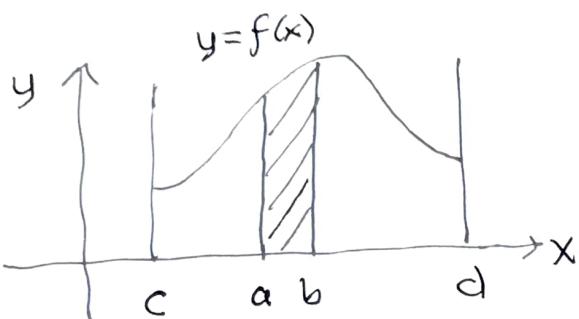


Probability Theory (1)

15.4b: 1

Review first Calc II 1-d probability distributions \rightarrow generalize to 2-d

$$P(a \leq x \leq b) = \int_a^b f(x) dx \quad \text{where } \int_c^d f(x) dx = 1, f(x) \geq 0$$



(c, d) closed interval or
semi-infinite: $[0, \infty)$
or infinite: $(-\infty, \infty)$

area under graph of f over interval
 $[a, b]$ equals probability that x assumes
a value in this interval
total area on allowed interval is 1.

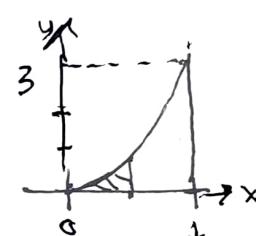
Fact: given any nonnegative function on a closed, semi-infinite or infinite interval can be made into a probability distribution function (PDF!) by simply dividing by its integral over that interval (provided that it is finite!).

$$\text{Example 1. } g(x) = x^2 \text{ on } 0 \leq x \leq 1. \quad \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

$$f(x) = \frac{x^2}{\frac{1}{3}} = 3x^2.$$

$$P(0 \leq x \leq \sqrt[3]{2}) = \int_0^{\sqrt[3]{2}} 3x^2 dx = x^3 \Big|_0^{\sqrt[3]{2}} = \frac{1}{8}$$

This gives more weight to larger values of x .



$$\text{Example 2. } g(x) = x(1-x) = x - x^2, 0 \leq x \leq 1, \quad \int_0^1 x - x^2 dx = \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{6}$$

$$f(x) = 6x(1-x)$$

$$P(\frac{1}{4} \leq x \leq \frac{3}{4}) = \int_{1/4}^{3/4} 6x(1-x) = \dots = \frac{11}{16} \approx 0.69$$

about $2/3$. 69% probability

This gives more weight to central values of interval.

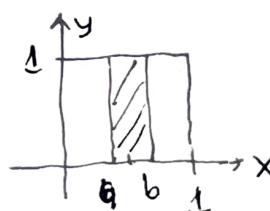


$$\text{Example 3. } g(x) = C = \text{constant}, \quad 0 \leq x \leq 1, \quad \int_0^1 C dx = Cx \Big|_0^1 = C$$

$$f(x) = 1$$

$$P(a \leq x \leq b) = \int_a^b 1 dx = b-a$$

equal weight at all values of x .



Probability Theory (2)

15.4b: 2

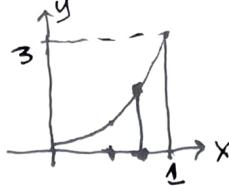
The "expected value" (mean value, average value) of the variable x with PDF $f(x)$ is the integral of x using the PDF as a weighting function:

$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x) dx$$

over entire allowed interval of x values

Example 1.

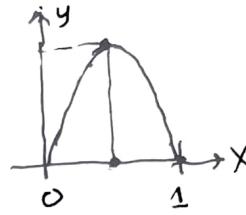
$$\langle x \rangle = \int_0^1 x(3x^2) dx = \frac{3}{7} = 0.75$$



expected value pushed to right of center

Example 2.

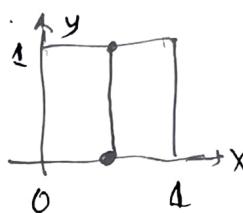
$$\langle x \rangle = \int_0^1 x(6x(1-x)) dx = \frac{1}{2}$$



expected value in center
(symmetry!)

Example 3.

$$\langle x \rangle = \int_0^1 x dx = \frac{1}{2}$$



ditto!

Poisson Distribution $g(x) = C e^{-kx}, 0 \leq x < \infty, C > 0, k > 0$

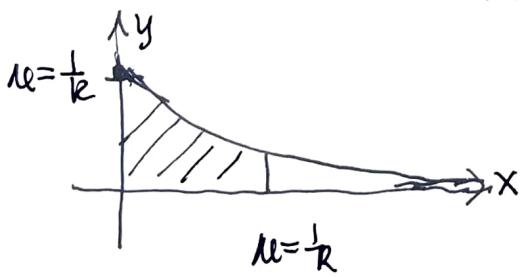
$$1 = \int_0^\infty C e^{-kx} dx = \frac{C}{-k} e^{-kx} \Big|_0^\infty = \frac{C}{k} (1 - e^{-k\infty}) = \frac{C}{k} \rightarrow C = k$$

$$f(x) = k e^{-kx}$$

$$\mu \equiv \langle x \rangle = \int_0^\infty x (k e^{-kx}) dx = -\frac{(kx+1)}{k} e^{-kx} \Big|_0^\infty = \frac{1}{k} (1 - \lim_{x \rightarrow \infty} \frac{(x+1)}{e^{kx}}) = \frac{1}{k}$$

$$\text{Rewrite } f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$$

directly represents PDF in terms of its expected value
instead of a constant without direct interpretation.



$$P(0 \leq x \leq \mu) = \int_0^\mu \frac{1}{\mu} e^{-\frac{x}{\mu}} dx = \dots = 1 - e^{-1} \approx 0.63$$

about 2/3 probability it assumes a value less than its expected value
(so about 1/3 probability its value exceeds μ)

Probability Theory (3)

15.4b: 3

change of variable for definite integrals can be very useful!

Poisson: $P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} e^{-x/\mu} \frac{dx}{\mu} = \int_{u_1}^{u_2} e^{-u} du$

define $u = \frac{x}{\mu}$ (use multiples of μ for tickmarks on graph)

$$du = \frac{dx}{\mu}, \quad x = x_1 \rightarrow u = \frac{x_1}{\mu} \equiv u_1$$

$$x = x_2 \rightarrow u = \frac{x_2}{\mu} \equiv u_2$$

$$= P(0_1 \leq u \leq u_2)$$

This is the standard Poisson distribution (equivalent to setting $\mu = 1$)

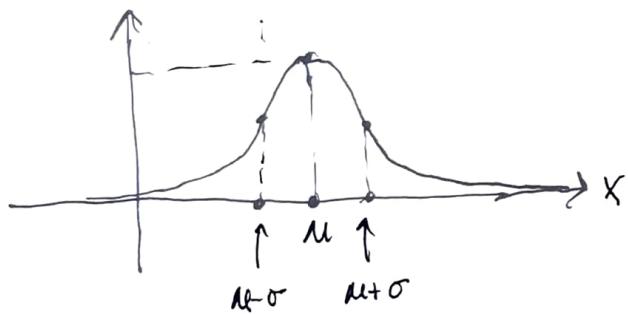
Any question about x can be answered in terms of the equivalent question about the corresponding "standard variable" u , on $0 \leq u < \infty$, same semi-infinite interval.

The normal distribution (bell curve)

$$\text{PDF} = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/\sigma^2}, \quad \sigma > 0, -\infty \leq \mu \leq \infty,$$

"standard deviation"
measures spread about center

expected value
locates peak of bell curve.



$$\text{standard variable: } u = \frac{x-\mu}{\sigma}$$

= distance from expected value measured in multiples of the standard deviation

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \text{PDF}(x; \mu, \sigma) dx = \int_{u_1}^{u_2} \underbrace{\text{PDF}(u, 0, 1)}_{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}} du \rightarrow \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}}_{\text{no simple antiderivative.}} du = 1$$

standard deviation (in general):

$$\sigma = \sqrt{\int (x-\mu)^2 \text{PDF}(x) dx} = \text{weighted average of squared distance from expected value.}$$

no simple antiderivative.

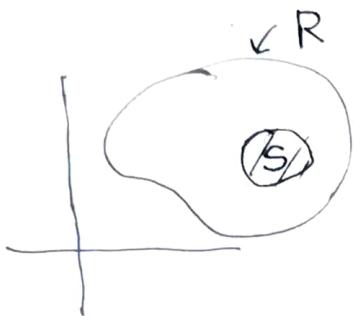
Probability Theory (4)

15.4b: 4

2-d case: 2 such "random variables" (x, y)

in closed region of xy plane, or $x > 0, y > 0$ (first quadrant)

or entire plane (usually one of these 3 cases)



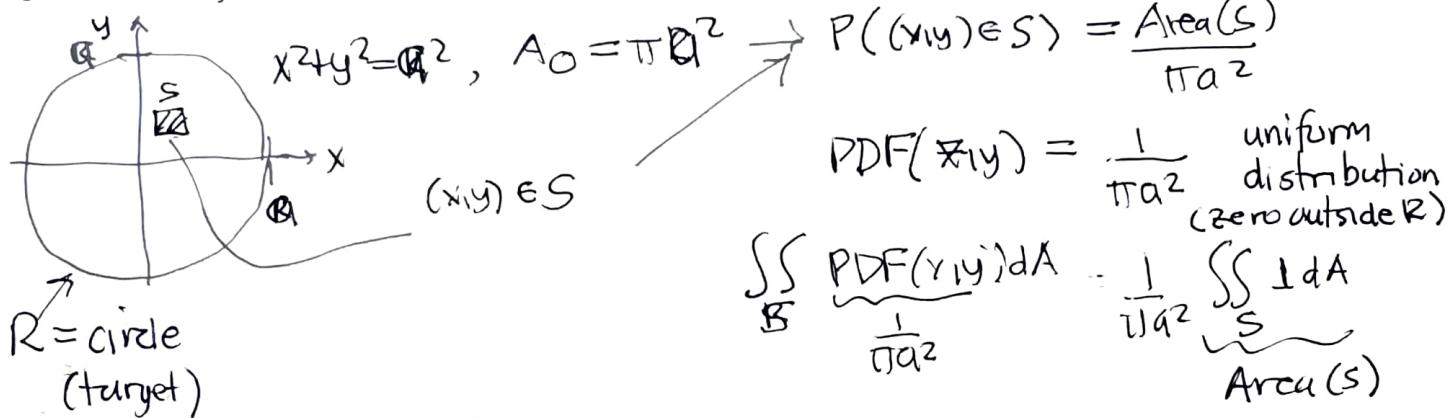
$$\text{allowed region: } R = \{(x, y) \mid (x, y) \in \mathbb{R}\}$$

$$P((x, y) \in S) = \iint_S \text{PDF}(x, y) dA$$

$$\iint_R \text{PDF}(x, y) dA = 1$$

EXAMPLE If you throw darts randomly at a circular target, what is the probability that you hit some subregion of the target given that the dart actually hits the target.

Answer. Obvious. If S is a part of the target, that probability should just be the fractional area of the target.



$$\langle (x, y) \rangle = \iint_R (x, y) \text{PDF}(x, y) dA = (0, 0)$$

by symmetry.

BUT easiest examples & just multiply two 1-d probability distributions

$$\text{PDF}(x, y) = \text{PDF}_1(x) \text{PDF}_2(y)$$

$$\iint_R \text{PDF}(x, y) dA = \underbrace{\int_{-\infty}^{+\infty} \text{PDF}_1(x) dx}_{=1 \text{ on allowed interval}} \underbrace{\int_{-\infty}^{+\infty} \text{PDF}_2(y) dy}_{=1 \text{ on allowed interval}} = 1 \cdot 1 = 1 \quad \checkmark$$

$$\langle (x, y) \rangle = \langle \mu_1, \mu_2 \rangle$$

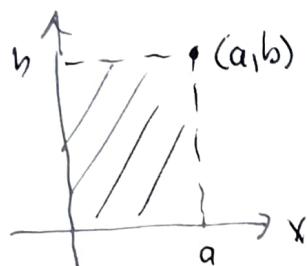
expected values of x & y are the same as their 1-d distributions

Probability Theory (5)

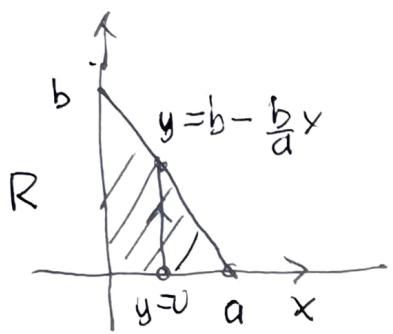
15.4b: 5

2-d Poisson $\text{PDF}(x_1, y) = \left(\frac{1}{\mu_1} e^{-\frac{x_1}{\mu_1}}\right) \left(\frac{1}{\mu_2} e^{-\frac{y}{\mu_2}}\right) = \frac{1}{\mu_1 \mu_2} e^{-\frac{x_1}{\mu_1} - \frac{y}{\mu_2}}$

$$\int_0^{\infty} \int_0^{\infty} \frac{1}{\mu_1 \mu_2} e^{-\frac{x_1}{\mu_1} - \frac{y}{\mu_2}} dy dx = 1.$$



$$\begin{aligned} P(0 \leq x_1 \leq a, 0 \leq y \leq b) &= \int_0^a \int_0^b \frac{1}{\mu_1 \mu_2} e^{-\frac{x_1}{\mu_1} - \frac{y}{\mu_2}} dy dx \\ &= \int_0^a \frac{1}{\mu_1} e^{-\frac{x_1}{\mu_1}} dx \int_0^b \frac{1}{\mu_2} e^{-\frac{y}{\mu_2}} dy \quad \text{constant limits!} \end{aligned}$$



$$P((x_1, y) \in R) = \int_0^a \int_0^{b - \frac{b}{a}x} \frac{1}{\mu_1 \mu_2} e^{-\frac{x_1}{\mu_1} - \frac{y}{\mu_2}} dy dx$$

nonconstant
limits
no factorization!

2-d normal

$$\text{PDF}(x_1, y) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1 - \mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y - \mu_2)^2}{2\sigma_2^2}}$$

etc, same deal, must use technology to evaluate these integrals while Poisson can be done by hand BUT we are only concerned with setting up the integrals—may as well use technology for them too.

14.5b probability

(6)

lightbulb & waiting time problems

These are applications of independent 1d exponential probability distributions for a nonnegative time variable $t \geq 0$.

example (video)

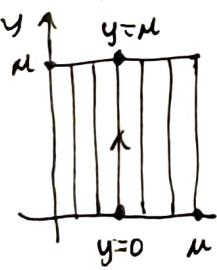
A lamp has two bulbs of a type with an average lifetime of 1000 hrs. Assuming that we can model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$,

- Find the probability that both of the lamp bulbs fail within 1000 hours.
- Find the probability that a bulb fails and then is replaced by another such bulb such that both bulbs fail within a total of 1000 hours,

In both cases the "PDF" is a product function of the two time variables designated $x \geq 0, y \geq 0$:

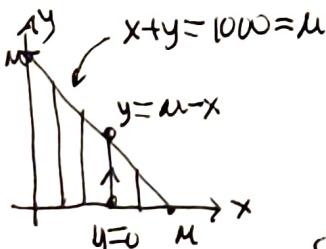
$$\text{PDF}(x, y) = \text{PDF}(x) \text{PDF}(y) \\ = (\mu^{-1} e^{-x/\mu})(\mu^{-1} e^{-y/\mu})$$

- a) $P(0 \leq x \leq 1000 \text{ and } 0 \leq y \leq 1000) =$ probability that $0 \leq x \leq \mu$ and $0 \leq y \leq \mu$
 which is a square in the first quadrant



$$= \int_0^\mu \int_0^\mu (\mu^{-1} e^{-x/\mu})(\mu^{-1} e^{-y/\mu}) dy dx \\ = \left(\int_0^\mu \mu^{-1} e^{-x/\mu} dx \right) \left(\int_0^\mu \mu^{-1} e^{-y/\mu} dy \right) \quad \begin{matrix} \text{coordinate below} \\ \sim \\ (1 - 2e^{-1} + e^{-2}) \end{matrix} \\ = \left(\int_0^\mu \mu^{-1} e^{-x/\mu} dx \right)^2 = (-e^{-x/\mu} \Big|_0^\mu)^2 = (1 - e^{-1})^2 \approx 0.3996$$

b) $P(0 \leq x+y \leq 1000)$



$$= \int_0^\mu \int_0^{μ-x} \mu^{-1} e^{-x/\mu} \cdot \mu^{-1} e^{-y/\mu} dy dx \\ = \int_0^\mu \mu^{-1} e^{-x/\mu} (1 - e^{-1} e^{-x/\mu}) dx = -e^{-x/\mu} - \mu^{-1} e^{-x} \Big|_0^\mu = -e^{-1} - e^{-1} + 1 \\ = 1 - 2e^{-1} \approx 0.2642$$

15.9b

probability

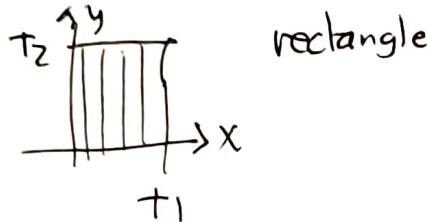
(7)

Waiting times — wait in two successive lines, what is probability that total waiting time less than or equal to ...
 $P(0 \leq x+y \leq T)$ similar



triangular region

or split waiting — ask for probability that waits in each line less than or equal to ...



rectangle

Normal distributions in two independent traits

$$\text{PDF } (x_1, y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$$

but not must use Maple to evaluate integrals

example $\mu_1 = 0 = \mu_2$, $\sigma_1 = \sigma_2 = \sigma$:

$$\left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} = \frac{1}{2\pi} e^{-\frac{1}{2}r^2}$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dy dx = \int_0^{\infty} \int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2} r dr d\theta \\ & \text{factors!} \quad \underbrace{\left(\frac{1}{2\pi} \int_0^{2\pi} d\theta\right)}_{=1} \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr = \int_0^{\infty} e^{-\frac{1}{2}u} \frac{du}{2} = \left[\frac{e^{-\frac{1}{2}u}}{-\frac{1}{2}} \right]_0^{\infty} = 1 - \left[e^{-\frac{1}{2}u} \right]_0^{\infty} = 1! \end{aligned}$$

$$= \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \right)^2$$

$$= 1 \rightarrow$$

$$\boxed{\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}}$$

$$\text{or } \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\frac{2\pi}{2}}$$

3-d integral evaluated through 2-d version!