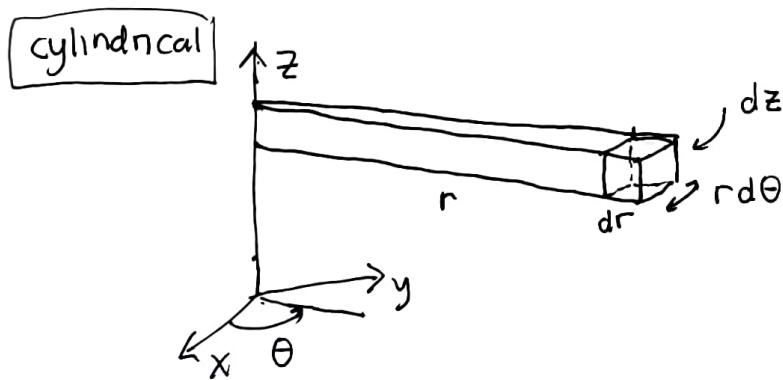
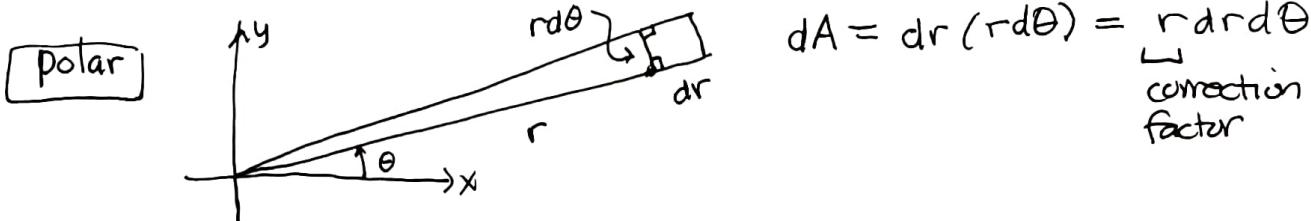
In each of the "orthogonal" coordinate systems in space:

Cartesian  $(x, y, z)$  cylindrical  $(r, \theta, z)$  spherical  $(\rho, \phi, \theta)$   
 as well as in polar coordinates  $(r, \theta)$  in the plane, the coordinate  
 "grids" formed by equally spaced coordinate lines meet at right angles,  
 so the grid boxes in the limit of "infinitesimal" side lengths look more  
 and more like rectangular boxes so the area or volume is just the  
 product of the orthogonal side arclengths which form the box.

This geometry allows us to evaluate the differential of area or volume as a correction factor times the product of the differentials of the coordinates. In Cartesian coordinates, the coordinates directly measure arclength, so no correction factor is needed:

$$dA = dx dy, \quad dV = dx dy dz$$

The remaining coordinate systems are a mix of arclength and angular variables, and the angular differentials must be multiplied by the [radius of the arc] along which they move.

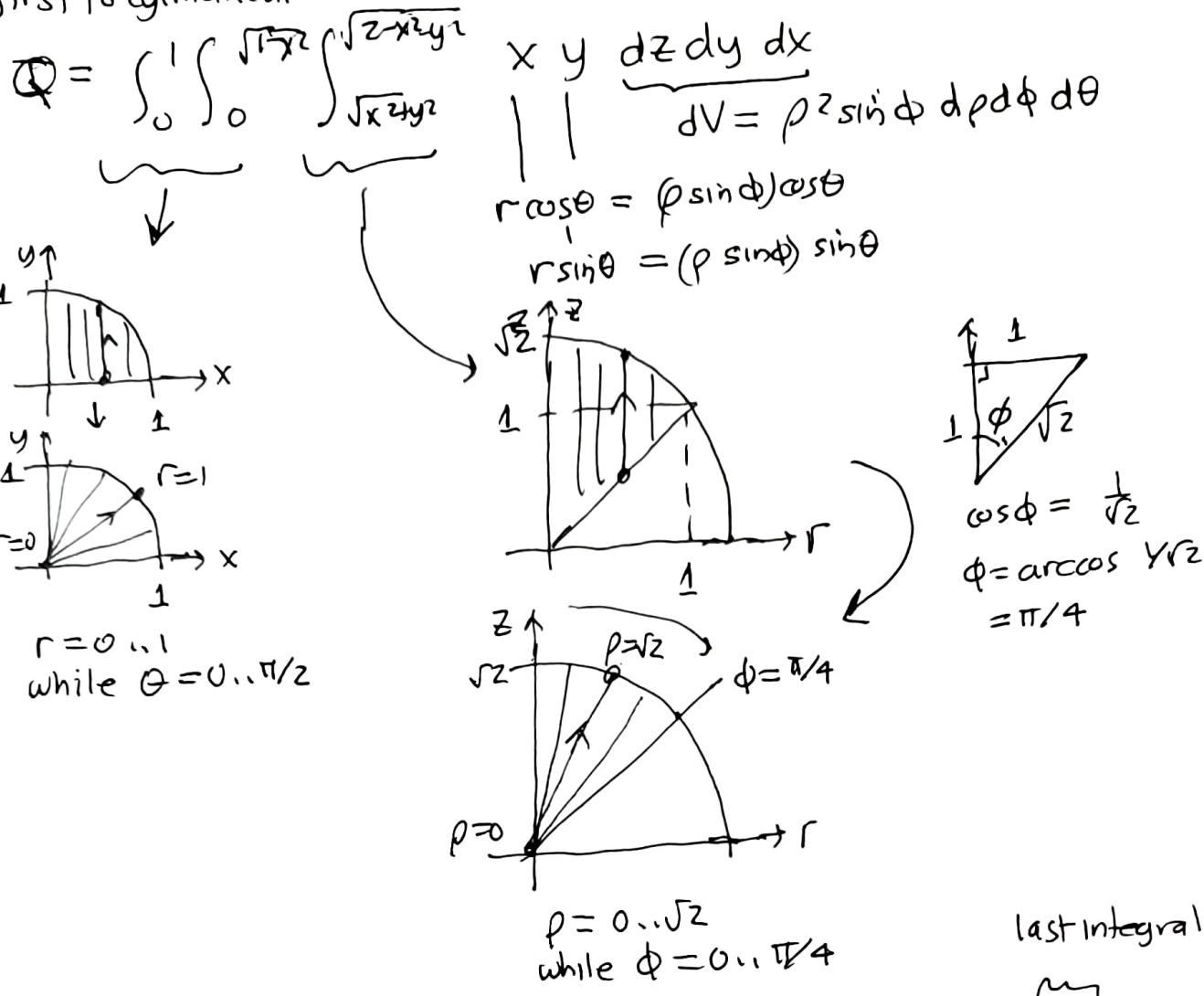


15.78b

cylindrical and spherical coordinate integration

(2)

Let's return to the previous lecture triple integral which we converted first to cylindrical coordinates:



so:

$$Q = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}}$$

$\underbrace{\hspace{10em}}$   $\underbrace{\hspace{10em}}$   $\underbrace{\hspace{10em}}$

constant limits

$\rho^4 \sin^3 \phi \sin \theta \cos \theta$   
factorizable integrand

$d\rho \ d\phi \ d\theta$   
usual best order  
 $\underbrace{\hspace{10em}}$   
r-z plane inner double integral

$$= \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta \cdot \int_0^{\frac{\pi}{4}} \sin^3 \phi d\phi \cdot \int_0^{\sqrt{2}} \rho^4 d\rho$$

$\underbrace{\hspace{10em}}$   $\underbrace{\hspace{10em}}$   $\underbrace{\hspace{10em}}$

Maple:  $\frac{1}{2}$       Maple:  $\frac{2}{3} - \frac{5\sqrt{2}}{12}$

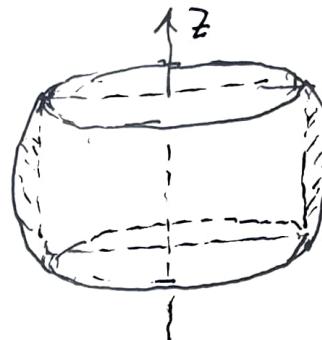
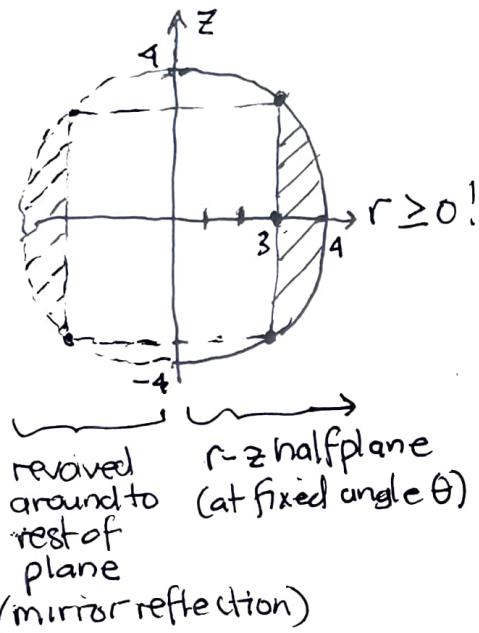
$$= \frac{4\sqrt{2}}{15} - \frac{1}{3} \approx 0.0438$$

# Wedding ring setup (1) (threaded bead for necklace?) 15.7-8b: 3

Drill a 3 unit radius hole through the center of a 4 unit radius sphere.  
Setup with center at origin.

$$\text{sphere: } x^2 + y^2 + z^2 = 4^2 \rightarrow r^2 = 4^2 \rightarrow r = 4 \quad r^2 + z^2 = 4^2$$

$$\text{cylinder(hole): } x^2 + y^2 = 3^2 \rightarrow r^2 = 3^2 \rightarrow r = 3 \quad r^2 + z^2 = 3^2$$



solid of revolution

$$0 \leq \theta \leq 2\pi$$

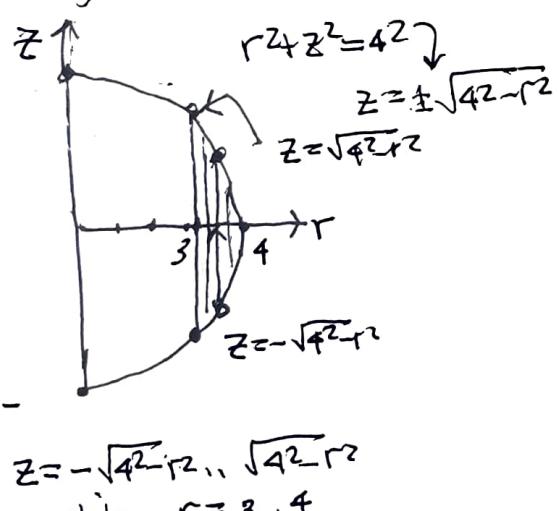
integrate last in triple integral

intersection points:

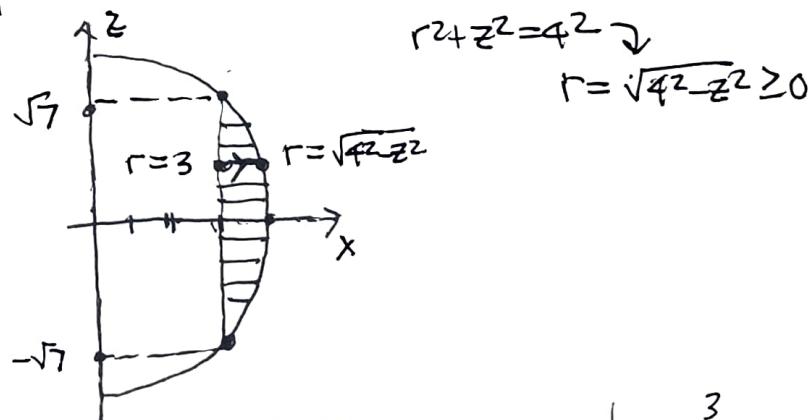
$$r^2 + z^2 = 4^2 \rightarrow z^2 = 4^2 - 3^2 = 7 \rightarrow z = \pm \sqrt{7}$$

$\nwarrow r=3$

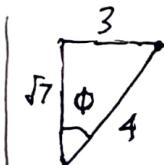
Integration order in  $r-z$  half plane:  $z$ -first (vertical) or  $r$ -first (horizontal)



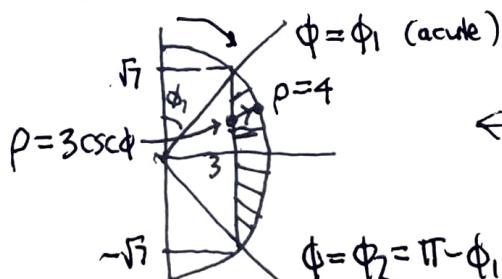
vertical linear cross-section  $\Rightarrow$  cylinder



horizontal linear cross-section  $\Rightarrow$  annulus



OR POLAR COORDS in  $r-z$  halfplane (spherical coords)



$\rho = 3 \csc \phi \cdot 4$ ,  
while  $\phi = \arcsin \frac{3}{4}, \pi - \arcsin \frac{3}{4}$   
while  $\theta = 0 \dots 2\pi$

$$z = \rho \cos \phi$$

$$\rho = \rho \sin \phi = 3$$

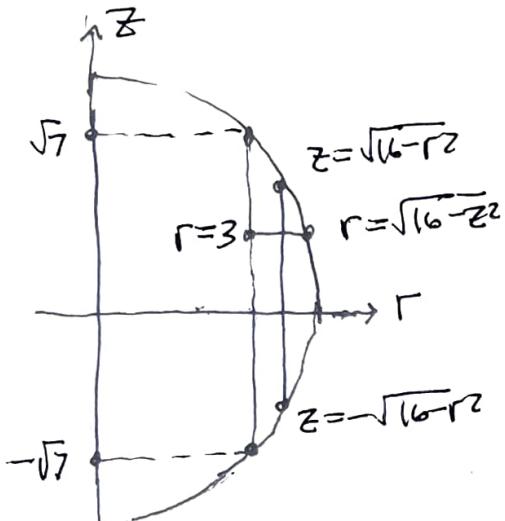
$$\frac{z}{\rho} = \tan \phi$$

$$\phi = \arctan \left( \frac{3}{\sqrt{7}} \right) = \arccos \left( \frac{4}{5} \right)$$

$$= \arcsin \left( \frac{3}{4} \right)$$

# Wedding Ring setup (2) : integration

15.7-8b: 4



innerable integral  
diagram

## $z$ -first (vertical)

$$\int_0^{2\pi} \int_3^4 \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}}$$

nothing  
depends on  $\theta$ ,  
factors out.  
do at any time  
 $\left( \int_0^{2\pi} 1 d\theta = 2\pi \right)$

$$f \underbrace{r dz dr d\theta}_{dV = dz dr (rd\theta)}$$

(if revolved  $360^\circ$ , otherwise limit)  
range of  $\theta$ )

## $r$ -first (horizontal)

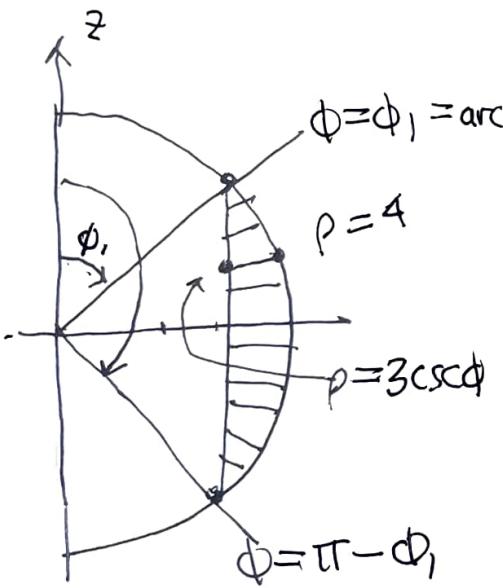
$$\int_0^{2\pi} \int_{-\sqrt{7}}^{\sqrt{7}} \int_3^{\sqrt{16-z^2}}$$

$$f \underbrace{r dr dz d\theta}_{dV = (dr)(dz)(rd\theta)}$$

$$\underbrace{dp(p d\phi)}_{p^2 \sin \phi} \underbrace{(p \sin \phi) d\theta}_{d\phi}$$

$$p^2 \sin \phi \, dp \, d\phi \, d\theta$$

geometric  
connection  
factor  
do  $p$  first  
(like do  $r$  first)  
 $= p \cdot r$  product of two arc radii.  
converting  $d\phi$  and  $dr$   
to arclength differentials.



## $\rho$ -first (radial)

$$\int_0^{2\pi} \int_{\arcsin 3/4}^{\pi - \arcsin 3/4} \int_{3 \csc \phi}^4$$

$$f \underbrace{\rho^3 \sin \phi \, d\rho \, d\phi \, d\theta}_T$$

radial integral  
always done first

since  $\rho = \rho(\phi)$  always defines  
curves of variable  $\phi$ .

always  
do  $\theta$  integral last

[15.78b] cylindrical and spherical coordinate integration

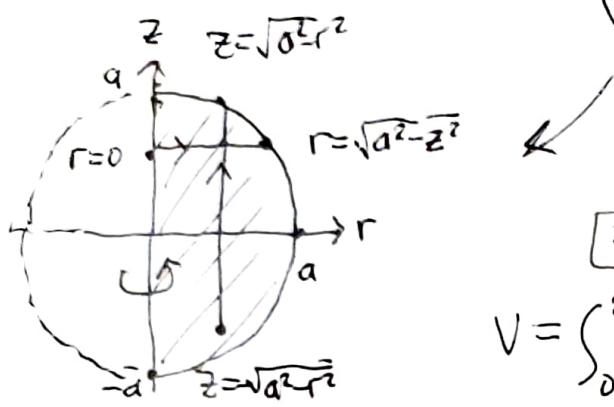
(5)

The simplest example for both these coordinates is a sphere. What is its volume for radius  $a$ ?

$$x^2 + y^2 + z^2 = a^2 \rightarrow r^2 + z^2 = a^2 \rightarrow \rho^2 = a^2 \quad \downarrow \rho = a$$

$$\downarrow z = \pm \sqrt{a^2 - r^2}$$

$$r = \sqrt{a^2 - z^2}$$



$$\theta = 0..2\pi$$

$$\uparrow \begin{cases} z = -\sqrt{a^2 - r^2} .. \sqrt{a^2 - r^2} \\ \text{while } r = 0..a \end{cases}$$

$$\rightarrow \begin{cases} r = 0..sqrt{a^2 - z^2} \\ \text{while } z = -a \text{ and } a \end{cases}$$

**[z-first]**

$$V = \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} r dz dr d\theta = \frac{4\pi}{3} a^3 \checkmark$$

$$= \int_0^{2\pi} d\theta \int_0^a r z \Big|_{z=-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} dr \quad \uparrow$$

$$\underbrace{2\pi}_{(2\pi)} \quad \underbrace{\int (a^2 - r^2)^{1/2} (2r dr)}_{\int u^{1/2} (-du)}$$

$$= \int u^{1/2} (-du) \quad \underbrace{-\frac{2}{3} (a^2 - r^2)^{3/2} \Big|_0^a}_{=\left(\frac{2}{3}\right) a^3}$$

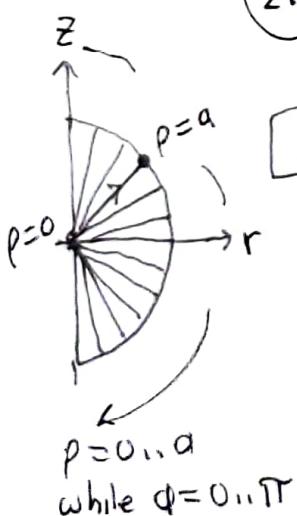
$$= -\frac{2}{3} u^{3/2} = -\frac{2}{3} (a^2 - r^2)^{3/2}$$

**[r-first]**

$$V = \int_0^{2\pi} \int_{-a}^a \int_0^{\sqrt{a^2 - z^2}} r dr dz d\theta = \frac{4\pi a^3}{3} \checkmark$$

$$= \int_0^{2\pi} d\theta \int_0^a \int_{r=0}^{\sqrt{a^2 - z^2}} \frac{r^2}{2} dz dr \quad \rightarrow \frac{a^2 z - z^3/3}{2} \Big|_0^a = \frac{a^3 - a^3/3}{2} = \frac{2a^3}{3}$$

$$\underbrace{2\pi}_{(2\pi)} \quad \underbrace{\frac{a^2 - z^2}{2} \cdot z}_{\text{odd}} \quad \uparrow \text{twice } \int_0^a$$



**[rho first]**

$$V = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \underbrace{\left(\int_0^{2\pi} d\theta\right)}_{2\pi} \left(\int_0^a \rho^2 d\rho\right) \left(\int_0^\pi \sin\phi d\phi\right) = \frac{4\pi}{3} a^3 \checkmark$$

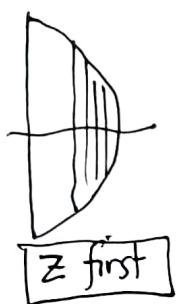
$$\underbrace{\rho^3 \Big|_0^a}_{\frac{a^3}{3}} = \frac{a^3}{3} \quad \underbrace{-\cos\phi \Big|_0^\pi}_{=2} = 2$$

15.78b

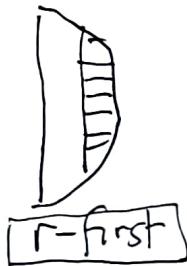
## cylindrical and spherical coordinate integration

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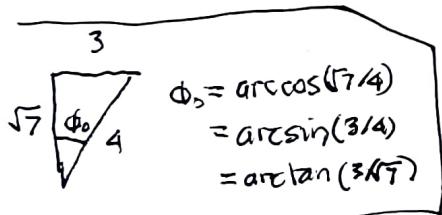
The wedding band / threaded bead integrals for their volume are a bit more involved.



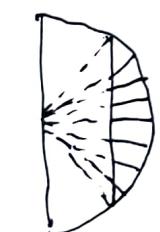
$$\begin{aligned}
 V &= \int_0^{2\pi} \int_3^4 \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} d\theta \int_3^4 r z \Big|_{z=-\sqrt{16-r^2}}^{z=\sqrt{16-r^2}} dr = 2\pi \left( -\frac{2}{3}(16-r^2)^{3/2} \right) \Big|_3^4 \\
 &\quad = \frac{4\pi}{3} \left[ (16-9)^{3/2} - (16-16)^{3/2} \right] = 7\sqrt{7} \\
 &\quad = \frac{28\sqrt{7}\pi}{3} \quad \boxed{\text{easiest}}
 \end{aligned}$$



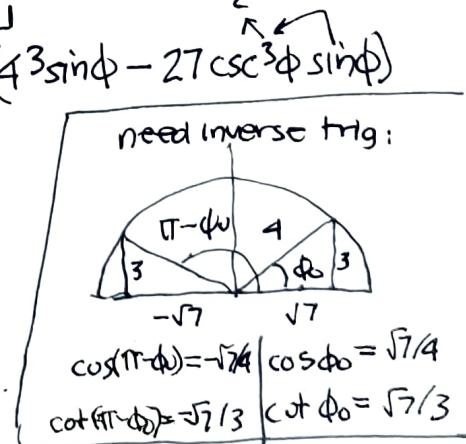
$$\begin{aligned}
 V &= \int_0^{2\pi} \int_{-\sqrt{7}}^{\sqrt{7}} \int_3^{\sqrt{16-z^2}} r \, dr \, dz \, d\theta = 2\pi \int_{-\sqrt{7}}^{\sqrt{7}} \frac{1}{2}(7-z^2) \, dz \\
 &= 2\pi \left[ \frac{1}{2}(7z - z^3/3) \right] \Big|_{z=-\sqrt{7}}^{z=\sqrt{7}} = 2\pi (7\sqrt{7} - 7\sqrt{7}/3) = \frac{4}{3}(2\pi) \cdot 7\sqrt{7} \\
 &= \frac{28\sqrt{7}\pi}{3} \quad \boxed{\text{easy}}
 \end{aligned}$$



$$\begin{aligned}
 \phi_0 &= \arccos(\sqrt{7}/4) \\
 &= \arcsin(3/4) \\
 &= \arctan(3/\sqrt{7})
 \end{aligned}$$



$$\begin{aligned}
 V &= \int_0^{2\pi} \int_{\arccos\sqrt{7}/4}^{\pi-\arccos\sqrt{7}/4} \int_{3\csc\phi}^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
 &= 2\pi \int_{\arccos\sqrt{7}/4}^{\pi-\arccos\sqrt{7}/4} \left[ \frac{\rho^3}{3} \sin\phi \right]_{\rho=3\csc\phi}^{\rho=4} = \frac{1}{3} (4^3 \sin\phi - 27 \csc^3\phi \sin\phi) \\
 &= 2\pi \left[ \frac{4^3}{3} \sin\phi + 9 \csc^2\phi \right]_{\arccos\sqrt{7}/4}^{\pi-\arccos\sqrt{7}/4} = 2\pi \left( -\frac{4^3}{3} \cos\phi + 9 \cot\phi \right) \\
 &= 2\pi \left( \frac{32}{3}\sqrt{7} - 6\sqrt{7} \right) = 2\pi \left( \frac{14}{3}\sqrt{7} \right) = \frac{28\sqrt{7}\pi}{3} \\
 &\quad \boxed{\text{Inversing! hard}}
 \end{aligned}$$



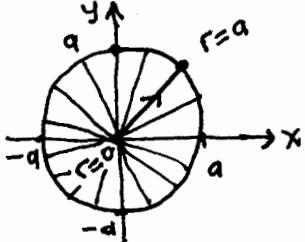
simple circles / cylinders / spheres in polar  $(r, \theta)$  / cylindrical  $(r, \theta, z)$  / spherical  $(\rho, \theta, \phi)$

assume  $a > 0$

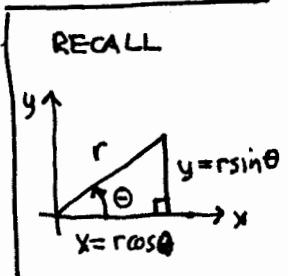
15.7-8b: 7

xy plane (circles in xy plane represent cylinders in xyz space)

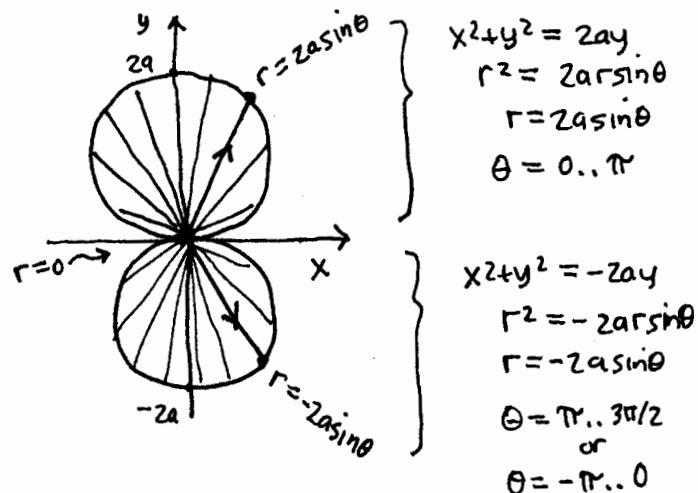
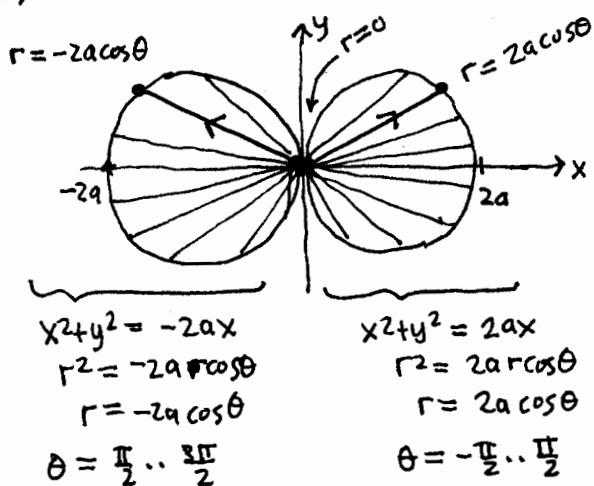
a) center at origin



$$\begin{aligned}x^2 + y^2 &= a^2 \\r^2 &= a^2 \\r &= a \\&\theta = 0..2\pi\end{aligned}$$

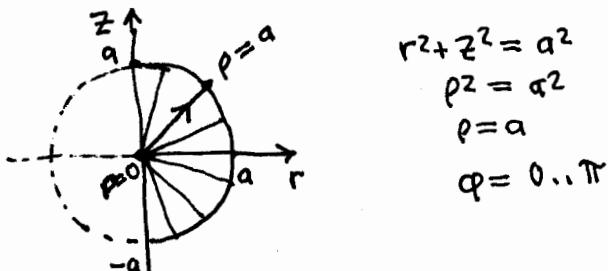


b) center on axis, tangent to origin ( $\theta$  interval between zeros of  $r$ )



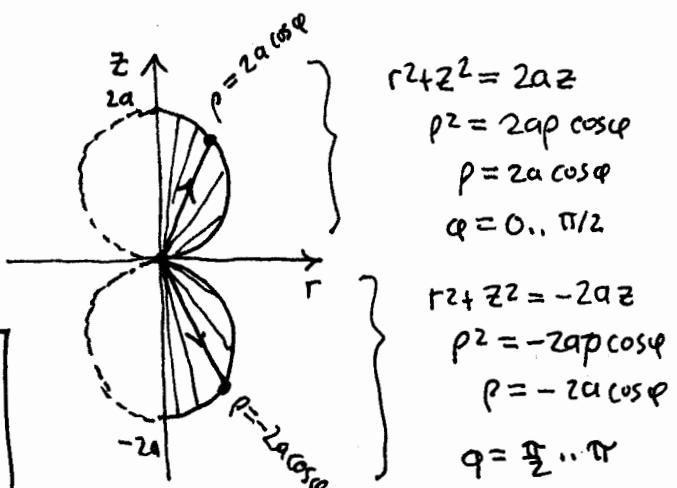
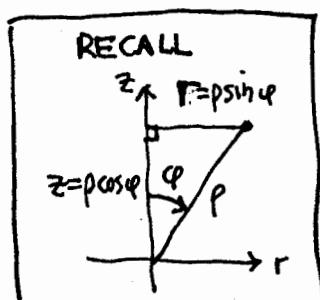
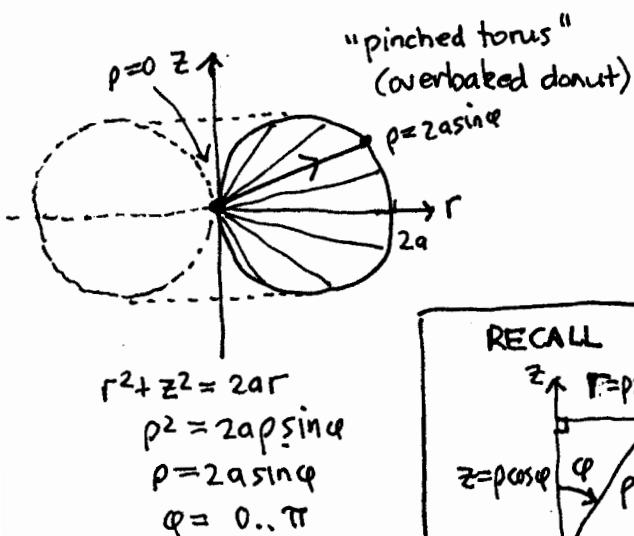
rz half plane ( $x^2 + y^2 + z^2 = r^2 + z^2$ )

a) sphere centered at origin



$$\begin{aligned}r^2 + z^2 &= a^2 \\&\rho^2 = a^2 \\\rho &= a \\&\phi = 0.. \pi\end{aligned}$$

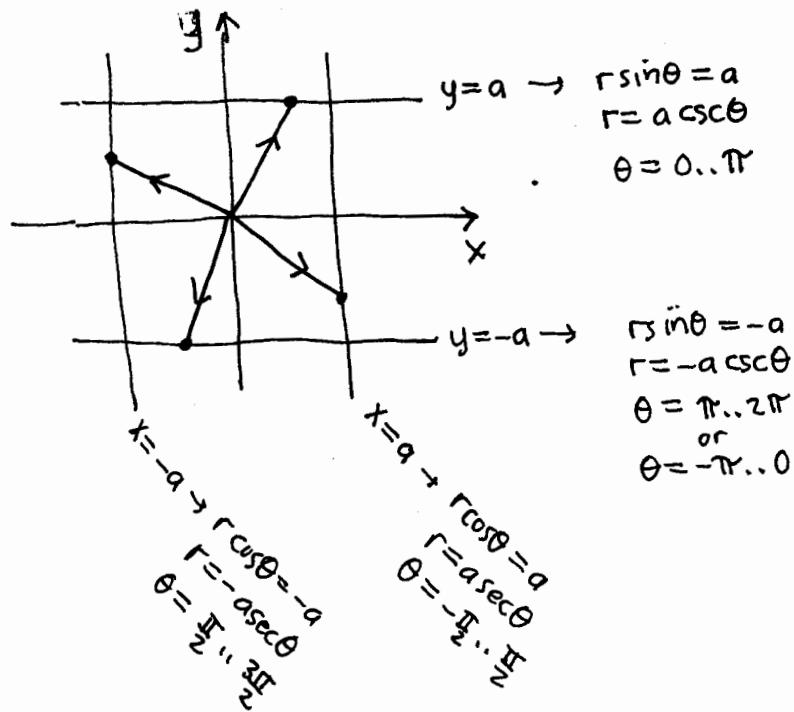
b) sphere centered on axis, tangent to origin ( $\phi$  interval between zeros of  $\rho$ )



simple lines/planes/cylinders in polar/cylindrical/spherical coordinates

assume  $a > 0$

xy plane (lines in xy space are planes in xyz space)



$r z$  half plane

