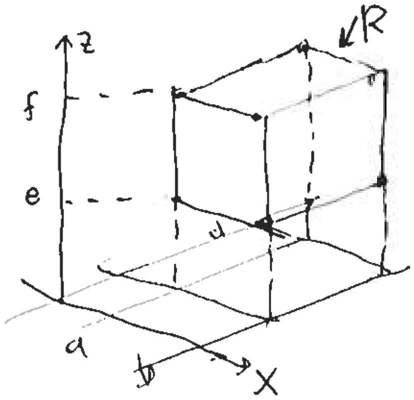
triple integrals (1)



R : rectangular box

$[a,b] \times [c,d] \times [e,f]$

$\sim x$: $i = 1..m$, $\Delta x = (b-a)/m$; $x_i = a + i\Delta x$; x_i^* in ith interval

$\sim y$: $j = 1..n$, $\Delta y = (d-c)/n$; $y_j = c + j\Delta y$; y_j^* in jth interval

$\sim z$: $k = 1..p$, $\Delta z = (f-e)/p$; $z_k = e + k\Delta z$; z_k^* in kth interval

Volume:

$\Delta V = \Delta x\, \Delta y\, \Delta z$

(x_i^*, y_j^*, z_k^*)

in i-j-k box

$$\iiint_R f(x,y,z)\,dV = \lim_{\substack{m\to\infty\\ n\to\infty\\ p\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\,\Delta V$$

interchange limits and sums

$$= \lim_{m\to\infty}\sum_{i=1}^{m}\Bigg(\underbrace{\lim_{n\to\infty}\sum_{j=1}^{n}\Big(\underbrace{\lim_{p\to\infty}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\,\Delta z}_{\int_e^f f(x_i^*, y_j^*, z)\,dz}\Big)\Delta y}_{\int_c^d \left(\int_e^f f(x_i^*, y, z)\,dz\right)dy}\Bigg)\Delta x$$

$$= \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

evaluate from inside out: nested integrals (order does not matter)

Example R: $0 \le x \le 1$, $-1 \le y \le 2$, $0 \le z \le 3$

$$\iiint_R xyz^2\,dV = \int_0^1 \int_{-1}^2 \int_0^3 xyz^2\,dz\,dy\,dx$$

$$\int_0^3 xyz^2\,dz = xyz^3/3 \Big|_{z=0}^{z=3} = 9xy$$

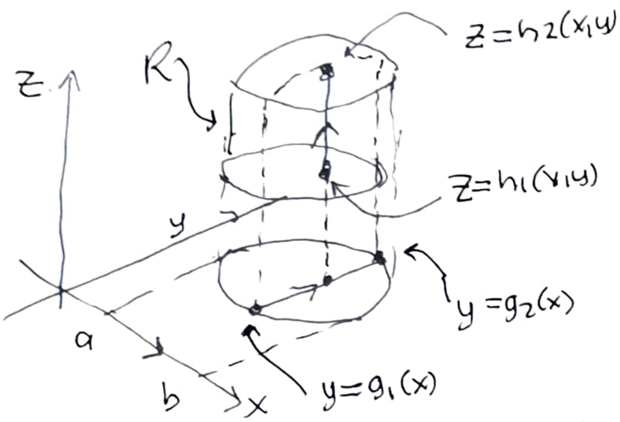
$$\int_{-1}^{2} 9xy\,dy = \frac{9xy^2}{2}\Big|_{y=-1}^{y=2} = \frac{9x}{2}(4-1) = \frac{27x}{2}$$

$$\int_0^1 \frac{27x}{2}\,dx = \frac{27x^2}{4}\Big|_0^1 = \boxed{\frac{27}{4}}$$

details not important — easy to do

triple integrals (2)

Nonconstant limits — inner integral limits can depend on remaining variables not yet integrated over



$z = h_1(x,y) \,..\, h_2(x,y)$
while $y = g_1(x) \,..\, g_2(x)$
while $x = a \,..\, b$

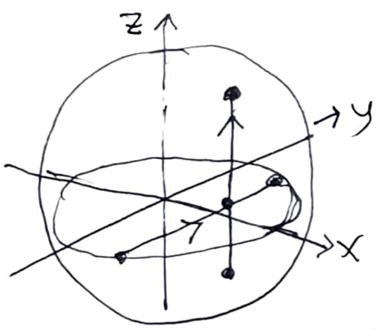
enclose space between 2 surface graphs
above the area between 2 curve graphs
over an interval of the final axis

$$\iiint_R f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

sweeps out vertical line cross-section
sweeps line cross-section to vertical plane cross-section
sweeps vertical plane cross-section across solid region

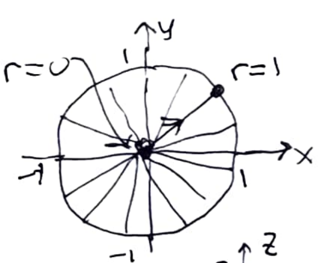
Example R is interior of sphere $\overbrace{x^2+y^2}^{r^2}+z^2=1 \leftrightarrow z=\pm\sqrt{1-x^2-y^2}$

projects to $x^2+y^2=1$ at $z=0$ projects to $x=\pm 1$ when $y=0$
$\hookrightarrow y=\pm\sqrt{1-x^2}$



$z = \overbrace{-\sqrt{1-x^2-y^2}}^{\text{floor}} \,..\, \overbrace{\sqrt{1-x^2-y^2}}^{\text{ceiling}}$ while $y=-\sqrt{1-x^2} \,..\, \sqrt{1-x^2}$
while $x=-1 \,..\, 1$

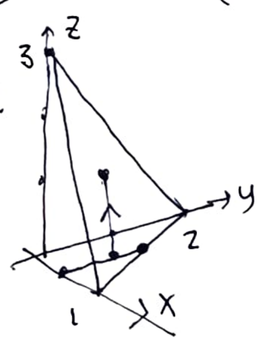
$$\iiint_R f\,dV = \underbrace{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}}_{\text{Symmetry}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \underbrace{f(xyz)}_{f(r\cos\theta,\, r\sin\theta,\, z) \,\equiv\, F(r,\theta,z)}\,dz\,dy\,dx$$



$$\begin{pmatrix} r^2+z^2=1 \\ z=\pm\sqrt{1-r^2} \end{pmatrix} \quad = \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} F(r,\theta,z)\,r\,dz\,dr\,d\theta$$

Example

$\iiint_R f\,dV$?



$\underbrace{x+\frac{y}{2}+\frac{z}{3}=1}_{\text{ceiling}}$, $\underbrace{x=0, y=0}_{\text{side walls}}$, $\underbrace{z=0}_{\text{floor}}$

$z=3(1-x-y/2)=0$
$\hookrightarrow x+\frac{y}{2}=1 \rightarrow y=2(1-x)$

$$\boxed{\int_0^1\int_0^{2(1-x)}\int_0^{3(1-x-y/2)} f\,dz\,dy\,dx}$$

triple integrals (4)

changing order of integration

some 3-d regions allow more than 1 integration order

maximum # re-orders is $3 \cdot 2 \cdot 1 = 3! = 6$ permutations of x, y, z

z-first then a region of x-y plane allowing either order

$$\underbrace{\iiint f\, dz}\ \underbrace{dy\, dx}_{\substack{\text{or} \\ dx\, dy}}$$

y-first then a region of x-z plane allowing either order

$$\underbrace{\iiint f\, dy}\ \underbrace{dz\, dx}_{\substack{\text{or} \\ dx\, dz}}$$

x-first then a region of y-z plane allowing either order

example: unit sphere: $\underbrace{x^2+y^2+z^2}=1$

symmetric under all permutations of x, y, z

all 6 orders of iteration possible

write down one, then just permute variables!

[previous example did $dz\,dy\,dx$ order]

example: cut off first octant by a plane: $\underbrace{\frac{x}{2}+\frac{y}{4}+\frac{z}{6}}=1$

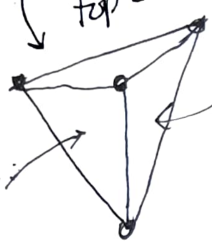
linear condition, all variables on same footing.

6 orders of integration possible

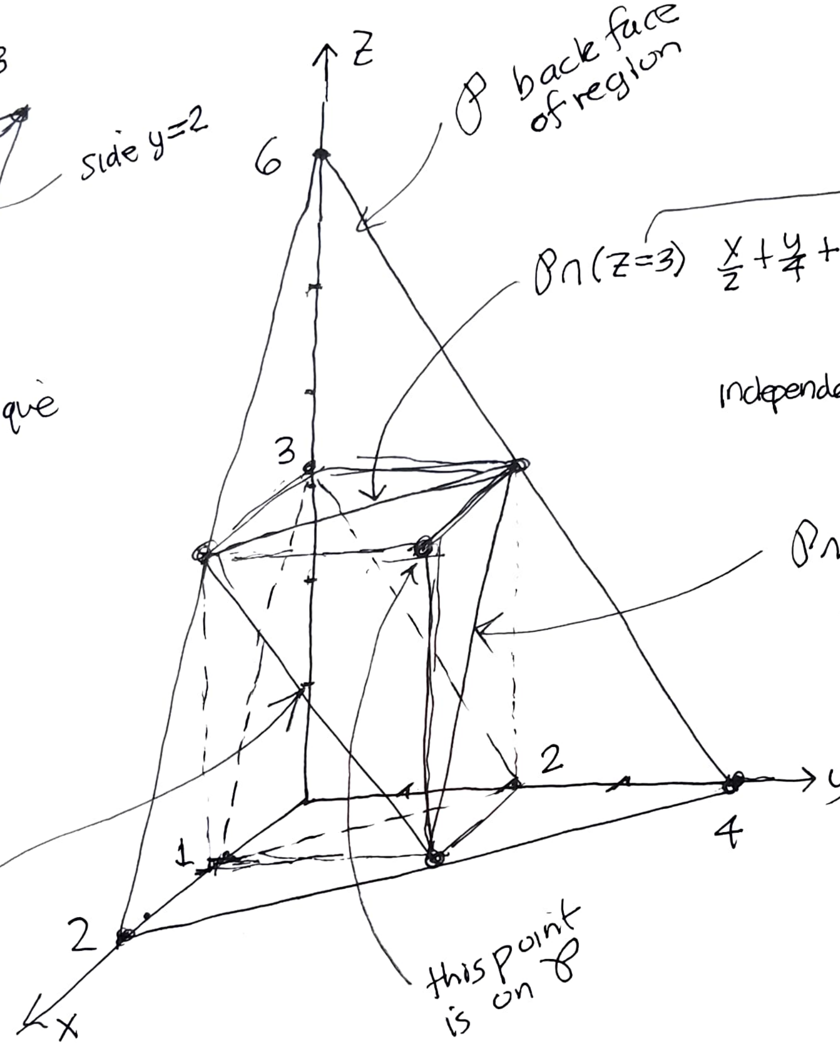
Example region bounded by 4 planes "tetrahedron" $x=1,\ y=2,\ z=3,\ \underbrace{\frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1}_{\text{"}P\text{"}}$

axis intercepts:
$x=2\ (y=z=0)$
$y=4\ (x=z=0)$
$z=6\ (x=y=0)$

P is the back



we need to describe 3 oblique edges to project onto the 3 coord planes



$P \cap (z=3)$: $\frac{x}{2}+\frac{y}{4}+\frac{3}{6}=1 \to \frac{x}{2}+\frac{y}{4}=\frac{1}{2}$

$x+\frac{y}{2}=1$

independent of z, projects onto xy plane

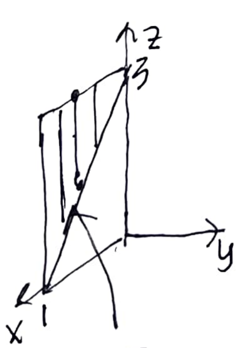
$P \cap (y=2)$:
$\frac{x}{2}+\frac{2}{4}+\frac{z}{6}=1$
$\frac{x}{2}+\frac{z}{6}=\frac{1}{2}$

$x+\frac{z}{3}=1$

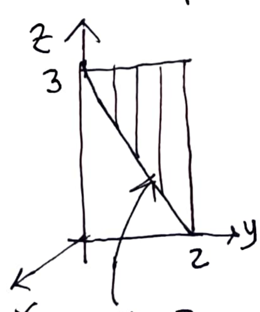
independent of y projects onto xz plane

$P \cap (x=1)$: $\frac{1}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{y}{4}+\frac{z}{6}=\frac{1}{2}$

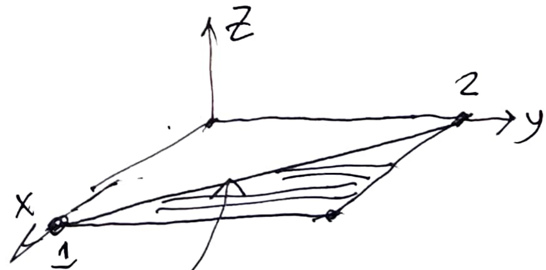
$\frac{y}{2}+\frac{z}{3}=1$ independent of x projects onto yz plane



$x+\frac{z}{3}=1$



$\frac{y}{2}+\frac{z}{3}=1$



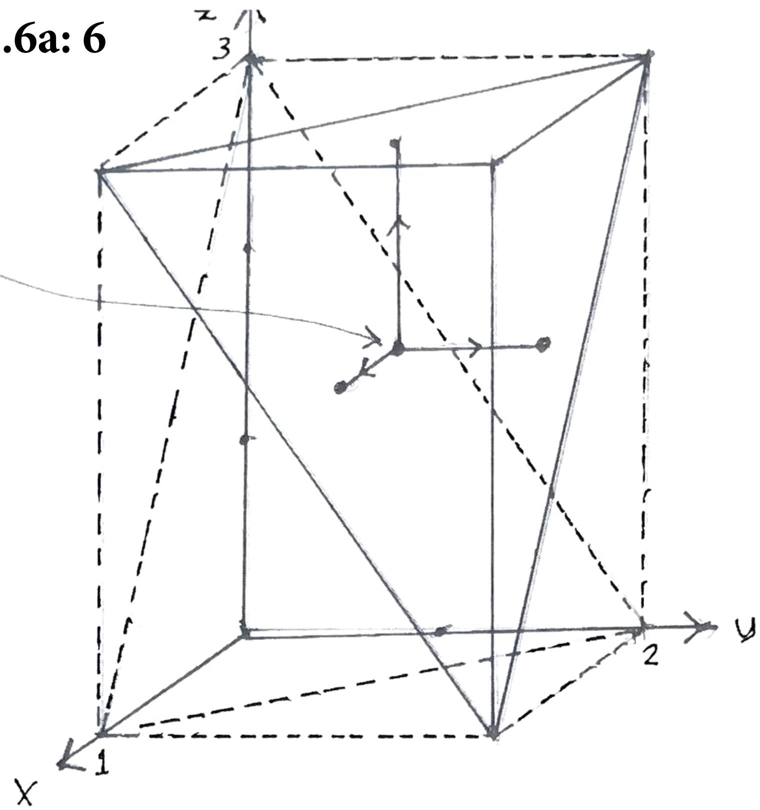
$x+\frac{y}{2}=1$

15.6a: 6

tetrahedron:
$x=1,\ y=2,\ z=3$
$\frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1$

solve for each variable in turn:
$x=2\left(1-\frac{y}{4}-\frac{z}{6}\right)$
$y=4\left(1-\frac{x}{2}-\frac{z}{6}\right)$
$z=6\left(1-\frac{x}{2}-\frac{y}{4}\right)$
starting values for 3 partial integrations

ending values:
$x=1$
$y=2$
$z=3$
these give lower & upper limits for innermost integration



6 way iteration example

innermost integral moves along coord axes from the oblique plane outward from origin

outer double integral is done over projection of solid to coordinate planes of remaining variables

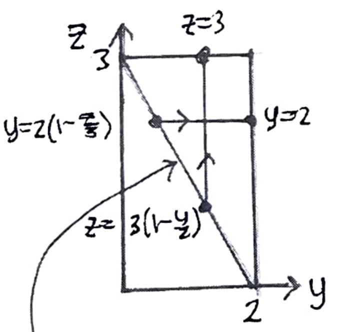
dashed triangles are those projections.

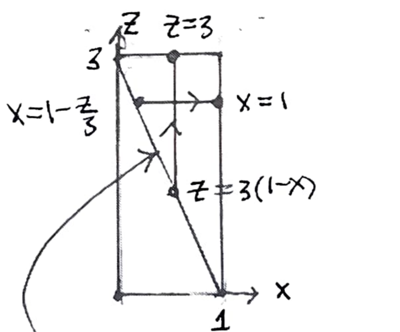
intersections of faces are common solutions of pairs of equations, eliminate one variable:

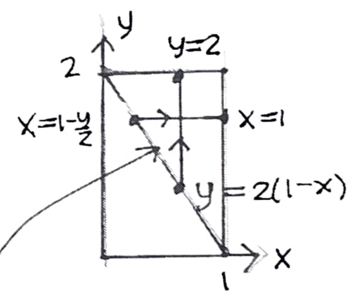
$x=1 \to \frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{1}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{y}{2}+\frac{z}{3}=1$ project solid onto yz plane

$y=2 \to \frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{x}{2}+\frac{2}{4}+\frac{z}{6}=1 \to x+\frac{z}{3}=1$ project solid onto xz plane

$z=3 \to \frac{x}{2}+\frac{y}{4}+\frac{z}{6}=1 \to \frac{x}{2}+\frac{y}{4}+\frac{3}{6}=1 \to x+\frac{y}{2}=1$ project solid onto xy plane







$\frac{y}{2}+\frac{z}{3}=1 \longrightarrow z=3\left(1-\frac{y}{2}\right)$, $\ y=2\left(1-\frac{z}{3}\right)$

$x+\frac{z}{3}=1 \longrightarrow z=3(1-x)$, $\ x=1-\frac{z}{3}$

$x+\frac{y}{2}=1 \longrightarrow y=2(1-x)$, $\ x=1-\frac{y}{2}$

$$\left(\int_{2\left(1-\frac{y}{4}-\frac{z}{6}\right)}^{1} f\,dx\right) \quad \left(\int_{4\left(1-\frac{x}{2}-\frac{z}{6}\right)}^{2} f\,dy\right) \quad \left(\int_{6\left(1-\frac{x}{2}-\frac{y}{4}\right)}^{3} f\,dz\right)$$

$$\int_0^2\int_{3\left(1-\frac{y}{2}\right)}^{3} (\quad)\,dz\,dy \qquad \int_0^1\int_{3(1-x)}^{3} (\quad)\,dz\,dx \qquad \int_0^1\int_{2(1-x)}^{2} (\quad)\,dy\,dx$$

$$\int_0^3\int_{2\left(1-\frac{z}{3}\right)}^{2} (\quad)\,dy\,dz \qquad \int_0^3\int_{1-\frac{z}{3}}^{1} (\quad)\,dx\,dz \qquad \int_0^2\int_{1-\frac{y}{2}}^{1} (\quad)\,dx\,dy$$

when $f(xyz)=1$, all integrals give the volume: 1.