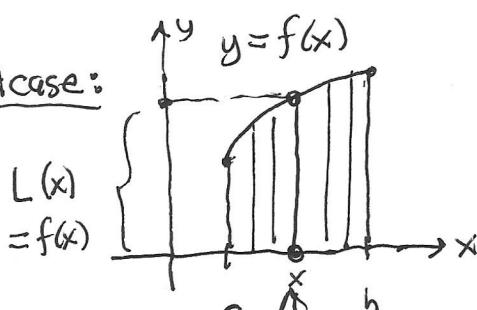


## Iterated Integrals

Why does  $\iint_R f(x,y) dA = \int_a^c \int_a^b f(x,y) dy dx = \int_a^b \int_c^d f(x,y) dy dx$  ?  
 where  $R = [a,b] \times [c,d]$  "R" for rectangle  
 (dont confuse the real numbers  $\mathbb{R}$ )

1d case:



If  $f(x)$  continuous on  $[a,b]$  then :

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

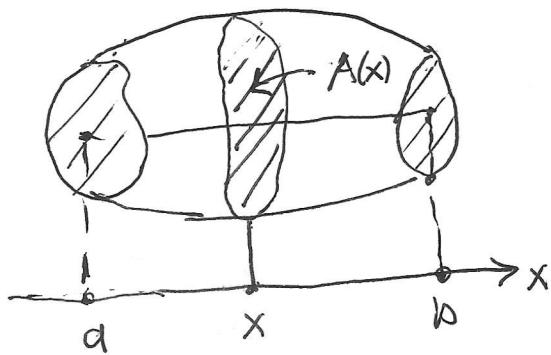
where  $F'(x) = f(x)$   
 antiderivative

typical vertical line segment  
 cross-section at  $x$

$$\int_a^b f(x) dx = \int_a^b L(x) dx$$

cross-sectional length differential of arclength  
 perpendicular to linear  
 cross-section

This generalizes to 2-d plane cross-sections of a 3-d solid region in space.



Consider solid region whose cross-sectional area is a function of a single variable  $x$  which measures arclength perpendicular to those plane cross-sections.

$$V = \int_a^b A(x) dx$$

cross-section area differential of arclength  
 orthogonal to planes.

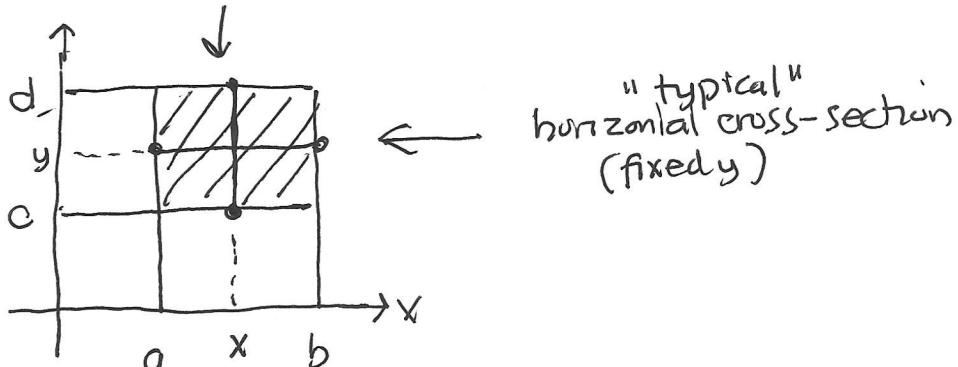
This volume evaluation is the key to interpreting iterated double integrals as volume under the graph.

## Iterated Integrals (2)

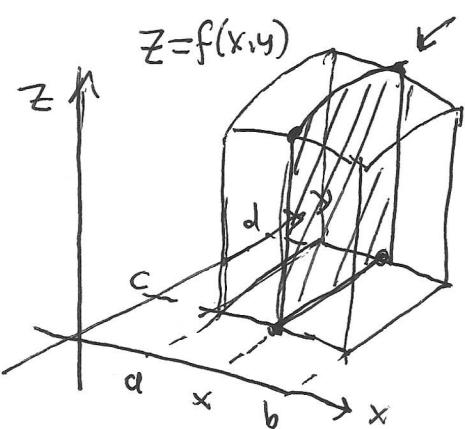
$\iint_R f(x,y) dA = \text{volume under graph (above } xy\text{ plane)}$

$\geq 0 \rightarrow$  for general  $f$ , this is the signed volume (volume above - volume below)

$$R = [a,b] \times [c,d]$$



"typical"  
vertical cross-section  
(fixed  $x$ )



typical vertical plane cross-section  
(fixed  $x$ )

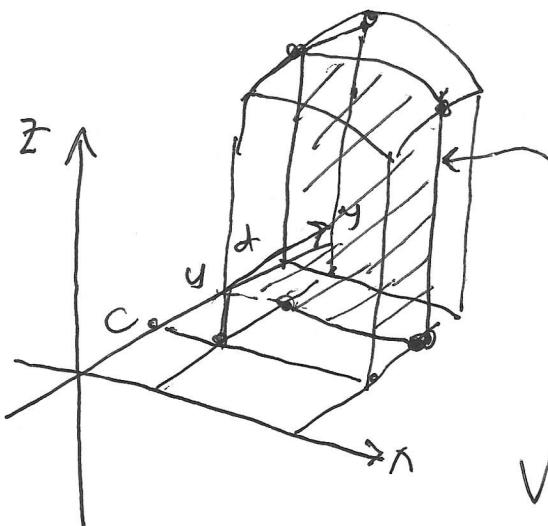
$$A(x) = \int_c^d f(x,y) dy$$

area under  
curve for  
1-d cross-section  
graph

= plane cross-section area

$$V = \int_a^b A(x) dx = \underbrace{\int_a^b \int_c^d f(x,y) dy dx}_{\iint_R f(x,y) dA}$$

iterated integral  
evaluates to  
this uniterated  
integral!



typical vertical plane cross-section  
( $y$  fixed)

$$A(y) = \int_a^b f(x,y) dx$$

area under curve  
for 1-d cross-section  
graph

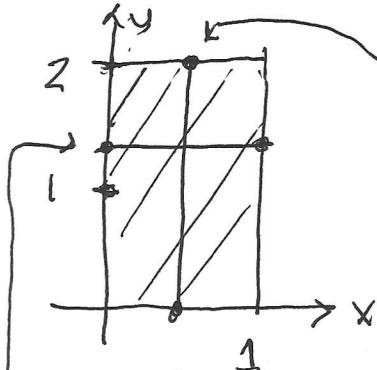
= volume cross-section area

$$V = \int_c^d A(y) dy = \underbrace{\int_c^d \int_a^b f(x,y) dx dy}_{\text{opposite order iterated integral gives previous result}}$$

opposite order iterated  
integral gives previous result

### Iterated Integrals (3)

$$z = 16 - x^2 - 2y^2 \text{ on } [0,1] \times [0,2]$$



integrate first over  $y=0..2$  while then  $x=0..1$

$$V = \iint_R 16 - x^2 - 2y^2 dA = \int_0^1 \int_0^2 16 - x^2 - 2y^2 dy dx$$

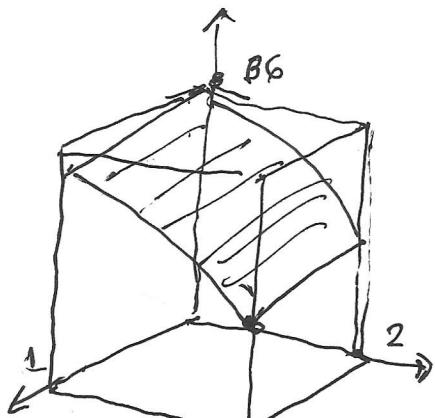
$$\begin{aligned} & 16y - x^2y - \frac{2}{3}y^3 \Big|_{y=0}^{y=2} = (16-x^2) \cdot 2 - \frac{2}{3}(8) \\ & = \int_0^1 32 - \frac{16}{3} - 2x^2 dx = (32 - \frac{16}{3})x - \frac{2}{3}x^3 \Big|_0^1 \\ & = 32 - \frac{16}{3} - \frac{2}{3} = 32 - 6 = 26 \checkmark \end{aligned}$$

integrate first  
over  $x=0..1$  and  
then  $y=0..2$

$$\begin{aligned} V &= \int_0^2 \int_0^1 16 - x^2 - 2y^2 dx dy \\ &\leftarrow (16x - \frac{x^3}{3} - 2y^2x) \Big|_{x=0}^{x=1} = (16 - \frac{1}{3} - 2y^2) \\ &= \int_0^2 (16 - \frac{1}{3} - 2y^2) dy = (16 - \frac{1}{3})y - \frac{2}{3}y^3 \Big|_0^2 \\ &= (16 - \frac{1}{3})(2) - \frac{2}{3}(8) = 32 - \frac{2}{3} - \frac{16}{3} = 32 - 6 = 26 \checkmark \end{aligned}$$

Cube has volume  $1 \times 2 \times 16 = 32$

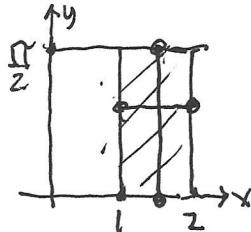
$\frac{26}{32} \approx .81 \sim 80\%$ , Maple 3d plot looks plausible



rough graphical  
estimate.

Iterated Integrals · In practice, order of integration matters:  
one order can lead to a difficult or impossible antiderivative problem

EX  $R = [1, 2] \times [0, \frac{\pi}{2}]$   $\iint y \sin(xy) dA$



y first:  $\int_1^2 \int_0^{\frac{\pi}{2}} y \sin(xy) dy dx$  requires integration by parts! no way!

product function of y

x first:  $\int_0^{\frac{\pi}{2}} \int_1^2 y \sin(xy) dx dy$  =  $\int_0^{\frac{\pi}{2}} -\cos(xy) \Big|_{x=1}^{x=2} dy$   
 ↑ simple as function of x, y held constant  $\cos y - \cos 2y$

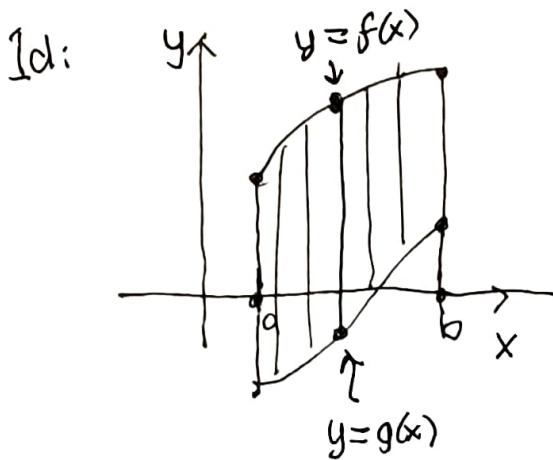
$$\begin{aligned} & \int a \sin(bx) dx \\ &= a \left( -\frac{1}{b} \cos bx \right) \\ &= \int_0^{\frac{\pi}{2}} \cos y - \cos 2y dy \\ &= \sin y - \frac{1}{2} \sin 2y \Big|_0^{\frac{\pi}{2}} \\ &= \sin \frac{\pi}{2} - \frac{1}{2} \sin \pi = 1 \end{aligned}$$

sure, we could have done this by integration by parts in y, but then the x integral is more complicated and requires another integration by parts — not worth the effort & more room for human error!

If we had started with  $\int_1^2 \int_0^{\frac{\pi}{2}} y \sin(xy) dy dx$  and seen the first difficulty we could have simply interchange the order of integration to see if it is easier to integrate.

(5)

## Volume between graphs



$L(x) = f(x) - g(x) \geq 0$  if  $f(x) \geq g(x)$   
on  $[a,b]$ .

cross-section length

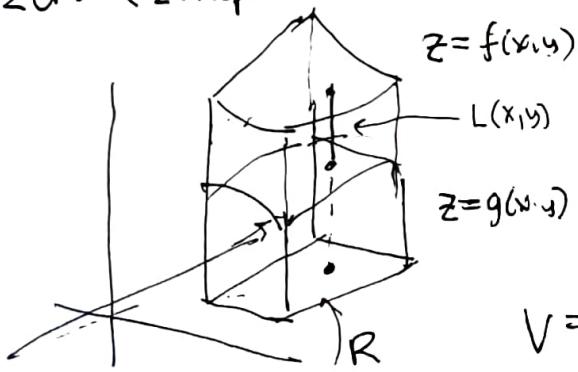
$$A = \int_a^b L(x) dx = \int_a^b f(x) - g(x) dx$$

$$\left( = \int_a^b f(x) dx - \int_a^b g(x) dx \right)$$

difference of signed areas

same deal for 2-d:  $V = \iint_R L(x,y) dA$   
length cross-section  $\perp$  (x-y plane)

2d: (2 independent variables)



$f(x,y) \geq g(x,y)$  over  $[a,b] \times [c,d]$

volume between graphs is just difference  
of volumes under each

linearity of integral

$$V = \iint_R f(x,y) dA - \iint_R g(x,y) dA = \iint_R \underbrace{f(x,y) - g(x,y)}_{L(x,y)} dA$$

EX Find volume between  $z = x+y$ ,  $z = \underbrace{4-x^2-y^2}_{\text{min value 2}}$ , over  $[0,1] \times [0,1]$

max value 2  
at  $x=y=1$

(plane) bottom  $\leq$  top (paraboloid)

$$V = \int_0^1 \int_0^1 \underbrace{(4-x^2-y^2) - (x+y)}_{L(x,y) \geq 0} dy dx \stackrel{\text{Maple}}{=} 7/3$$

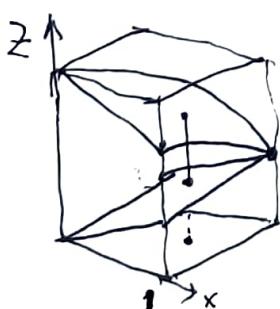
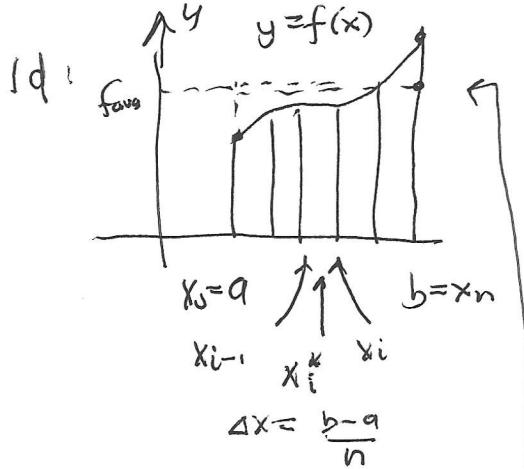


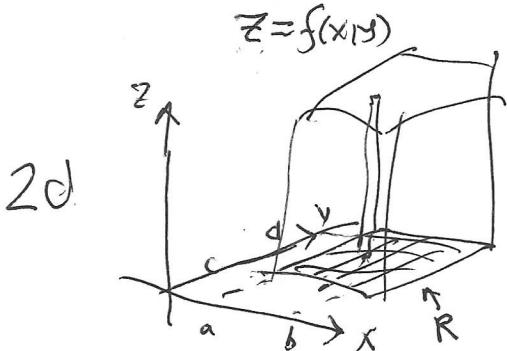
diagram doesn't  
need to be art!  
just clarify top &  
bottom surfaces.

integrate cross-sectional line  
segment length over perpendicular  
2-d region

## Riemann and average value of a function



Interpretation: replace  $f$  by  $f_{\text{avg}}$ , get same integral over  $[a, b]$ .  
rectangle has same area.



$$R: [a, b] \times [c, d]$$

$$\Delta x = \frac{b-a}{n}, \Delta y = \frac{d-c}{m}$$

$$A = (b-a)(d-c)$$

= area( $R$ )

$$= \iint_R 1 \, dA$$

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

$$= \frac{b-a}{n}$$

$$= (b-a) \lim_{n \rightarrow \infty} \left( \frac{\sum_{i=1}^n f(x_i^*)}{n} \right)$$

average of sampled values

$\equiv f_{\text{avg}}$  over interval

$$f_{\text{avg}} = \underbrace{\frac{1}{b-a}}_{\text{divide integral by length of interval of integration}} \int_a^b f(x) \, dx$$

divide integral by length of interval of integration

$$\iint_R f(x, y) \, dA$$

$$= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x \Delta y$$

$$\left( \frac{b-a}{n} \right) \left( \frac{d-c}{m} \right) = \frac{A}{nm}$$

$$= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{1}{nm} \left( \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \right) \cdot A$$

number of sampled values

avg of sampled values

$\equiv f_{\text{avg}}$

$$f_{\text{avg}} = \frac{1}{A} \iint_R f(x, y) \, dA = \frac{\iint_R f(x, y) \, dA}{\iint_R 1 \, dA}$$

divide integral by area of rectangle of integration

(in 3d divide by volume of rectangular box)  
of integration

Interpretation: replace  $f$  by  $f_{\text{avg}}$ , get same integral over rectangle.

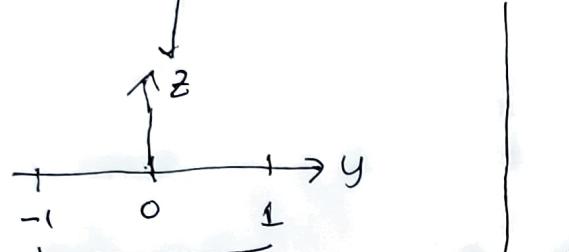
$f_{\min} \leq f_{\text{avg}} \leq f_{\max}$  always!

## double integrals and symmetry

example

$$\int_0^1 \int_{-1}^1 xye^{-x^2-y^2} dy dx$$

$f(x,y)$   
 $f(y,-y) = -f(y,y)$  odd function of  $y$



limits symmetric about origin

$$\begin{aligned} & \int_0^1 f(x,y) dy \\ &= - \int_{-1}^0 f(x,y) dy \\ & \underbrace{\int_0^1 f(x,y) dy + \int_{-1}^0 f(x,y) dy}_{\int_{-1}^1 f(x,y) dy} = 0 \end{aligned}$$

just a calc'ld result for partial integral —  
no surprise