

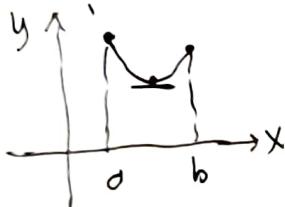
14176

max-min word problems and boundaries

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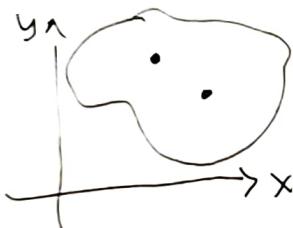
Calc 1 word problems are numerous! Many involve geometry in order to serve as toy problems in reaching understanding of how to convert the words into a well formulated max-min problem that we can then solve, and then return to "answer" the word problem which usually poses some question to be answered. Applications of this technique in STEM fields is also important. 2d max-min problems are similar.

Draw a diagram when possible, introduce appropriate variable names and translate the words into mathematical expressions and statements.

Boundaries

When we look for the global max and min of a calc 1 function on a closed interval,

- 1) we check for critical points in the interior and their function values.
- 2) we check the values of the function at the endpoints.
- 3) we pick the smallest and largest such function values



If we have a 2-d region of the plane confining the allowed domain of a function $f(x, y)$ we must do the same procedure but not checking the extreme values on the boundary to compare with those in the interior.

Constraints

Many interesting problems start with a function of 3 independent variables on which a constraint is imposed that can be used to solve for one of the 3 variables and eliminate it from the problem, reducing to a function of the two remaining variables:

Extremize $\underbrace{F(x_1, y, z)}$ subject to the constraint $\underbrace{C(x_1, y, z) = 0}$
 expression for "objective function"
 to be extremized equation
 ↓ ↓
 don't confuse these (or any constant!)

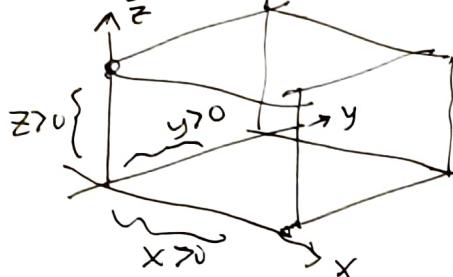
14.7b

max-min word problems and boundaries

(2)

Example A rectangular box with no lid is to be made from 12 m^2 of cardboard. Find the maximum volume of the box.

Draw diagram
assign variables



$$\text{Volume: } V = xyz$$

$$\text{Area: } A = xy + 2xz + 2yz$$

$$\text{constraint: } A = xy + 2z(x+y) = 12$$

eliminate z ! (easiest)

$$z = \frac{12 - xy}{2(x+y)} > 0 \rightarrow 12 > xy \\ xy < 12$$

$$V(x,y) = xy \left(\frac{12 - xy}{2(x+y)} \right), \quad \underbrace{x > 0, y > 0, xy < 12}_{\text{boundary of set missing closing points at infinity}}$$

Because $V=0$ on boundary which is excluded from possible answers (dimensions > 0 !) and $V > 0$ in the interior, the maximum value must occur at a critical point in the interior. If there is only one, it must be the global maximum

Find critical points:

$$V = \frac{xy(12 - xy)}{2(x+y)}$$

$$V_x = \dots = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} = 0$$

$$V_y = \dots = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2} = 0$$

$$\begin{aligned} 12 - 2xy - x^2 &= 0 & 12 - 2xy - y^2 &= 0 \\ -: 0 + y^2 - x^2 &= 0 \rightarrow y = x & & \end{aligned}$$

$\xrightarrow{x^2 = 12/3 = 4}$
 $\xrightarrow{y = x}$
 $(2,2)$

No need to check 2nd derivatives but

$$V_{xx} = \frac{-y^2(y^2+12)}{(x+y)^3} < 0, \quad V_{yy} = \frac{-x^2(x^2+12)}{(x+y)^3} < 0$$

$$V_{xy} = \frac{y^2(x^2+4xy+y^2-36)}{(x+y)^3}$$

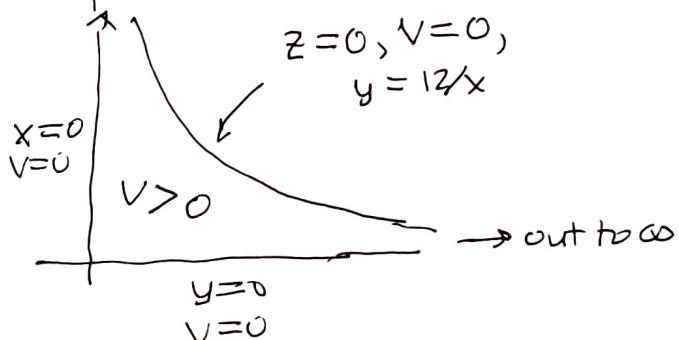
$$V_{xx}(2,2) = -1 \neq V_{yy}(2,2), \quad V_{12}(2,2) = -1/2, \quad V_{xx}(2,2)V_{yy}(2,2) - V_{xy}^2(2,2) = 1 - (-\frac{1}{2})^2 = \frac{3}{4} > 0$$

Answer: The largest box has base 2m and 2m height 1m, and volume $\frac{4}{3} \text{ m}^3$

↑
length
units:
meters

$$x > 0, y > 0, xy < 12$$

boundary of set missing closing points at infinity



note obvious symmetry between x and y but not z
(since the top is missing)

$$z = \frac{12 - (2x^2)}{2(2+x^2)} = \frac{12}{2(2+4)} = 1$$

$$V = 2 \cdot 2 \cdot 1 = V(2,2) = 4$$

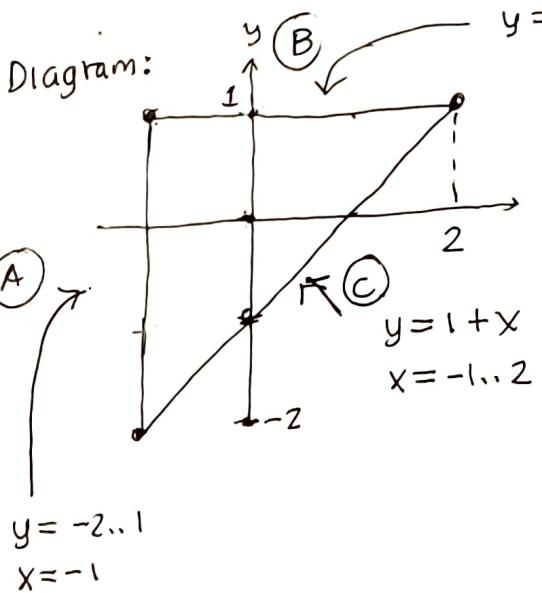
(only makes sense for
Maple not by hand!)

14.7b 2d max-min word problems and boundaries

(3)

Example Extremize $f(x,y) = x^2 + 2xy + 3y^2$ on the triangle with vertices $(-1, -2), (-1, 1), (2, 1)$.

Diagram:



$$y = 1, x = -1 \dots 2$$

$$f(x,y) = x^2 + 2xy + 3y^2$$

$$f_x(y_1, y) = 2x + 2y = 2(x+y) = 0 \rightarrow y = -x$$

$$f_y(y_1, y) = 2x + 6y = 2(x+3y) = 0 \rightarrow x - 3x = 0$$

$$x = 0$$

$$(0, 0)$$

only critical pt.

$$f(0, 0) = 0$$

$$f_{xx}(x, y) = 2 > 0 \Leftrightarrow \text{local min}$$

$$f_{yy}(x, y) = 6 > 0 \Leftrightarrow \text{local min}$$

$$f_{xy}(x, y) = 2$$

$$f_{xx}(0, 0) f_{yy}(0, 0) - f_{xy}(0)^2 = 2(6) - 2^2 > 0 \text{ confirms local min}$$

Boundary has 3 sides, each leads to a 1-d max-min problem.

$$\textcircled{A} \quad x = -1, -2 \leq y \leq 1 \quad f(-1, y) = 1 - 2y + 3y^2 = g(u) \quad g\left(\frac{1}{2}\right) = 1 - 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 = \frac{2}{3}$$

$$g'(u) = -2 + 6u = 0 \rightarrow u = \frac{1}{3}$$

$$g''\left(\frac{1}{3}\right) = 4\left(\frac{1}{3}\right) = 2 > 0 \Leftrightarrow \text{local min}$$

$$\textcircled{B} \quad y = 1, -1 \leq x \leq 2 : \quad f(x, 1) = x^2 + 2x + 3 = h(x)$$

$$h'(x) = 2x + 2 = 0 \rightarrow x = -1 \rightarrow h(-1) = 1 - 2 + 3 = 2 \text{ at endpt}$$

$$h''(x) = 2 > 0 \Leftrightarrow \text{local min}$$

$$\textcircled{C} \quad y = x - 1, -1 \leq x \leq 2 : \quad f(x, x-1) = x^2 + 2x(x-1) + 3(x-1)^2$$

$$= x^2 + 2x^2 - 2x + 3 - 6x + 3x^2$$

$$= 6x^2 - 8x + 3 = k(x)$$

$$k'(x) = 12x - 8 = 0 \rightarrow x = 12/8 = 3/2 \quad k\left(\frac{3}{2}\right) = \dots = \frac{1}{3}$$

$$k''(x) = 12 > 0 \Leftrightarrow \text{local min} \quad \downarrow y = 3/2 - 1 = 1/2$$

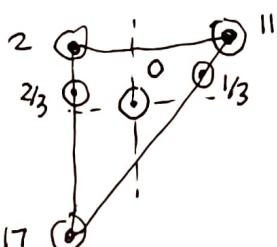
must also check:

Three vertices

$$f(-1, -2) = (-1)^2 + 2(-1)(-2) + 3(-2)^2 = 1 + 4 + 12 = 17$$

$$f(-1, 1) = (-1)^2 + 2(-1)(1) + 3(1)^2 = 1 - 2 + 3 = 2$$

$$f(2, 1) = 2^2 + 2(2)(1) + 3(1)^2 = 4 + 4 + 3 = 11$$



min value 0 at $(0, 0)$
max value 17 at $(-1, -2)$

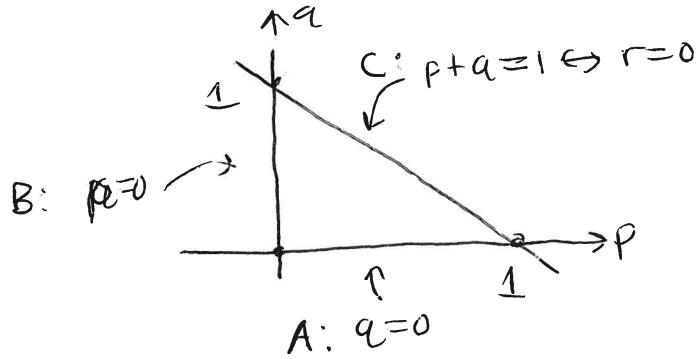
HW remarks 14.7.58

word problem, YadaYada. \rightarrow Math extremization problem

Maximize $P = 2pq + 2pr + 2rq$ subject to constraints $\begin{cases} p > 0, q > 0, r > 0 \\ p + q + r = 1 \end{cases}$

solve constraint to eliminate one variable by expressing it in terms of the others: $r = 1 - p - q > 0 \rightarrow p + q < 1$

so $P(R) = 2pq + 2(p+q)(1-p-q)$



Maximize inside triangle or on boundary

$$\begin{aligned} A: q=0, p=0 \dots &\rightarrow P = 2p(1-p) \\ B: p=0, q=0 \dots &\rightarrow P = 2q(1-q) \\ C: p+q=1 \rightarrow q=1-p, p=0 \dots &\rightarrow P = 2pq = 2p(1-p) \end{aligned}$$

} 1-d extremization
on closed intervals
 $f \sim$ downward parabola,
max at midpoint
between zeros
etc.

Interior: $P(p,q) \rightarrow$ find crits, use second derivative test.

58. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p , q , and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

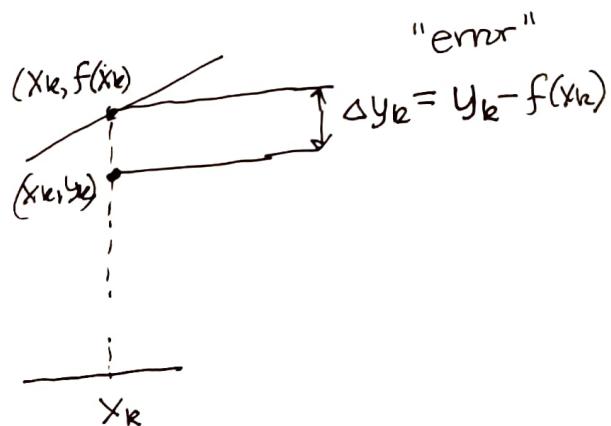
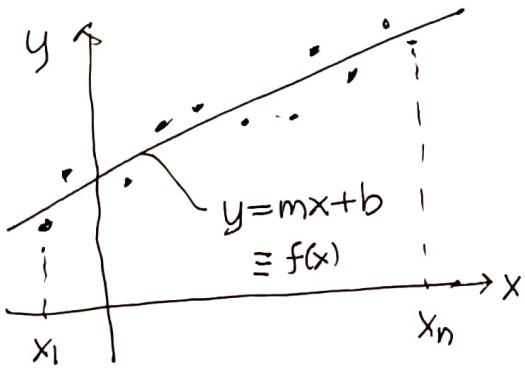
14.7b 2d max-min word problems and boundaries

(5)

method of least squares data fitting (Excel!)

Given n data points $(x_1, y_1), \dots, (x_n, y_n)$ in the plane, find the "line of best fit":

This is one of the most useful applications of this 2-d max-min technique and generalizes to higher order polynomial fits for $n > 2$ or fitting other families of functions depending on n parameters.



data vectors:

$$\vec{x} = \langle x_1, \dots, x_n \rangle$$

$$\vec{y} = \langle y_1, \dots, y_n \rangle$$

$$\vec{f} = \langle f(x_1), \dots, f(x_n) \rangle$$

$$\vec{\Delta y} = \langle y_1 - f(x_1), \dots, y_n - f(x_n) \rangle$$

2 points in the $2n$ -dim space of n data points

$$\vec{\Delta y} = \langle y_1 - (mx_1 + b), \dots, y_n - (mx_n + b) \rangle \Rightarrow \text{minimize square of distance in that space}$$

$$\vec{\Delta y} \cdot \vec{\Delta y} = \sum_{k=1}^n (y_k - (mx_k + b))^2 = F(m, b)$$

extremize!

$$0 = F_m(m, b) = F_b(m, b) \rightarrow \text{one critical point (global minimum)}$$

see problem 14.7. 59 for details
if interested

log-log fits $y = Cx^m \rightarrow \underbrace{\ln y}_{\vec{Y}} = m \underbrace{\ln x}_{\vec{X}} + \underbrace{\ln C}_b \Leftrightarrow \vec{Y} = m \vec{X} + b$

"power law" functions are straight lines on ln-ln graphs, i.e., transform to the corresponding logarithmic data, fit with straight line, then transform back.