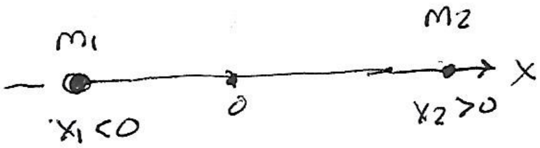
$\boxed{15.4}$ center of mass / centroid (geometric center)

Archimedes (1d)



Balance if $\underbrace{\overbrace{m_1|x_1|}^{-m_1x_1}}_{} = \underbrace{m_2 x_2}_{}$

moment arms about origin

$$m_1 x_1 + m_2 x_2 = 0$$

more masses: $\sum_i m_i x_i = 0$ "center of mass" at origin

Suppose we want to find the center of mass ?. $x = \bar{x}$ (location)

$$\sum_i m_i(x_i - \bar{x}) = 0 \rightarrow \sum m_i x_i - \underbrace{\sum m_i \bar{x}}_{(\underbrace{\sum m_i}_{M \text{ total mass}})\bar{x}} = 0$$

$$\sum m_i x_i = M\bar{x}$$

as though all mass placed at center of mass
same total moment about origin

$$\bar{x} = \frac{\sum_i m_i x_i}{M} = \sum_i \left(\frac{m_i}{M}\right) x_i$$

$$\sum \frac{m_i}{M} = \frac{\sum m_i}{M} = \frac{M}{M} = 1$$

fractional masses

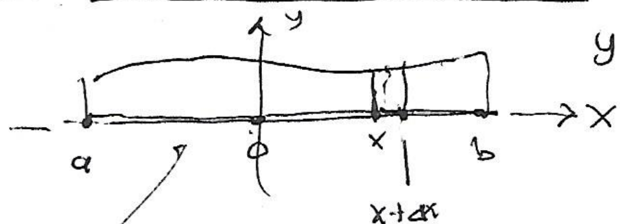
if $m_i = m$ all same $M = n m_i$

$$\frac{m_i}{M} = \frac{1}{n}$$

get ordinary average of position

$\bar{x}$ = weighted average of position, weighted by fractional mass

15.4 center of mass / centroid



"inhomogeneous wire"
variable mass density

$y = \rho(x)$ linear distribution of mass

mass/unit length

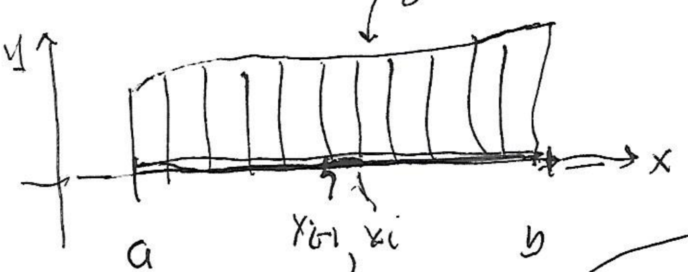
$\rho(x)\,\Delta x \approx$ amount of mass from x to $x+\Delta x$

$$\int_a^b \rho(x)\,dx = M \quad \text{total mass}$$

uniform distribution: $\rho(x) = \rho_0$

$$M = \int_a^b \rho_0\,dx = \rho_0 x\Big|_a^b = \rho_0 \underbrace{(b-a)}_{\text{length}}$$

Riemann $y = \rho(x)$



$\Delta m_i \approx \rho(x_i^*)\Delta x$

$$\sum_i \underbrace{\Delta m_i x_i^*}_{\text{ith moment about origin}} \approx M\bar{x}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\, x_i^*\, \Delta x = M\bar{x}$$

$$\int_a^b \rho(x)\, x\, dx = M\bar{x}$$

$$\bar{x} = \frac{\int_a^b \rho(x)\, x\, dx}{M} = \int_a^b \underbrace{\frac{\rho(x)}{M}}_{\text{fractional mass distribution}} x\, dx$$

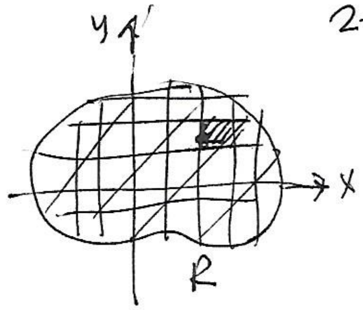
$$= \frac{\int_a^b \rho(x)\, x\, dx}{\int_a^b \rho(x)\, dx}$$

weighted average of distance from origin weighted by fractional mass

$$\text{uniform: } \bar{x} = \frac{\int_a^b \rho_0\, x\, dx}{\int_a^b \rho_0\, dx = (b-a)} = \frac{1}{b-a}\underbrace{\int_a^b x\, dx}_{\text{average of } x \text{ over interval}}$$

$$\frac{1}{2}(a+b)! = \frac{1}{b-a}\,\frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2}$$

$\boxed{15.4}$ center of mass / centroid

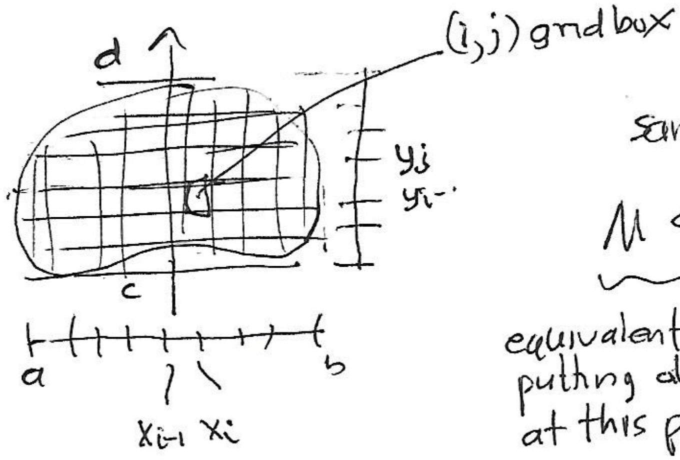


2-d distribution of mass in plane (lamina)

$\rho(x,y) =$ mass / unit area.

$\rho(x,y) \underbrace{\Delta x \Delta y}_{= \Delta A} \approx$ mass in small grid box

$\iint_R \rho(x,y)\, dA = M$ total mass



same deal:

$$\underbrace{M \langle \bar{x}, \bar{y} \rangle}_{\text{equivalent to putting all mass at this position}} \approx \sum_i \sum_j \underbrace{\Delta m_{ij}}_{\rho(x_i^*, y_j^*) \underbrace{\Delta x \Delta y}_{\Delta A}} \langle x_i^*, y_j^* \rangle$$

$$= \sum_i \sum_j \rho(x_i^*, y_j^*) \langle x_i^*, y_j^* \rangle \Delta A$$

$$\downarrow \lim \quad \iint_R \rho(x,y) \langle x, y \rangle\, dA$$

$$= \langle \underbrace{\iint_R \rho x\, dA}_{\substack{M_y = \text{moment} \\ \text{about y axis} \\ = \text{distance from} \\ \text{y axis}}}, \underbrace{\iint_R \rho y\, dA}_{\substack{M_x = \text{moment} \\ \text{about x axis} \\ = \text{distance from} \\ \text{x-axis}}} \rangle$$

$$\langle \bar{x}, \bar{y} \rangle = \frac{1}{M} \langle \iint_R \rho x\, dA, \iint_R \rho y\, dA \rangle = \frac{\langle M_y, M_x \rangle}{M}$$

M: $\iint_R \rho\, dA$

integrate x & y against ρ
divide by integral of ρ

do 3 integrals, divide 2 by third to get coords of center of mass

15.4 center of mass / centroid

$$\langle \bar{x}, \bar{y} \rangle = \frac{\langle \iint_R \rho x \, dA, \iint_R \rho y \, dA \rangle}{\iint \rho \, dA}$$

if $\rho = \rho_0$ homogeneous solid

$$= \frac{\langle \iint \cancel{\rho_0} x \, dA, \iint \cancel{\rho_0} y \, dA \rangle}{\iint \cancel{\rho_0} \, dA}$$

$$= \frac{\langle \iint x \, dA, \iint y \, dA \rangle}{\iint dA}$$

center of area distribution "centroid" (geometric center)