

15.1a Iterated integrals & Riemann integration

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[technique] Finally multivariable integration.
The basic tool of 1-d integration is:

"limits of integration"
(upper/lower)

matching delimiters enclosing integrand

variable of integration (important!)

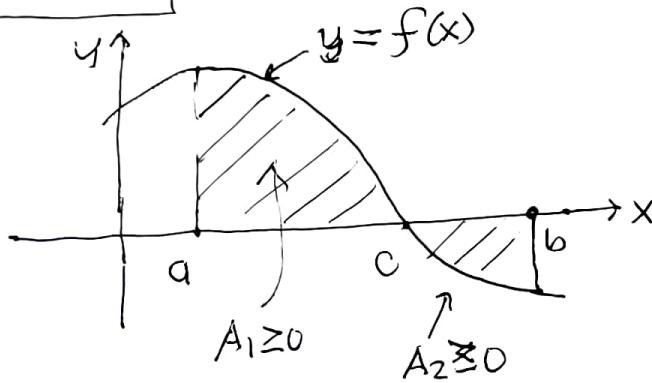
integrand

$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$

antiderivative:
 $F'(x) = f(x)$

"Fundamental Thm of Calculus"

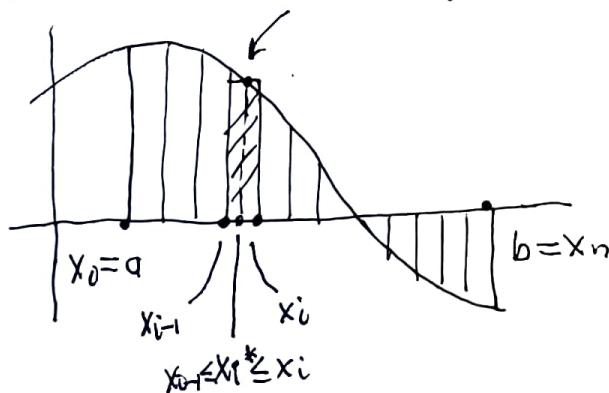
interpretation



$$\int_a^b f(x) dx = A_1 - A_2$$

= signed area between graph and horizontal axis

justification



$$\sum_{i=1}^n A_i = \sum_{i=1}^n f(x_i^*) \Delta x$$

$\downarrow \lim_{n \rightarrow \infty}$

$$A = \int_a^b f(x) dx$$

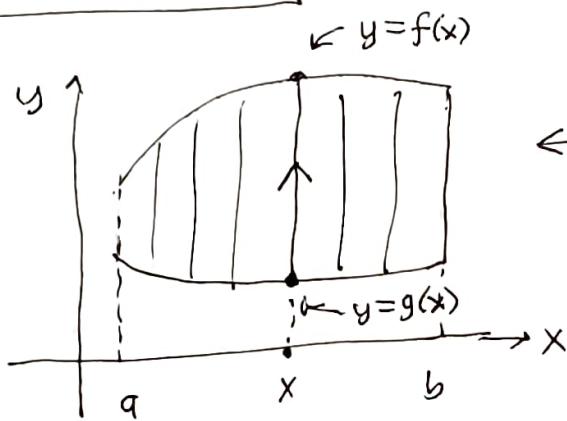
equal subdivision: $\Delta x = \frac{b-a}{n}$

$x_i = a + i \Delta x$

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one step further } cross-section line segments



region R of plane between
two graphs: $f(x) \geq g(x)$
for $a \leq x \leq b$

$L(x)$ = length of cross-section
(line segment)

difference area:

$$\int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b \underbrace{f(x) - g(x)}_{L(x)} dx$$

$$\text{Area} = \int_a^b L(x) dx > 0$$

cross-section perpendicular
to direction of integrating
variable

The diagram shows a "typical" cross-section as it moves from left to right to sweep out entire region.

The arrow on the cross-section line segment indicates the increasing direction of the vertical variable y .

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"Iterating" 1-d integration to multiple integrals

nested
integrals:

$$\int_c^d \left(\int_a^b f(x,y) dx \right) dy$$

$= F(y)$ no longer function of x

← "double integral"

$$= \int_c^d F(y) dy = \text{constant}$$

no longer function of y

The first integral in this sequence (working from inside out) is a "partial integration" of $f(x,y)$.

The second in this sequence is an ordinary integral of the remaining variable but we can call it a "2nd partial integral" of $f(x,y)$.

We can do it for $f(x,y,z)$ too!

$$\int_e^f \int_c^d \left(\int_0^b f(x,y,z) dx dy dz \right)$$

$= g(y,z)$

$= h(z)$

$= \text{constant}$

← "triple integral"

Each time you integrate over a variable it becomes a function of the remaining variables

Partial Integration

function of multiple variables $f(x_1, y, \dots)$

partial differentiation with respect to any one of these variables means treat all the other variables as constants and apply the rules of differentiation w.r.t. the chosen variable.

partial integration with respect to any one of these variables means treat all the other variables as constants and integrate with respect to the chosen variable

Indefinite partial integration: find an anti-derivative, add an arbitrary constant

definite partial integration: subtract the values of the antiderivative at the upper & lower limits

The order of two successive such operations does not matter

$$\frac{\partial^2 f}{\partial x \partial y}(x_1, y, \dots) = \frac{\partial^2 f}{\partial y \partial x}(x_1, y, \dots) \quad \leftarrow \text{iterated Calc I derivatives}$$

$$\int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} f(x_1, y, \dots) dx \right) dy = \int_{x_1}^{x_2} \left(\int_{y_1}^{y_2} f(x_1, y, \dots) dy \right) dx$$

These are called "iterated" integrals, just successive Calc II operations you know how to do.

The calc III aspect is understanding the meaning of these iterated integrals.

Evaluating them is not the problem.

partial derivatives / integrals "commute"

$$f_x(x,y) = \frac{\partial}{\partial x}(xy^2) = y^2 \rightarrow f_{xy}(x,y) = \frac{\partial}{\partial y}(y^2) = 2y$$

!! !

$$f(x,y) = xy^2$$

$$f_y(x,y) = \frac{\partial}{\partial y}(xy^2) = 2xy \rightarrow f_{yx}(x,y) = \frac{\partial}{\partial x}(2xy) = 2y$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

nested integrals (double integrals)

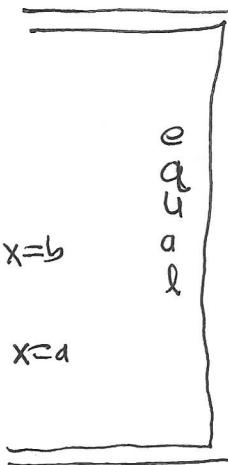
$$\int_a^b f(x,y) dx = \int_a^b xy^2 dx = \left. \frac{x^2 y^2}{2} \right|_{x=a}^{x=b} = (b^2 - a^2) \frac{y^2}{2}$$

$$\int_c^d \int_a^b f(x,y) dx dy = \int_c^d \left. \frac{b^2 - a^2}{2} y^2 \right|_{y=c}^{y=d} dy = \left. \frac{b^2 - a^2}{2} \frac{y^3}{3} \right|_{y=c}^{y=d}$$

$$= \left(\frac{b^2 - a^2}{2} \right) \left(\frac{d^3 - c^3}{3} \right)$$

$$\int_c^d f(x,y) dy = \int_c^d xy^2 dy = \left. \frac{xy^3}{3} \right|_{y=c}^{y=d} = \frac{x}{3} \left(\frac{d^3 - c^3}{3} \right)$$

$$\int_a^b \int_c^d f(x,y) dy dx = \int_a^b \left. \frac{x}{3} \left(d^3 - c^3 \right) \right|_{x=a}^{x=b} dx = \left. \frac{d^3 - c^3}{3} \frac{x^2}{2} \right|_{x=a}^{x=b}$$

$$= \left(\frac{d^3 - c^3}{3} \right) \left(\frac{b^2 - a^2}{2} \right)$$


evaluating double and triple integrals (definite integrals)

you already know how to do — it is just a matter of successive Calc II integrations with respect to distinct variables.

These are "nested" integrals like "nested" parentheses.

must match in pairs

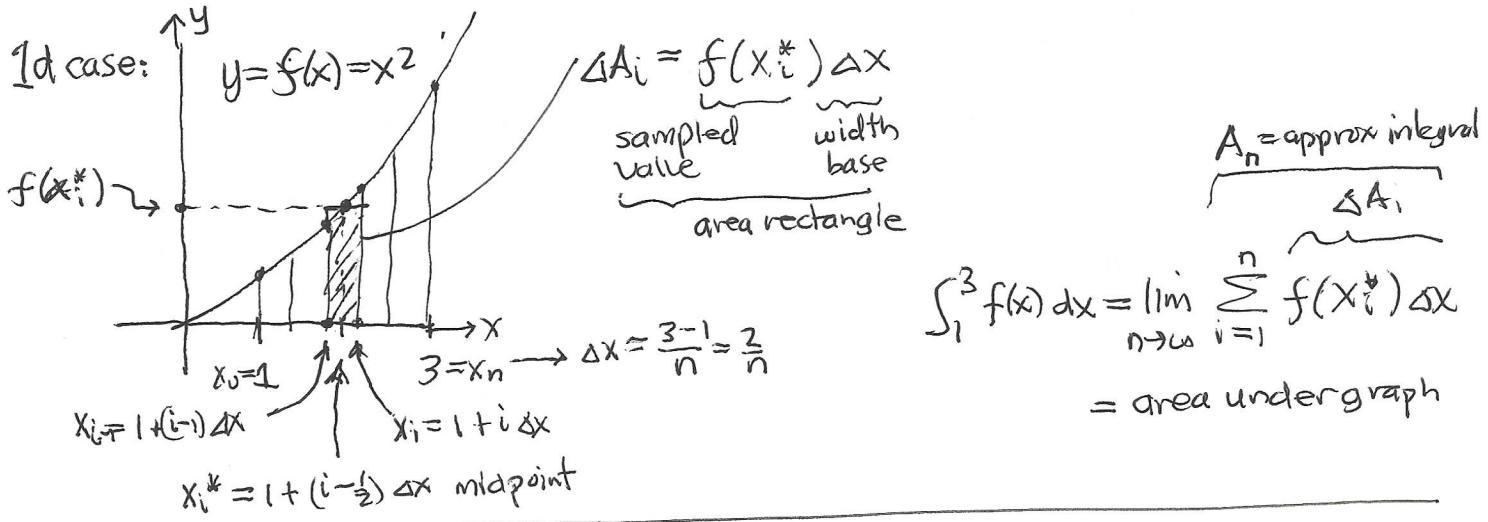


$\int \left\{ \dots \right. dy dx$

ditto

now more than ever these differentials MATTER!

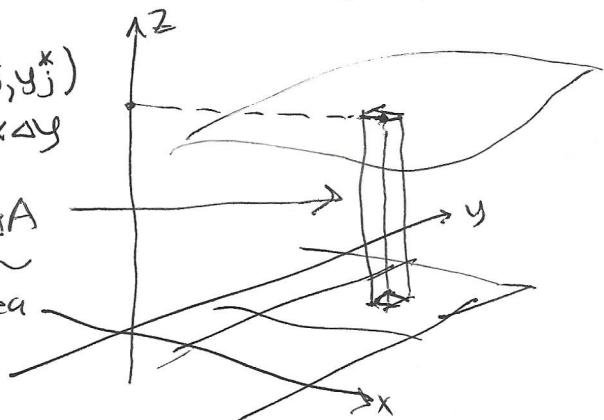
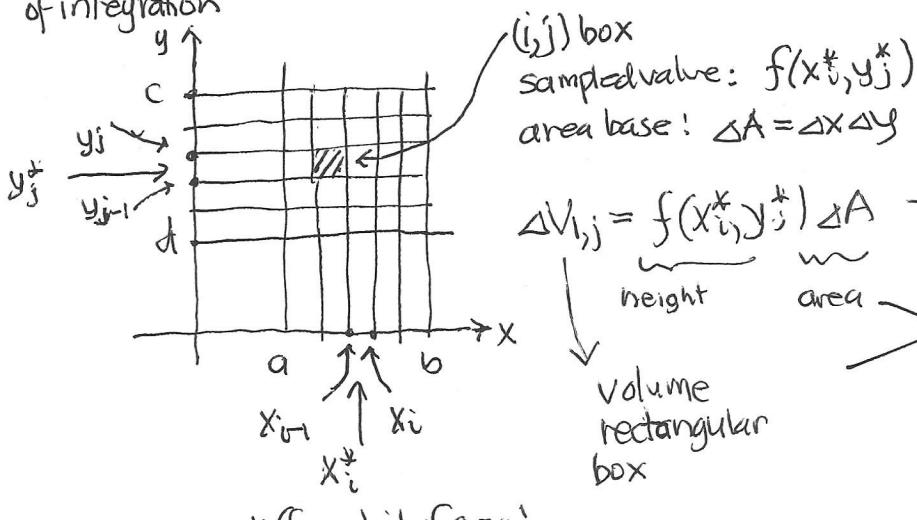
Riemann limit gives volume interpretation to double integral



2d case:

rectangular region $R: [a, b] \times [c, d]$

of integration



differential of area!

$$\iint_R f(x, y) dA = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \underbrace{\sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x \Delta y}_{V_{(n,m)} \text{ approximate integral}}$$

integral of f w.r.t. area of base

volume under graph
(signed volume when f not non-negative)

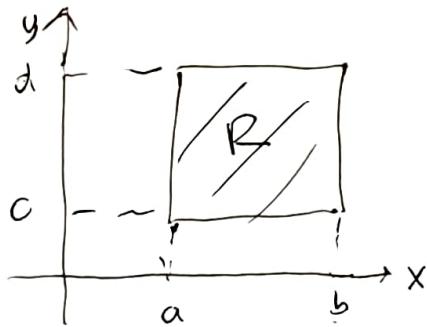
ApproximateIntTutor helps you evaluate approximate integrals

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so how are the Riemann integrals related to the iterated double integrals?



$$\iint_R f(x,y) dA$$

integral of f wrt
differential of area

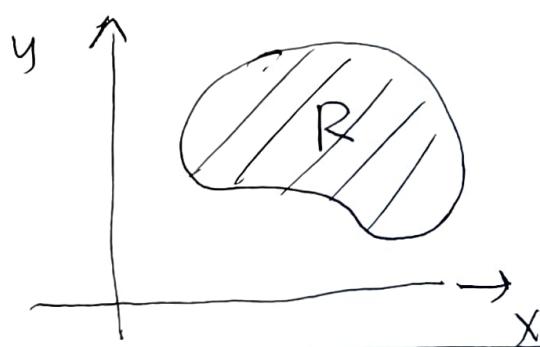
interpretation
(geometric definition)

$$= \int_c^d \int_a^b f(x,y) dy dx = \int_a^b \int_c^d f(x,y) dx dy$$

↑ "iterated integrals"
justified next time
using cross-section idea

↑ evaluation
(computational definition)

then



generalize to integrals over
nonrectangular regions of
plane & space
(14.2)

3-d



then



(14.6)