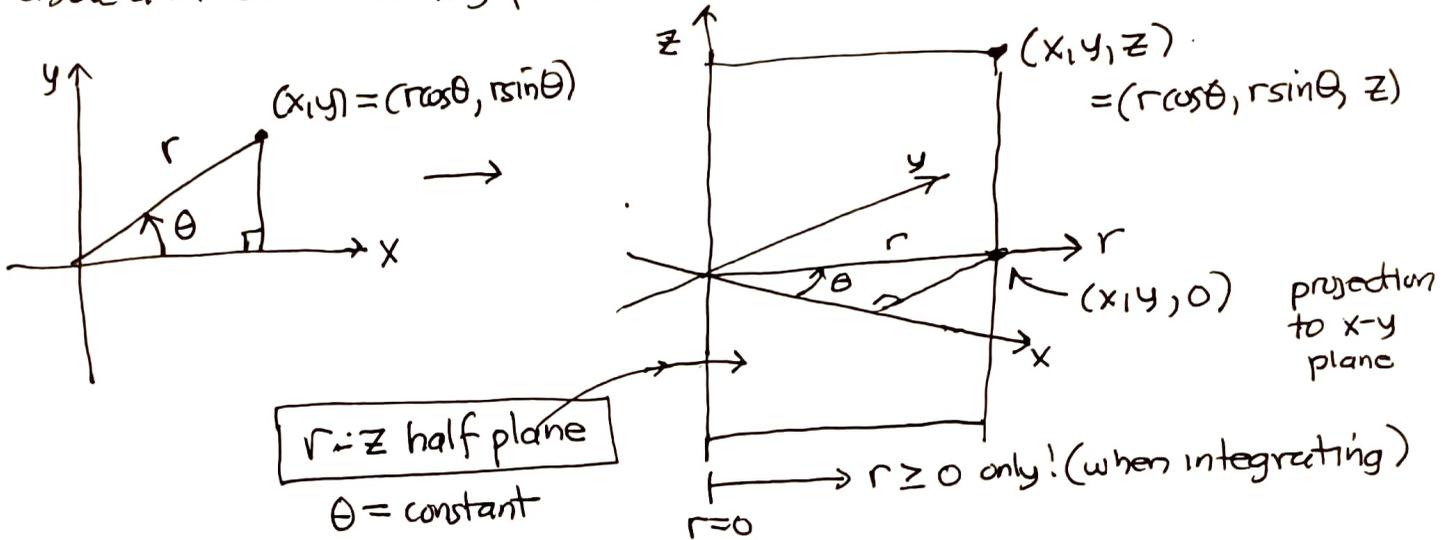


15.7-8a cylindrical and spherical coordinate integration

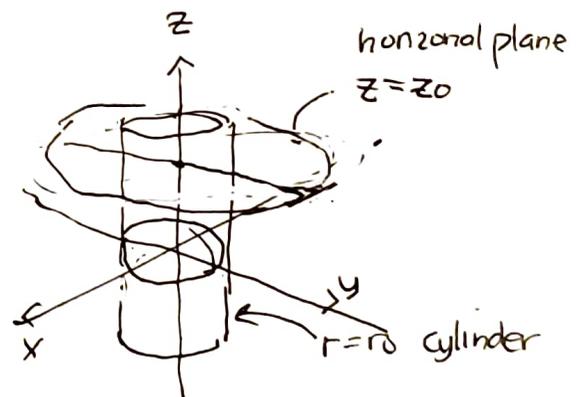
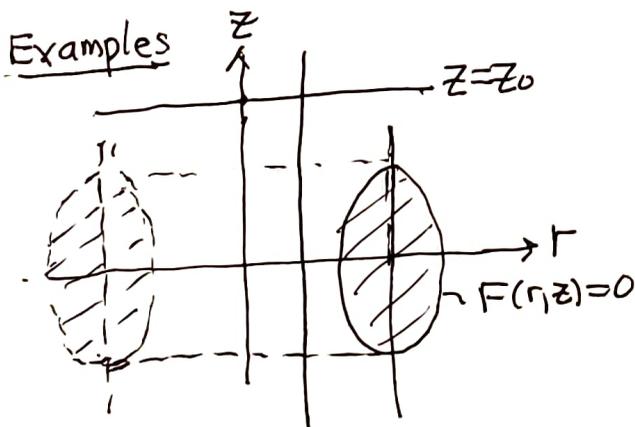
①

Polar coordinates in the plane remain valid in space in the sense that they have the same definitions in terms of (x,y) but now as functions of (x,y,z) which are independent of z , i.e., extend vertically above and below the $x-y$ plane $z=0$

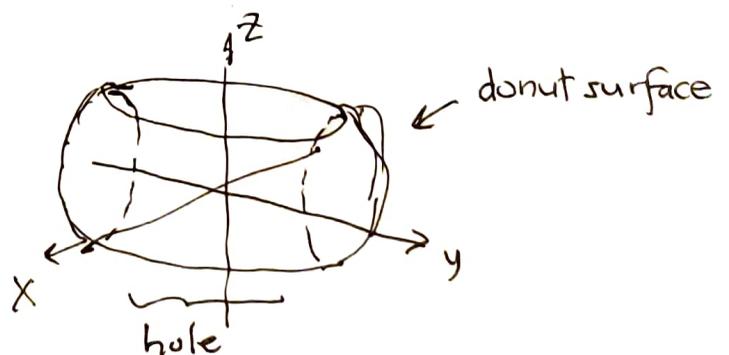


The triplet (r, θ, z) are called the cylindrical coordinates of (x,y,z) .
 ↑ ↑ we choose either $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$.
 ↓ we only allow $r \geq 0$

Any surface which is invariant under rotation about the vertical z -axis is independent of the angle θ , and so determined by a curve $F(r,z)=0$ in the $r-z$ half plane.



$r=r_0$
 reflected image for $\theta \Rightarrow \theta \pm \pi$
 $r \geq 0, \theta$ fixed



15.7-8a cylindrical and spherical integration

(2)

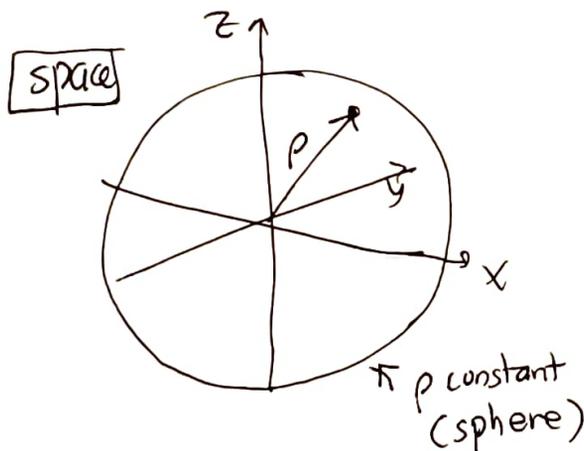
We have already been using cylindrical coordinates when we converted a triple integral

$$\iiint f(x, y, z) dz dy dx = \iiint f(r \cos \theta, r \sin \theta) \underbrace{r dz dr d\theta}_{dA}$$

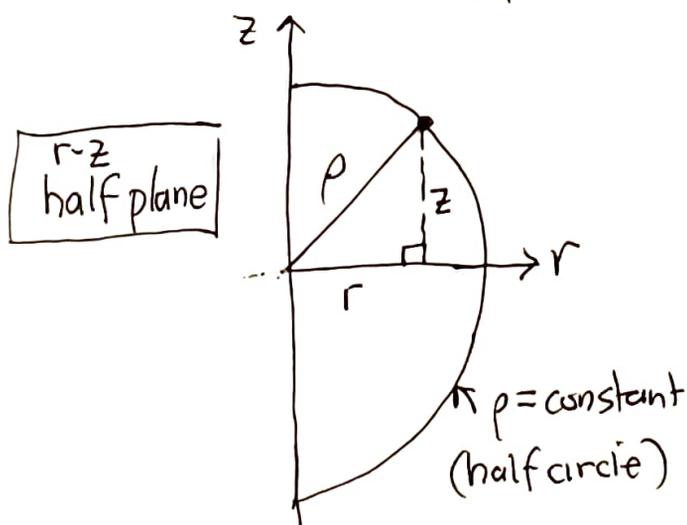
Cylindrical coordinates are appropriate to describe functions and regions of space which have simple representations in those coordinates because of simple properties of their rotation about the z-axis.

An alternative to cylindrical coordinates (adapted to rotations ABOUT AN AXIS) are spherical coordinates adapted to rotations ABOUT A POINT (the origin).

The new radial coordinate $\rho \geq 0$ measures distance from the origin instead of (horizontal) distance from the vertical z-axis like $r \geq 0$ does.



$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} && \text{distance formula} \\ &= \sqrt{r^2 + z^2} && \text{Pythag Thm in } r\text{-}z \text{ plane} \end{aligned}$$

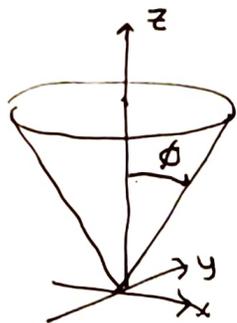


ρ is the radial coordinate in the r-z half plane

but what angle do we use?

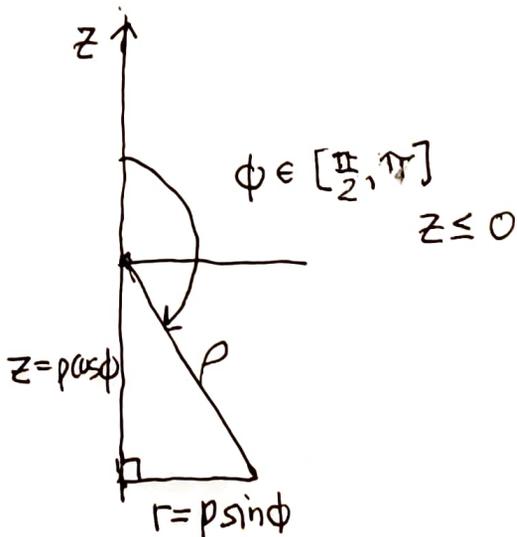
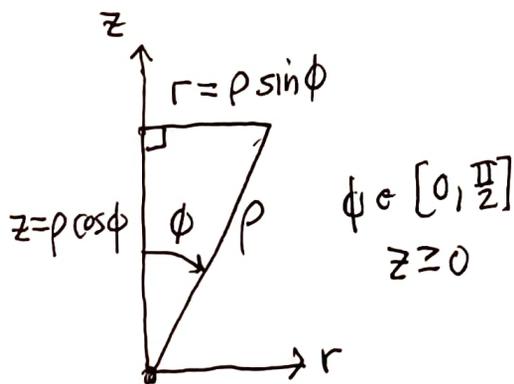
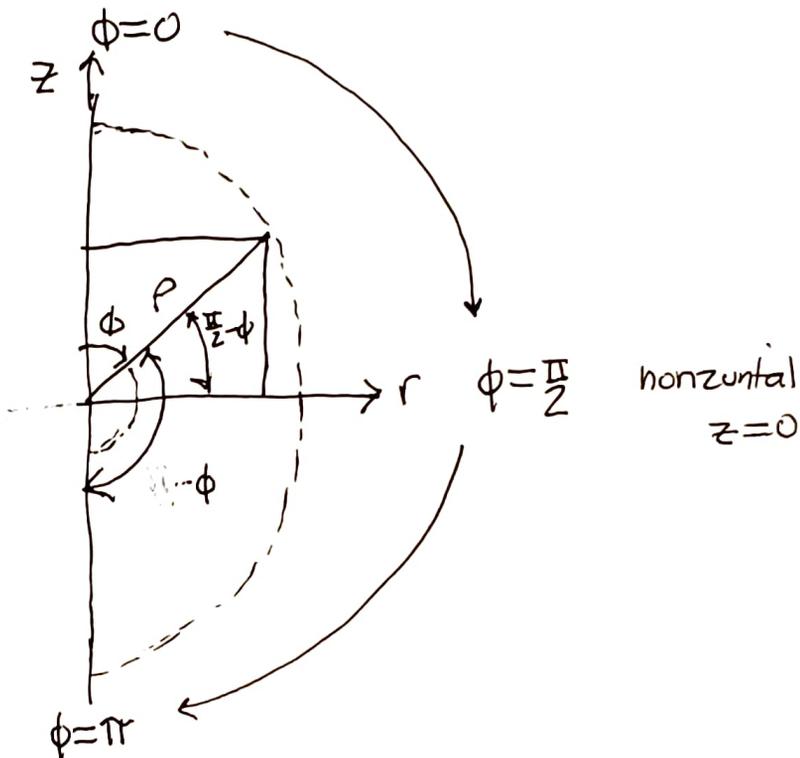
15.7-8a cylindrical and spherical coordinate integration (3)

Since the upward vertical direction is unique but there are 360° of different horizontal directions, it makes sense to introduce an angle with respect to 'up'.



$\phi = \text{constant}$ (cone)
 $0 \leq \phi \leq \pi$
 angle down from the upward vertical direction

In the r - z plane we just measure the angle in the clockwise direction from the upper vertical axis.



note $\cos \phi = z/\rho$
 $\phi = \arccos(z/\rho) \in [0, \pi] !!$

15.7-8 a cylindrical and spherical coordinate integration ④

spherical coordinates are just double polar coordinates in space
 The passage from Cartesian to cylindrical to spherical in two steps:

Cartesian	cylindrical	spherical
$x = r \cos \theta$	$= r \cos \theta$	$= \rho \sin \phi \cos \theta$
$y = r \sin \theta$	$= r \sin \theta$	$= \rho \sin \phi \sin \theta$
$z = z$	$= z$	$= \rho \cos \phi$

$x = r \cos \theta$
 $y = r \sin \theta$
 from
 "horizontal"
 (in x-y plane)

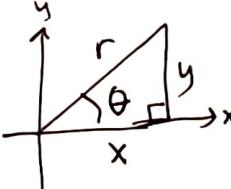
$\rho = \rho \sin \phi$
 $z = \rho \cos \phi$
 from
 vertical
 (in r-z plane)

Use these to re-express integrands and expressions/eqns from Cartesian to cylindrical and spherical coordinates.

$$f(x, y, z) = F(r, \theta, z) = G(\rho, \phi, \theta).$$

To go backwards we must invert the relationships.

$$r = \sqrt{x^2 + y^2}$$

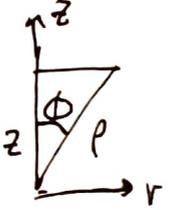
$$\tan \theta = \frac{y}{x}$$


$$\theta = \arctan(y, x) \in [-\pi, \pi]$$

or

$$\arctan\left(\frac{y}{x}\right) + \begin{cases} 0 & \text{I} \\ \pi & \text{II} \\ -\pi & \text{III} \\ 0 & \text{IV} \end{cases}$$

$$\rho = \sqrt{r^2 + z^2}$$

$$\cos \phi = \frac{z}{\rho}$$


$$\phi = \arccos\left(\frac{z}{\rho}\right) \in [0, \pi]$$

$\left(\sqrt{x^2 + y^2 + z^2}\right)$

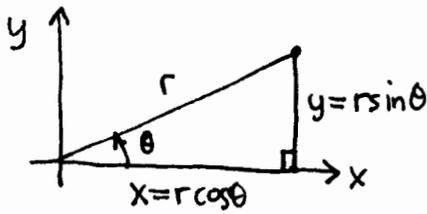
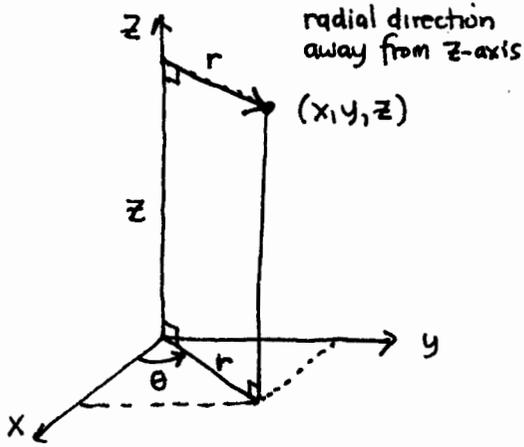
15.7-8a cylindrical and spherical coordinate integration

⑤

BUT we don't need formulas to find the cylindrical or spherical coordinates of a point (x, y, z) .

Give me 3 nonzero integers between 1 and 5 & my whiteboard ...

cylindrical coordinates (r, θ, z)



$r \geq 0, 0 \leq \theta \leq 2\pi$ (or $-\pi \leq \theta \leq \pi$)

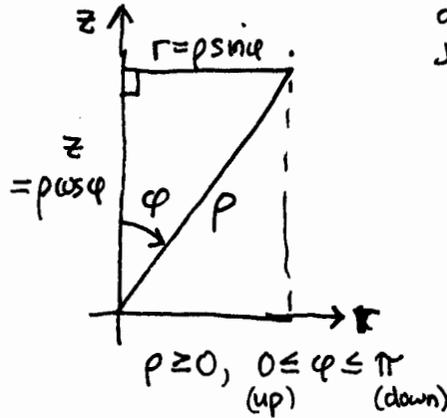
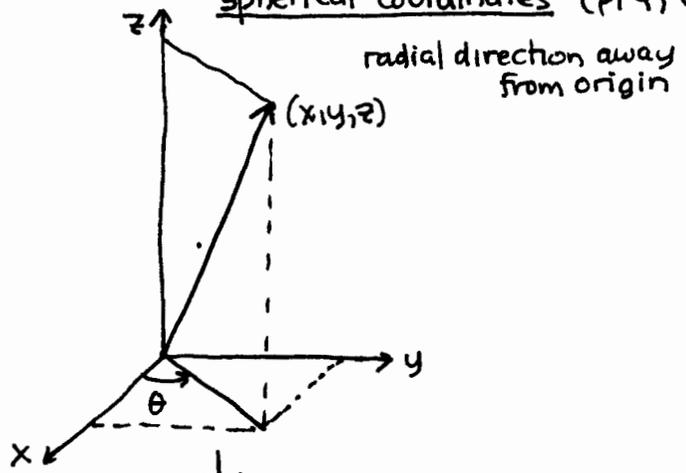
→ keep z , polar coords in xy plane

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned}$$

$x^2 + y^2 = r^2 \rightarrow r = \sqrt{x^2 + y^2} \geq 0$

$\frac{y}{x} = \tan \theta \rightarrow \theta = \arctan \frac{y}{x} + \begin{cases} 0; & \text{I, IV} \\ \pi; & \text{II} \\ -\pi; & \text{III} \end{cases}$ (quad)

spherical coordinates (ρ, φ, θ)



don't memorize just remember diagram

$$\begin{aligned} r &= \rho \sin \varphi \\ z &= \rho \cos \varphi \end{aligned}$$

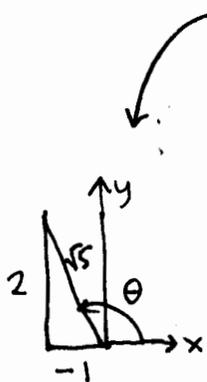
$$\begin{aligned} &= (\rho \sin \varphi) \cos \theta \\ &= (\rho \sin \varphi) \sin \theta \\ &= (\rho \cos \varphi) \end{aligned}$$

$x^2 + y^2 + z^2 = \rho^2 \rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$

$\cos \varphi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

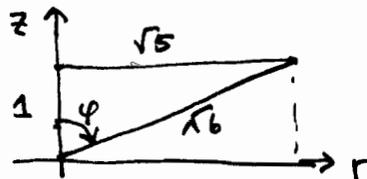
$\varphi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

Example. Find cyl/sph coords of $(x, y, z) = (-1, 2, 1)$.



$\tan \theta = \frac{2}{-1}$
 $\theta = \pi - \arctan 2$
 $(\approx 116.6^\circ)$

$(-1, 2, 1) \rightarrow \rho = \sqrt{1+4+1} = \sqrt{6}$
 or $\rho = \sqrt{5+1} = \sqrt{6}$
 $r = \sqrt{1+4} = \sqrt{5} \rightarrow z = 1$

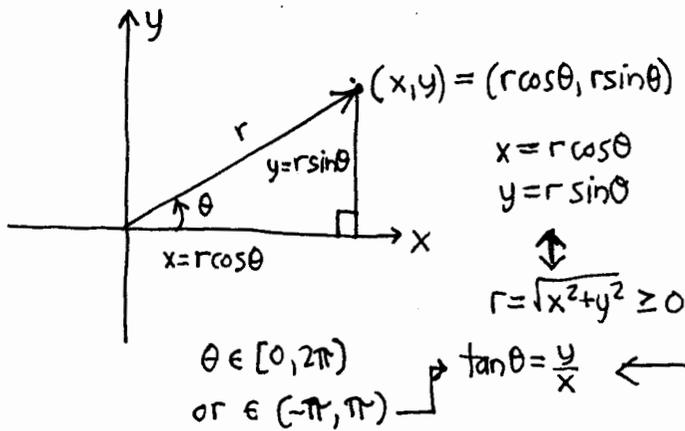


visualize think, use simple trig

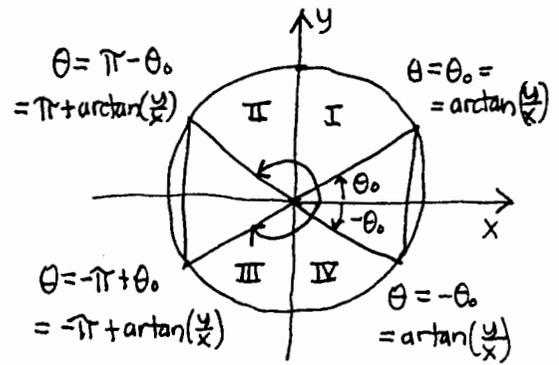
$\cos \varphi = \frac{1}{\sqrt{6}}$
 $\varphi = \arccos \frac{1}{\sqrt{6}}$
 $(\approx 65.9^\circ)$

polar, cylindrical, spherical coordinates

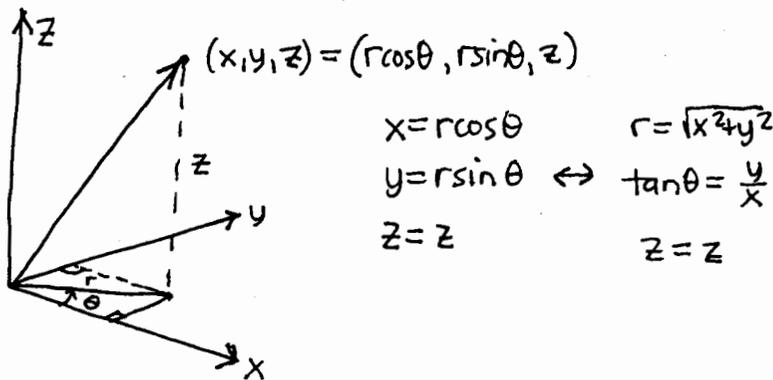
polar coords in plane



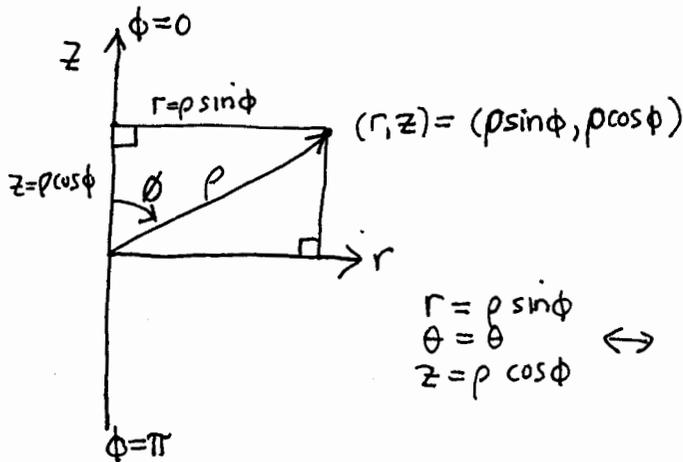
MAPLE: $\theta = \arctan(y, x) \in (-\pi, \pi]$



cyl coords



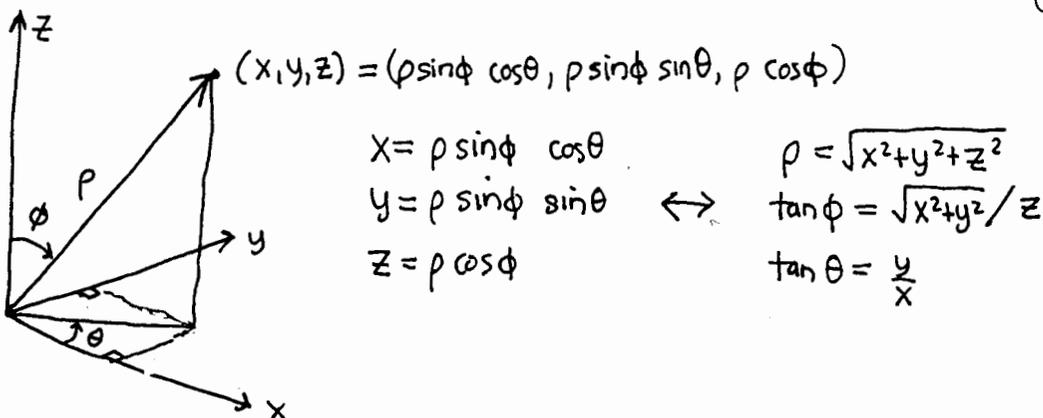
cyl to sphere



introduce polar coords in r - z plane (halfplane $r \geq 0$) of fixed θ but measured from the z -axis:
 $0 \leq \phi \leq \pi$

$\rho = \sqrt{r^2 + z^2} \geq 0$
 $\theta = \theta$
 $\tan \phi = \frac{r}{z} \leftrightarrow \phi = \begin{cases} \arctan(\frac{r}{z}) \in [0, \frac{\pi}{2}], & z > 0 \\ \pi + \arctan(\frac{r}{z}) \in (\frac{\pi}{2}, \pi], & z < 0 \end{cases}$
 $= \text{arccot}(\frac{z}{r}) \in [0, \pi)$

sph coords

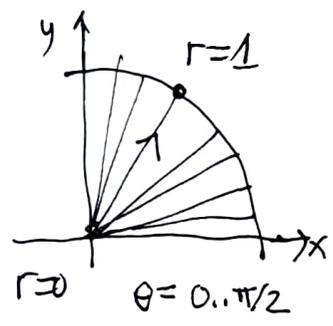
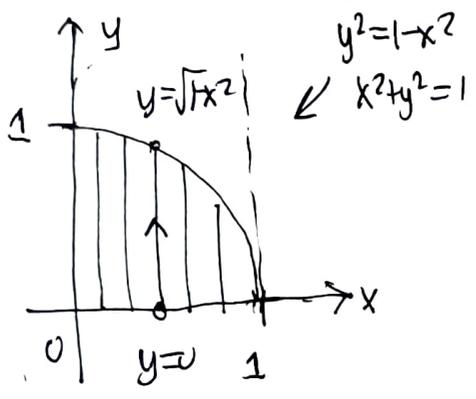


15.7-8a cylindrical and spherical coordinate integration

$$Q = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$$

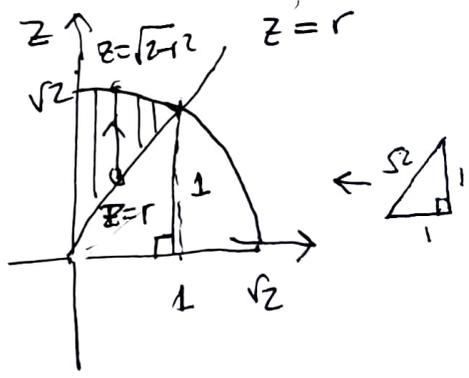
$$Q = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$$

$\downarrow \quad \downarrow \quad \downarrow$
 $r \cos \theta \quad r \sin \theta \quad r^2 dr d\theta$



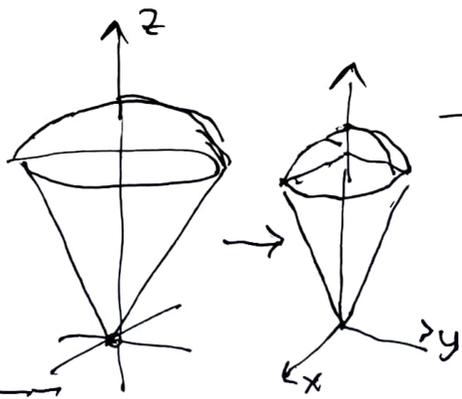
$z^2 = 2 - x^2 - y^2$
 $x^2 + y^2 + z^2 = 2$
 $\rho^2 = 2$
 $\rho = \sqrt{2}$
 Sphere (ZZU)
 (hemisphere)

$z^2 + r^2 = 2$
 Semikreis in
 r-z plane
 $z \geq 0$



1 quarter of:

Italian ke?
snow cone?



$$Q = \int_0^{\pi/2} \int_0^1 \int_r^{\sqrt{2-r^2}} r^3 \sin \theta \cos \theta \, dz \, dr \, d\theta$$

Dune!