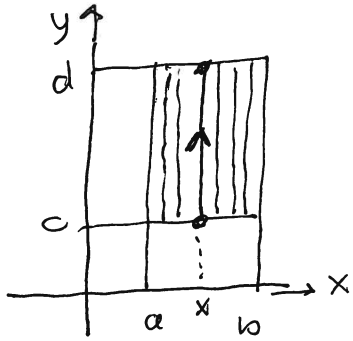
"Relaxing" the constant limits of integration

integration region:

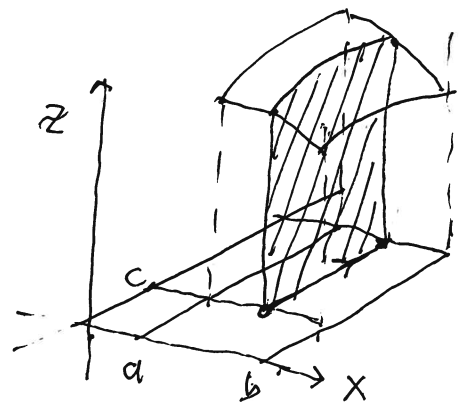


typical vertical cross-section for "y first" integration

$y = c..d$ while $x = a..b$

$$\int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{\text{"y first"}}\ \underset{\text{"then x"}}{dx}$$

graph diagram above: $f(x,y) \geq 0$

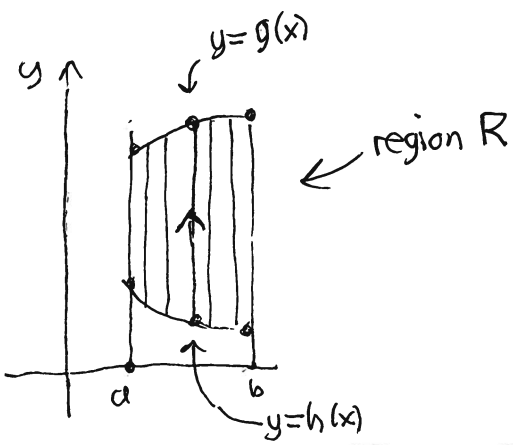


vertical plane cross-section area:

$$A(x) = \int_c^d f(x,y)\,dy$$

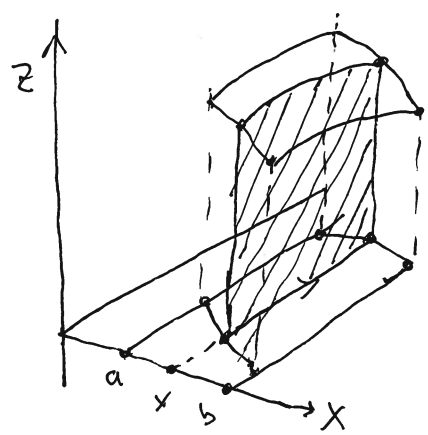
then $\int_a^b \ldots dx$ sweeps plane cross-section across solid region

$\boxed{\text{now allow limits } y = y(x) \text{ for innermost integral}}$



$\boxed{y = h(x)..g(x) \text{ while } x = a..b}$

$$\int_a^b \underbrace{\int_{h(x)}^{g(x)} f(x,y)\,dy}_{\text{"y-first", result function only of outer variable x}}\ dx$$



vertical plane cross-section area

$$A(x) = \int_{h(x)}^{g(x)} f(x,y)\,dy$$

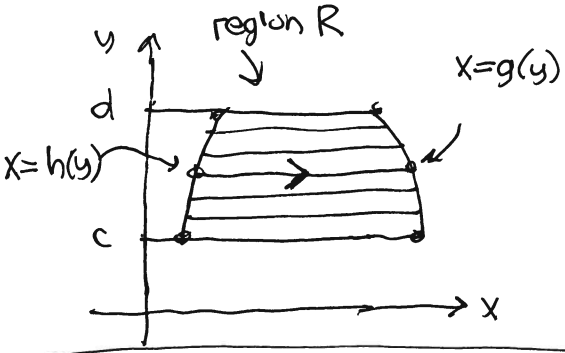
$\int_a^b \ldots dx$ sweeps cross-section across solid region

(for general f integral gives signed volume)

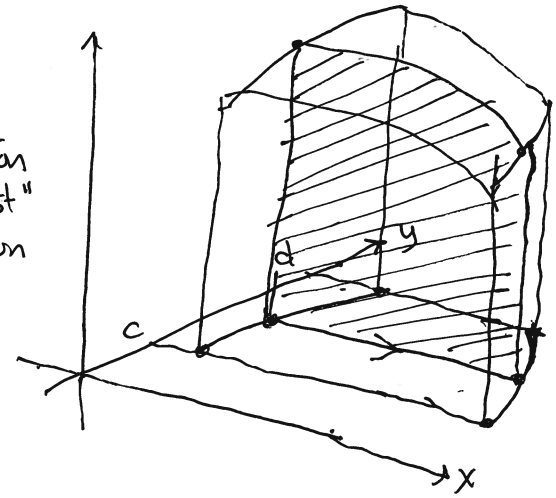
arrow on "typical" bullet point terminated vertical line cross-section points in direction of increasing variable y along that direction

"Relaxing" limits of integration (2)

same discussion holds if interchange x and y



typical horizontal cross-section for "x-first" integration



$$\boxed{x = h(y) \,..\, g(y) \text{ while } y = c..d}$$

$$\int_c^d \underbrace{\int_{h(y)}^{g(y)} f(x,y)\,dx}_{\text{"x-first" produces function only of outer variable } y} \, dy$$

then y

vertical plane cross-section area:

$$A(y) = \int_{h(y)}^{g(y)} f(x,y)\,dx$$

then $\int_c^d \ldots\, dy$ sweeps plane cross-section across solid region

The inner limits of integration can only depend on the outer variable of integration to have this interpretation

$$\iint_R f(x,y)\,dA$$

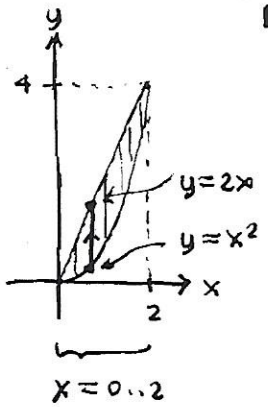
$$\left(\int_y^{y^2} \int_{xy}^{x^2} xy\,dx\,dy \text{ can be evaluated but it does not correspond to } \iint_R xy\,dA \text{ for any region } R \right)$$

arrow on "typical" bullet point terminated horizontal line cross-section points in direction of increasing variable x along that direction.

By labeling the bullet point endpoints, one identifies the starting and stopping values of the variable which increases along it, from "lower" to "upper" value.

(See handout example)

double integrals : it's really about describing a region of the plane



■ Consider the region R between the graphs $y=x^2$ and $y=2x$.
These curves intersect at : $x^2=2x$ or $0=x^2-2x=x(x-2) \to x=0,2$
$\to y=0,4$
points $(0,0)$ and $(2,4)$.

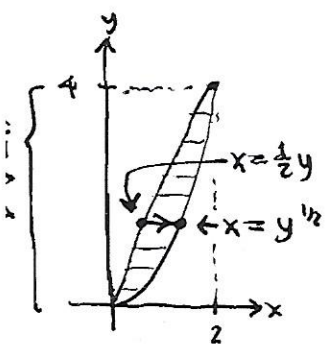
The diagram shows a "typical" vertical cross-section and its "direction" (increasing y) with endpoints labeled by start/stop values

R : $y=x^2..2x$ (increasing bot. to top)
as $x=0..2$ (increasing from left to right)

This describes the region as a "type I" region (y first, then x)

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx = \int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$$

$y=x^2..2x$ as $x=0..2$



■ But we can also describe the bounding curves with x expressed as a function of y, a "type II" region (x first, then y)

$y=x^2 \xrightarrow{x\ge 0} x=y^{1/2}$

$y=2x \longrightarrow x=\frac{1}{2}y$

R : $x=\frac{1}{2}y..y^{1/2}$ (increasing left to right)
as $y=0..4$ (increasing bot. to top)

The diagram shows a "typical" horizontal cross-section and its "direction" (increasing x) with endpoints labeled by start/stop values

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=4}\int_{x=\frac{1}{2}y}^{x=y^{1/2}} f(x,y)\,dx\,dy = \int_0^4\int_{\frac{1}{2}y}^{y^{1/2}} f(x,y)\,dx\,dy$$

$x=\frac{1}{2}y..y^{1/2}$ as $y=0..4$

technique : changing the order of integration

For a region that allows either choice above, we can start with one order of integration, make a diagram of the region of integration and its bounding curves, then re-express them as above and determine the new limits of integration

$$\int_0^2\int_{x^2}^{2x} \underbrace{f(x,y)}_{z=f(x,y)}\,dy\,dx = \int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx \longrightarrow \text{make diagram} \to \text{re-describe}$$

then go backwards to the new double integral

REMARK

$z=f(x,y)$ helps visualize integral in 3D as signed volume between graph and xy plane.

integrand is function whose graph in 3D leads to solid associated with 2D integral

"Deconstructing" a double integral to reverse order of integration

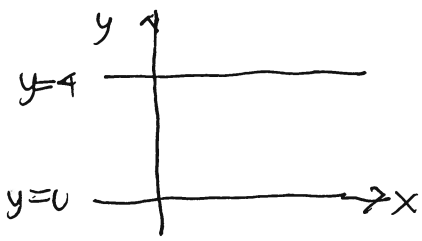
$$\int_0^4 \int_{\frac{1}{2}y}^{y^{1/2}} f(x,y)\,dx\,dy \longrightarrow \text{"}x=\tfrac{1}{2}y..y^{1/2} \text{ while } y=0..4\text{"}$$

↓ label limits of integration by corresponding variables

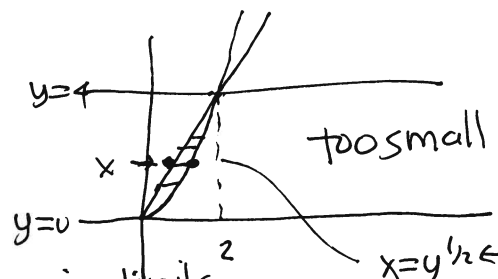
$$\underbrace{\int_{y=0}^{y=4} \int_{x=\frac{1}{2}y}^{x=y^{1/2}}}\, f(x,y)\,dx\,dy$$

get 4 eqns of curves bounding region R of integration

$x=\frac{1}{2}y \Leftrightarrow y=2x$



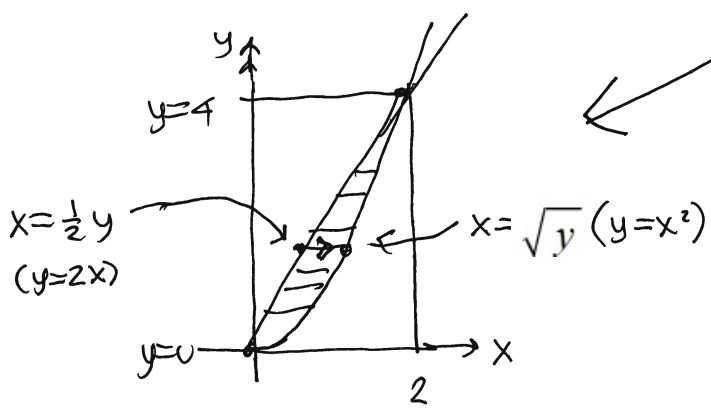
outer limits



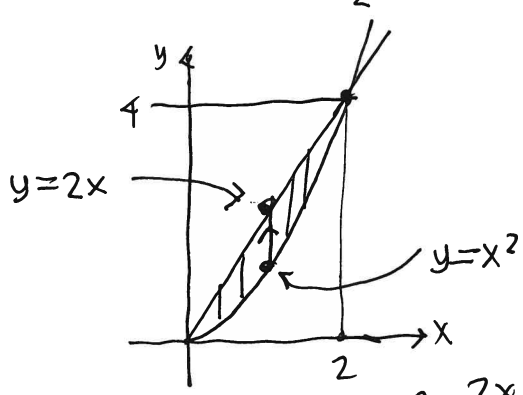
add inner limits

too small, retry

$x=y^{1/2} \Leftrightarrow y=x^2$



invert functions to relable with x as the independent variable



"$y=x^2..2x$ while $x=0..2$"

$$\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$$

the arrowhead correlates with "from lower to upper" limits of integration

Reversing order of integration only works on compatible regions

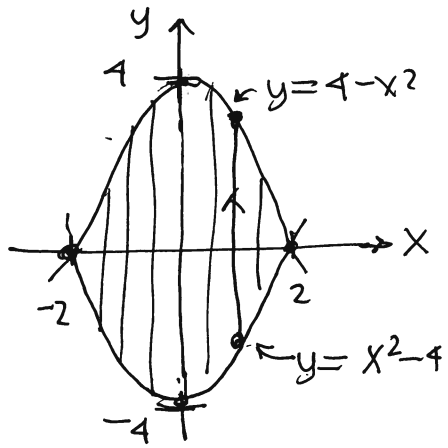
otherwise we need to break up the region into subregions and add the resulting integrals.

Example Integrate $f(x,y) = x^2+y^2$ over the region R enclosed by the two curves $y = x^2-4$, $y = 4-x^2$.

intersection pts: set equal $x^2-4 = 4-x^2$ solve

$2x^2 = 8$, $x^2 = 4$, $x = \pm 2$

$y = (\pm 2)^2 - 4 = 0$

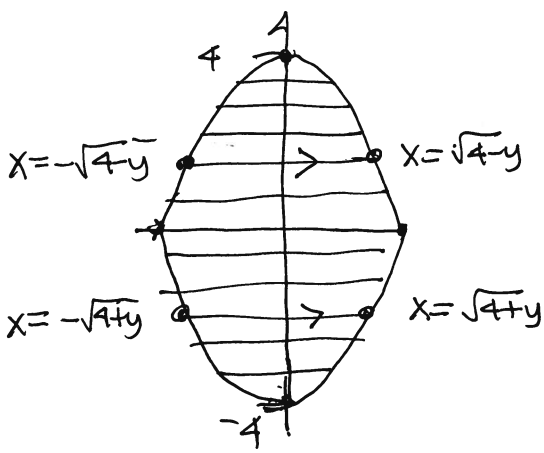


R: $y = x^2-4 \,..\, 4-x^2$
while $x = -2..2$

$$\iint_R x^2+y^2\,dA = \int_{-2}^{2}\int_{x^2-4}^{4-x^2} x^2+y^2\,dy\,dx$$

invert: $y = 4-x^2 \to x^2 = 4-y \to x = \pm\sqrt{4-y}$

$y = x^2-4 \to x^2 = y+4 \to x = \pm\sqrt{4+y}$



$$\int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}} (x^2+y^2)\,dx\,dy$$

$$+$$

$$\int_{-4}^{0}\int_{-\sqrt{4+y}}^{\sqrt{4+y}} (x^2+y^2)\,dx\,dy$$

$$= \iint_R x^2+y^2\,dA$$

cannot do as a single iterated double integral in this order of integration

The book calls compatible regions type I (y-first) and type II (x-first).

Setting up double integral to get volume of a solid

Find the volume of the tetrahedron (4 sided) (Stewart Example 15.2.4)

bounded by: $x+2y+z=2$, $x=2y$, $x=0$, $z=0$

plane $z=2-x-2y$

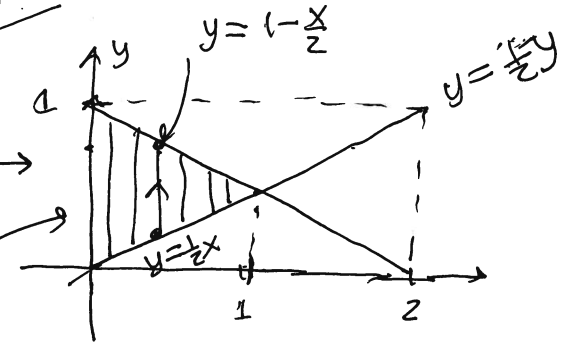
$=0$ intersects $z=0$:

$x+2y=2$

$y=\frac{2-x}{2}=1-\frac{x}{2}$

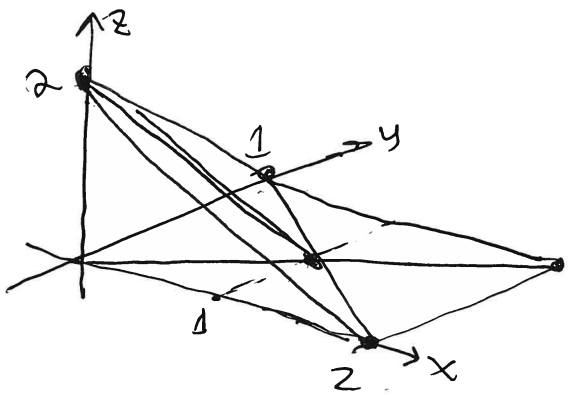
axis intercepts:

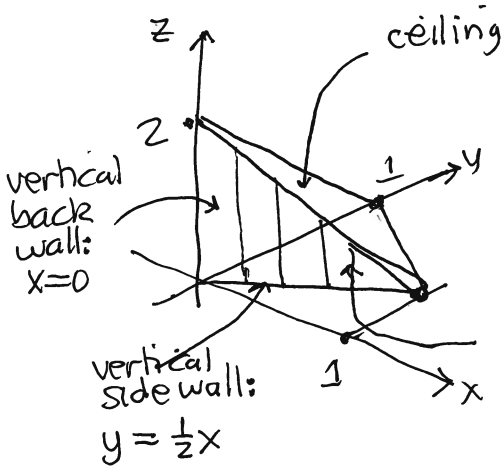
$z=2$, $x=2$, $y=1$



region R of plane

$x=0$ intersection of 3 faces with 4th face $z=0$





ceiling: $z=2-x-2y$ (over)

want volume under graph of $f(x,y)=2-x-2y$

floor: $z=0$ (under)

$$V=\iint_R 2-x-2y\,dA=\int_0^1\int_{\frac{1}{2}x}^{1-\frac{1}{2}x}(2-x-2y)\,dy\,dx \overset{\text{Maple}}{=} \left(\text{see worksheet}\right)$$

15.2 double integrals over general 2d regions

Example Find the volume between the 2 surfaces:

$z = x^2 + 2y^2$ (paraboloid) and $x + y + z = 4$. (plane) $\longrightarrow (z = 4 - x - y)$

bot: $f(x,y) = x^2 + 2y^2$
top: $g(x,y) = 4 - x - y$ $\Big] \rightarrow$ $x^2 + 2y^2 = 4 - x - y$

$x^2 + x + 2y^2 + y = 4$ (ellipse)

solve for x or y, we choose y:

$$2y^2 + y + (x^2 + x - 4) = 0$$

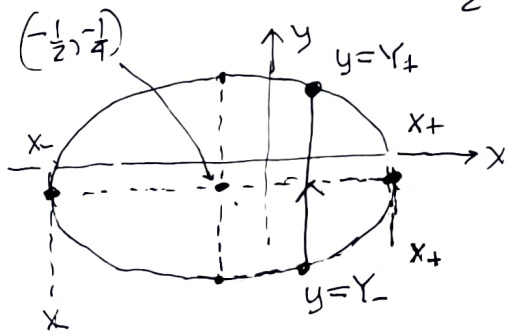
$$y = \frac{-1 \pm \sqrt{1 - 4(2)(x^2+x-4)}}{2(2)} = \frac{-1 \pm \sqrt{33 - 8x^2 - 8x}}{4} = Y_\pm(x)$$

This has real solns as long as $33 - 8x^2 - 8x \geq 0$

with endpoints: $8x^2 + 8x - 33 = 0$

$$x = \frac{-8 \pm \sqrt{64 - 4(8)(-33)}}{2(8)} = -\frac{1}{2} \pm \frac{1}{4}\sqrt{\cancel{4}+66}$$

$$= -\frac{1}{2} \pm \frac{1}{4}\sqrt{70} = X_\pm$$



$$V = \int_{x=X_-}^{x=X_+} \int_{y=Y_-}^{y=Y_+} \underbrace{g(x,y)}_{top} - \underbrace{f(x,y)}_{bot} \, dy\, dx$$

$$= \int_{-\frac{1}{2}-\frac{1}{4}\sqrt{70}}^{-\frac{1}{2}+\sqrt{70}} \int_{-\frac{1}{4}-\frac{1}{4}\sqrt{33-8x^2-8x}}^{-\frac{1}{4}+\frac{1}{4}\sqrt{33-8x^2-8x}} 4 - x - y - x^2 - 2y^2 \, dy\, dx$$

$$\underset{\text{Maple}}{=} \frac{1225\sqrt{2}\pi}{256} \approx 21.2599 \approx 21.26$$

Setting up this double integral is what we need to do. In practice we cannot actually perform the integration evaluation ourselves, nor can Maple do so exactly in general BUT it can give reliable high precision numerical values.