Center of mass, centroid (geometric center) (1)

mass moments

1-d

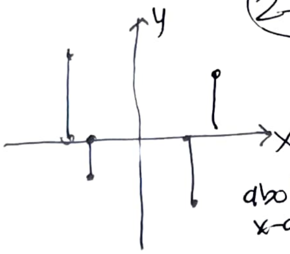
moment about origin, distance from $x=0$

product of masses times separation distances from point, axis, plane summed together.

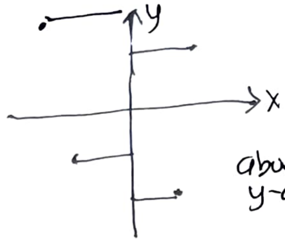
↓

continuous distributions integrate mass density against separations

2-d



about x-axis

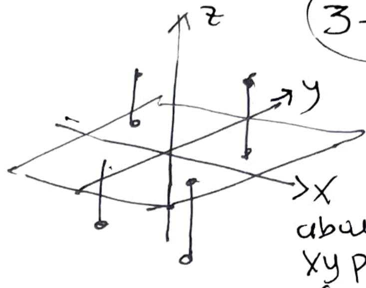


about y-axis

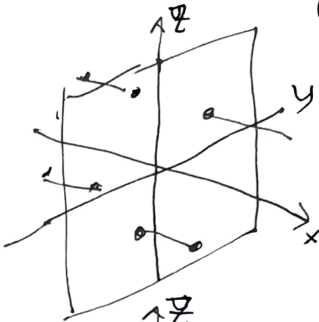
moments about axes

distances from $y=0$, $x=0$

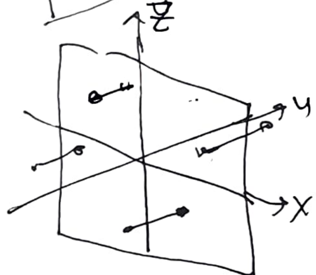
3-d



about xy plane ($z=0$)



about yz plane ($x=0$)



about xz plane ($y=0$)

distances from $z=0$, $y=0$, $x=0$

$M=\int \rho\, dx$ linear density: mass/length

$M=\iint \rho\, dA$ surface density: mass/area

$M=\iiint \rho\, dV$ volume density: mass/volume

3-d mass moments

$$M_{yz}=\iiint \rho\, x\, dV, \quad M_{xz}=\iiint \rho\, y\, dV, \quad M_{xy}=\iiint \rho\, z\, dV$$

about $x=0$, about $y=0$, about $z=0$

center of mass:

weighted average of position vector against fractional mass density

$$\iiint \left(\frac{\rho}{M}\right) dV = \frac{\iiint \rho\, dV}{M} = \frac{M}{M} = 1 \qquad \text{(similar in 1-d, 2-d)}$$

$$\langle \bar{x},\bar{y},\bar{z}\rangle = \iiint \langle x,y,z\rangle \frac{\rho}{M}\, dV = \frac{\iiint \langle x,y,z\rangle \rho\, dV}{\iiint \rho\, dV}$$

$\rho=\rho_0$ constant $\rightarrow$ ρ_0 cancels out of quotient, get geometric center = centroid

equivalent to set $\rho_0=1$:

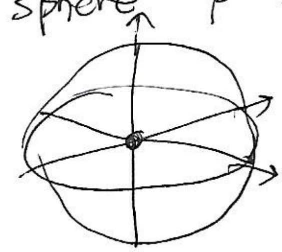
$$\langle V, M_{yz}, M_{xz}, M_{xy}\rangle = \iiint \langle 1,x,y,z\rangle\, dV$$

divide these by first integral to get centroid

c.o.m.: $\langle \bar{x},\bar{y},\bar{z}\rangle = \dfrac{\langle M_{yz}, M_{xz}, M_{xy}\rangle}{M}$, $M \longrightarrow V$ for centroid

Center of mass, Centroid (2)

homogeneous sphere $\rho = \rho_0 \rightarrow$ centroid = "center" of sphere (obvious!)



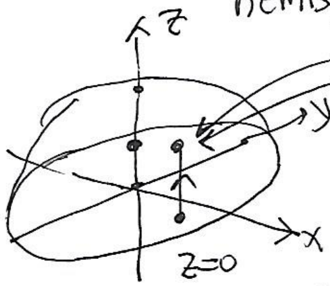
$\iiint x \, dV = 0$ since $x > 0$ balances $x < 0$ by symmetry

etc.

$x^2 + y^2 + z^2 = a^2$, radius a.

hemisphere $H \longrightarrow$ expect centroid to still be on z-axis but below halfway point to pole.

$z = \sqrt{a^2 - x^2 - y^2}$

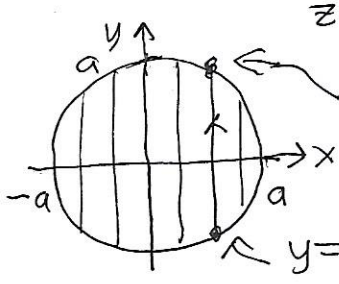


$z = 0$

$$\iiint f \, dV$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} f \, dz \, dy \, dx$$

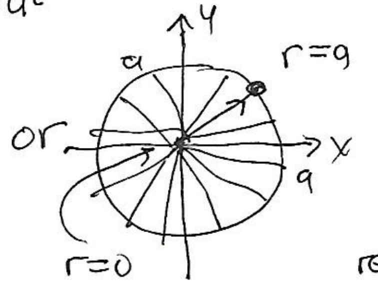
$z = 0$: $x^2 + y^2 = a^2$



$y = \sqrt{a^2 - x^2}$

$y = -\sqrt{a^2 - x^2}$

or



$r = a$

$r = 0$

$$= \int_0^{2\pi} \int_0^{a} \int_0^{\sqrt{a^2 - r^2}} f \, r \, dz \, dr \, d\theta$$

result: $\langle \bar{x}, \bar{y}, \bar{z} \rangle = \langle 0, 0, \frac{3a}{8} \rangle$

$= \langle 0, 0, 0.375a \rangle$

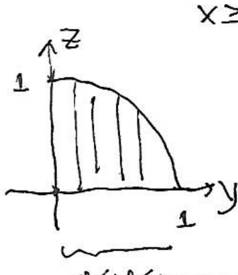
less than half as expected

$\boxed{15.6.23}$

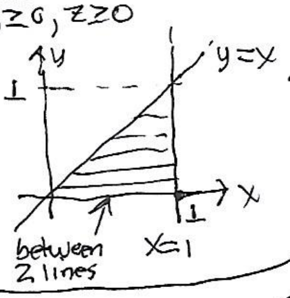
wedge cut from cylinder : $\rho = 1 \rightarrow$ centroid

Region in first octant bounded by cylinder $y^2 + z^2 = 1$ and the planes $y = x$, $x = 1$.

$x \geq 0, y \geq 0, z \geq 0$

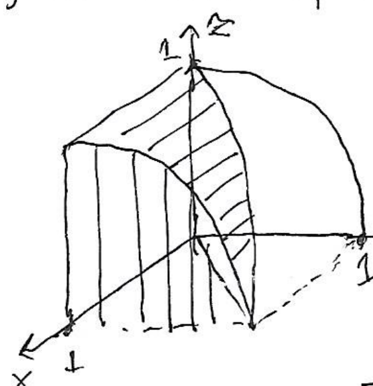


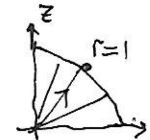
$0 \leq y \leq 1$



between 2 lines

$x = 1$



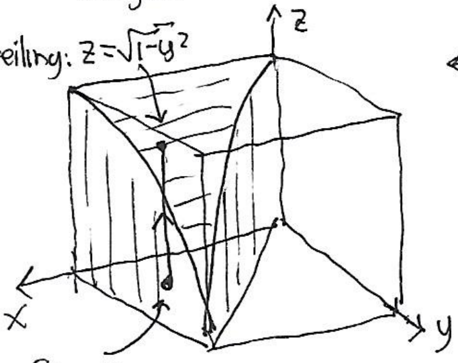


$y = r\cos\theta$

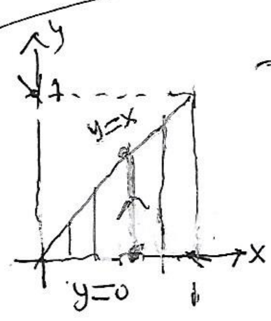
$z = r\sin\theta$

rotate view

ceiling: $z = \sqrt{1 - y^2}$



floor $z = 0$



$$\iiint_R f \, dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} f \, dz \, dy \, dx$$

$$\text{OR} \quad = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_y^1 f \, dx \, dz \, dy$$

(x first order)

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_{r\cos\theta}^1 f \, r \, dx \, dr \, d\theta$$

$z = 0 .. \sqrt{1-y^2}$ while $y = 0 .. x$

while $x = 0 .. 1$

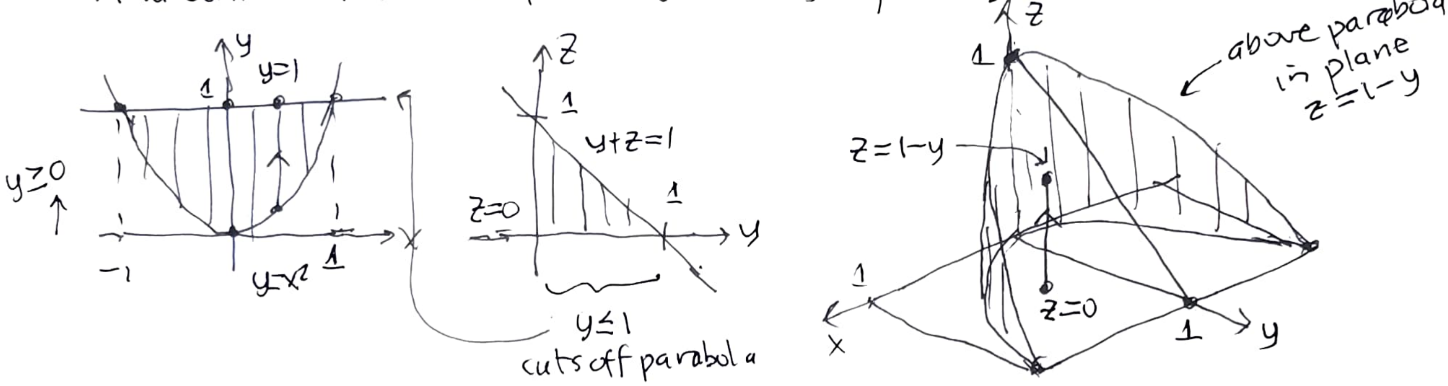
result: $\langle \bar{x}, \bar{y}, \bar{z} \rangle \approx \langle .45, .30, .46 \rangle$

see Maple worksheet

values make sense

Center of Mass, Centroid (3)

Region R: enclosed by $y=x^2$, $z=0$, $y+z=1$
Find centroid and center of mass if density $\rho \propto y$. ($\to \rho = ky$)



reflection symmetry across vertical plane $x=0$ (above y axis) of both region & density function. ∴ must lie in $x=0$ plane.
Expect centroid to be closer to $y=0$ plane, but density pushes outward in y direction.

$z=0..1-y$ while $y=x^2..1$ while $x=-1..1$

$$\iiint_R f\,dV = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f\,dz\,dy\,dx$$

"V" for volume moments!

centroid:

$$\langle V, V_{yz}, V_{xz}, V_{xy}\rangle = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} \langle 1, x, y, z\rangle \; dz\,dy\,dx$$

$$= \left\langle \tfrac{8}{15}, 0, \tfrac{8}{35}, \tfrac{16}{105}\right\rangle$$

$$\langle \bar{x}, \bar{y}, \bar{z}\rangle_c = \frac{\langle V_{yz}, V_{xz}, V_{xy}\rangle}{V} = \left\langle 0, \tfrac{3}{7}, \tfrac{2}{7}\right\rangle$$

$$\approx \langle 0, 0.43, 0.29\rangle$$

center of mass:

$$\langle M, M_{yz}, M_{xz}, M_{xy}\rangle = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} \langle 1, x, y, z\rangle\, ky\, dz\,dy\,dx$$

$$= k\left\langle \tfrac{8}{35}, 0, \tfrac{8}{63}, \tfrac{16}{315}\right\rangle$$

$$\langle \bar{x}, \bar{y}, \bar{z}\rangle = \frac{\langle M_{yz}, M_{xz}, M_{xy}\rangle}{M} = \left\langle 0, \tfrac{5}{9}, \tfrac{2}{9}\right\rangle$$

$$\approx \langle 0, 0.56, 0.22\rangle$$

0.43 → 0.56, 0.29 → 0.22

centroid moves along y, down (because of wedge) makes sense.

(see Maple Worksheet)