

14.7a max-min math (2d)

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Calculus provides tools to understand how functions behave. An important aspect of their behavior is where they exhibit extreme values: local maximum or minimum values.

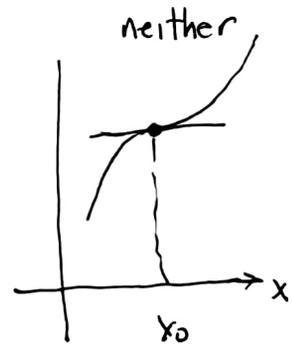
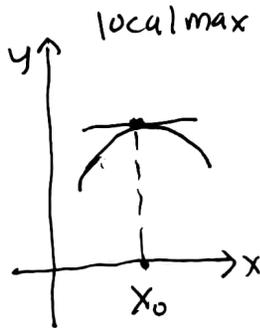
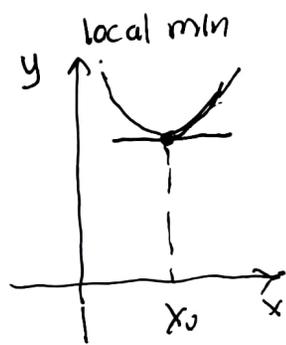
"Extremizing" or "optimizing" functions is important for practical applications.

Here we only consider max-min problems in 2 independent variables to give a taste of the general situation in higher dimensions that require linear algebra for $n > 2$ independent variables.

Calc 1 max-min theory excluding nondifferentiable extrema

local extrema occur at horizontal tangent lines

critical points have $f'(x_0) = 0$



second derivative test

$f''(x_0) > 0$
concave up

$f''(x_0) < 0$
concave down

$f''(x_0) = 0$
 f'' changes sign

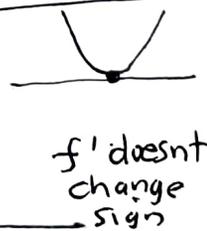
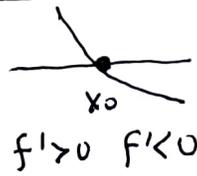
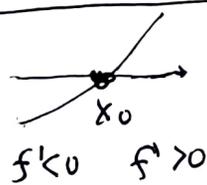
"icon" {

happy face up

sad face down

inflection pt.

first derivative test



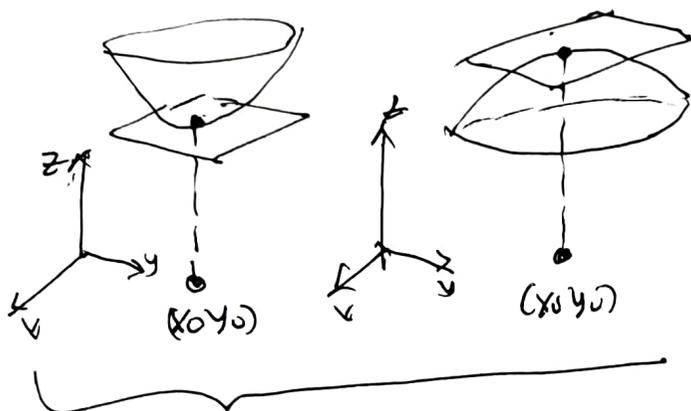
exclude nondifferentiable critical points



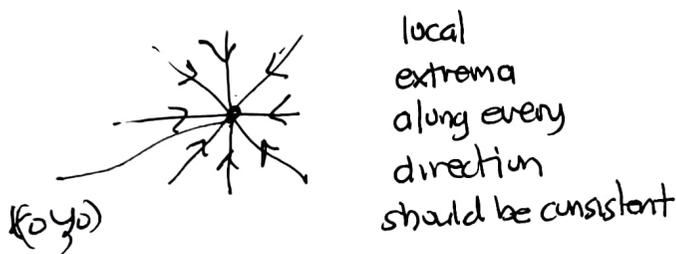
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$n=2$ extrema occur at a horizontal tangent plane



$z = f(x, y)$
horizontal tangent plane?



in order to be a local extrema as a function of the two variables
[we assume f differentiable at (x_0, y_0)]

2-d critical points

tangent plane: $z = L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$
 $\qquad\qquad\qquad = f(x_0, y_0) \qquad\qquad\qquad = 0$

horizontal plane thru pt on graph

$\rightarrow \frac{\partial f}{\partial x}(x_0, y_0) = 0 = \frac{\partial f}{\partial y}(x_0, y_0)$

$\leftrightarrow \boxed{\vec{\nabla} f(x_0, y_0) = 0}$

true in any dimension

2nd derivative test?

We need to calculate the 2nd directional derivative at (x_0, y_0) :

$D_{\hat{u}}^2 f = D_{\hat{u}}(D_{\hat{u}} f) = \frac{d^2 f(\vec{r}(t))}{dt^2}$ for $\vec{r} = \vec{r}_0 + t \hat{u}$ for all \hat{u}

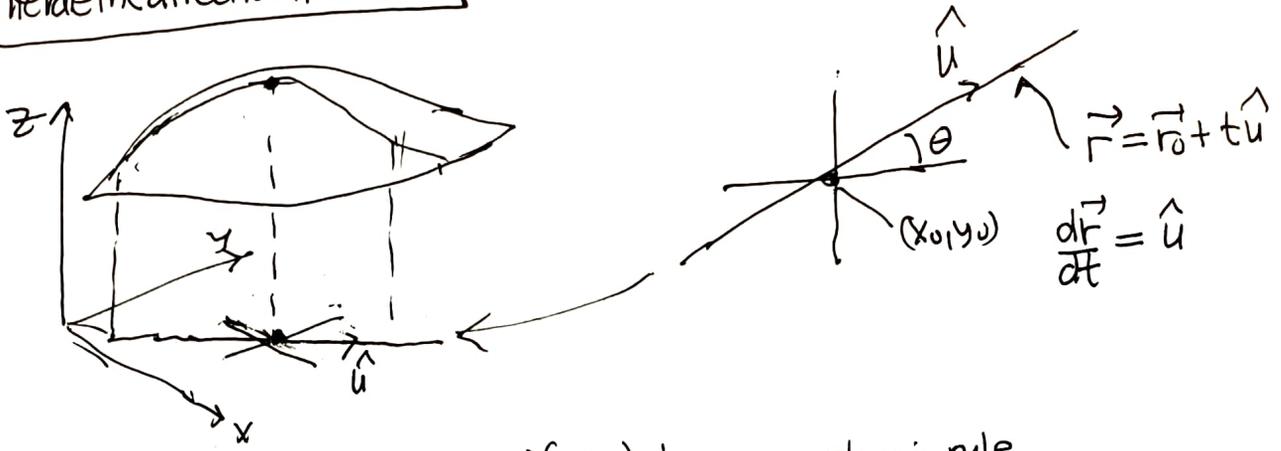
this should be the same sign in every direction

- $\left\{ \begin{array}{l} > 0 : \text{local min} \\ < 0 : \text{local max} \end{array} \right.$

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Heuristics: directional derivative



chain rule

$$\frac{d}{dt} f(\vec{r}(t)) = \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) \Big|_{\vec{r}(t)}$$

$$= \vec{r}'(t) \cdot \nabla f(\vec{r}(t)) = \hat{u} \cdot \nabla f(\vec{r}(t)) = D_{\hat{u}} f(\vec{r}(t))$$

$$= u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y}$$

iterate:

$$D_{\hat{u}}^2 f = D_{\hat{u}} (D_{\hat{u}} f) = u_1 \frac{\partial}{\partial x} (u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y}) + u_2 \frac{\partial}{\partial y} (u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y})$$

$$= u_1^2 \frac{\partial^2 f}{\partial x^2} + u_1 u_2 \frac{\partial^2 f}{\partial x \partial y} + u_2 u_1 \frac{\partial^2 f}{\partial y \partial x} + u_2^2 \frac{\partial^2 f}{\partial y^2}$$

order doesn't matter: $f_{xy} = f_{yx}$

$$= f_{xx} u_1^2 + f_{yy} u_2^2 + 2 f_{xy} u_1 u_2$$

$\hat{u} = \hat{i} : D_{\hat{u}}^2 f = f_{xx}$
 $\hat{u} = \hat{j} : D_{\hat{u}}^2 f = f_{yy}$
 reduces to 2nd partials along axes

mixed partials needed to determine 2nd derivatives along in between directions

We want $D_{\hat{u}}^2 f$ to have the same sign for every direction so that all 1-d critical points have an extremum of the same kind.

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$$f_{xx} u_1^2 + f_{yy} u_2^2 + 2f_{xy} u_1 u_2$$

must have same sign for all \hat{u} at a critical pt where $\vec{\nabla} f(x_0, y_0) = 0$

so CANNOT CHANGE SIGN

so no directions where $= 0$

$$= u_2^2 \left[\underbrace{f_{xx} \left(\frac{u_1}{u_2}\right)^2 + f_{xy} \left(\frac{u_1}{u_2}\right) + f_{yy}}_a \quad b \quad c \right]$$

cannot change sign

$= 0$ cannot have solutions

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

discriminant $b^2 - 4ac < 0$ means no real roots

$$b^2 - 4ac = (2f_{xy})^2 - 4f_{xx}f_{yy} = 4(f_{xy}^2 - f_{xx}f_{yy}) < 0$$

$$\boxed{f_{xx}f_{yy} - f_{xy}^2 > 0}$$

is the condition that all 1d extrema are consistent.

if $f_{xy} = 0$

f_{xx} and f_{yy} must have same sign for consistent extrema along the axis directions so must be same sign so $f_{xx}f_{yy} > 0$ is necessary but not sufficient

This quantity must remain positive after subtracting the mixed derivative term.

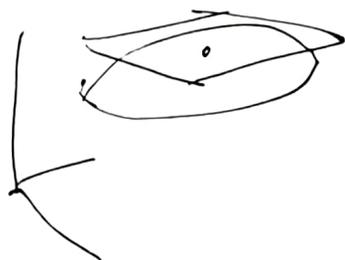
If $f_{xx}f_{yy} < 0$ go no further! since $f_{xx}f_{yy} - f_{xy}^2$ can only be more negative

$\begin{matrix} + & - \\ - & + \end{matrix}$
must be of opposite signs

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What is the takeaway? How do we use this?



Find critical points where $\vec{\nabla} f(x_0, y_0) = \vec{0}$

Check that $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) > 0$
must be same sign!

If so, next check that $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0$

If so, this confirms local extrema suggested by sign of consistent 2nd partials along axes.

If $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0$ then "saddle pt"
concave up in some directions
concave down in other directions

If $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 = 0$ "Inconclusive"
just like in calc 1.

example: $f(x) = x^4$
 $f'(x) = 4x^3 = 0 \rightarrow x=0$ critical pt
 $f''(x) = 12x^2$ $f''(0) = 0$ 2nd derivative test fails

need analysis of higher derivatives!

$f_{xx} f_{yy} - f_{xy}^2$
 $= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$
need linear algebra
for $n > 2$
determinants
etc

Taylor series: $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{6} f'''(x_0)(x-x_0)^3 + \dots$
value unimportant
need to look at first nonzero term after constant term

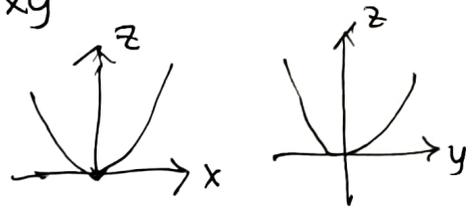
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easy example

$$z = f(x,y) = x^2 + y^2 - 3xy$$

along axes: $z = x^2$ ($y=0$)
 $z = y^2$ ($x=0$)



both local mins
 maybe 2d local min?

$$f_x = 2x - 3y \begin{cases} f_{xx} = 2 > 0 \quad \checkmark \\ f_{xy} = -3 \end{cases}$$

$$f_x = 2y - 3x \begin{cases} f_{yx} = -3 \\ f_{yy} = 2 > 0 \quad \checkmark \end{cases}$$

$f_{xx} f_{yy} > 0$
 ~~\checkmark~~ in all directions?

constants so no need to evaluate at (0,0)

$$\begin{aligned} 2x - 3y = 0 \\ 2y - 3x = 0 \rightarrow y = \frac{3x}{2} \end{aligned} \quad \rightarrow \quad 2x - 3\left(\frac{3x}{2}\right) = 0 \rightarrow \left(2 - \frac{9}{2}\right)x = 0 \rightarrow x = 0$$

$y = 0$
 (0,0)
 only critical pt

$$\begin{aligned} f_{xx} f_{yy} - f_{xy}^2 &= 2(2) - (-3)^2 \\ &= 4 - 9 < 0 \quad (\text{value unimportant!}) \end{aligned}$$

saddle point!

in fact along $y=x$: $z = f(x,x) = x^2 + x^2 - 3x^2 = -x^2$



local max!

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example $f(x,y) = x^4 + y^4 - 4xy + 1$

$$f_x = 4x^3 - 4y = 4(x^3 - y) = 0 \rightarrow y = x^3$$

$$f_y = 4y^3 - 4x = 4(y^3 - x) = 0 \rightarrow (x^3)^3 - x = 0$$

$$x^9 - x = 0$$

$$x(x^8 - 1) = 0$$

$$\downarrow x=0 \rightarrow y=0$$

$$\downarrow x^2=1 \rightarrow x = \pm 1$$

$$\downarrow y = (\pm 1)^3 = \pm 1$$

make table:

	(0,0)	(1,1)	(-1,-1)
$f_{xx} = 12x^2$	0	$12 > 0$	ditto (only depends on x^2, y^2)
$f_{xy} = -4$	-4	-4 local min?	
$f_{yy} = 12y^2$	0	$12 > 0$	
$f_{xx}f_{yy} - f_{xy}^2$	$0 - (-4)^2 < 0$	$(12)^2 - 4(-4)^2 > 0$	
	saddle.	confirms local min	ditto

$f(0,0) = 1$

$$f(\pm 1, \pm 1) = (\pm 1)^4 + (\pm 1)^4 - 4(\pm 1)(\pm 1) + 1$$

$$= 1 + 1 - 4 + 1 = -1$$

minimum value

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example 9

$$f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4 \quad x=0$$

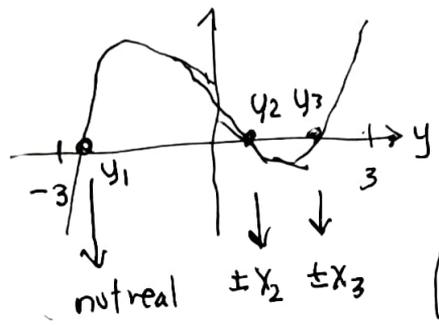
$$f_x = 20xy - 10x - 4x^3 = 4x(5y - 5/2 - x^2) = 0 \quad x^2 = 5y - 5/2$$

$$f_y = 10x^2 - 8y - 8y^3 = -2(4y^3 + 2y - 5x^2)$$

$$x=0: 4y^3 + 4y - 5x^2 = 0 \rightarrow 4y(1+y^2) = 0 \rightarrow y=0 \quad (0,0)$$

$$x^2 = 5y - 5/2: -5x^2 = -5y + 25/2: 4y^3 + 4y - 25y + 25/2 = 0$$

$$4y^3 - 21y + 25/2 = 0$$



cubic: 3 roots (numerical): y_1, y_2, y_3

backsub to $x^2 = 5y - 5/2$:

get only $\pm x_2, \pm x_3$

$$(0,0), (y_2, \pm x_2), (y_3, \pm x_3)$$

($x_1^2 < 0$)

5 critical points symmetric about y axis.

Evaluate:

(x,y)	$(0,0)$	$(\pm x_2, y_2)$	$(\pm x_3, y_3)$
$f(x,y)$	0	-1.48	8.50

so pbt $z = -2...10$

classify critical points

$$f_{xx} = 20y - 10 - 8x^2 = 2(10y - 5 - 4x^2)$$

$$f_{yy} = -8 - 24y^2 = -8(1+y^2) < 0 \quad \text{local max}$$

$$f_{xy} = 20x$$

$(x,y) = (0,0):$

$$f_{xx} = -10 < 0$$

$$f_{yy} = -8 < 0$$

$$f_{xy} = 0$$

confirms **local max**

$$y^2 = 5y - 5/2$$

$$\downarrow$$

$$= 2(10y - 5 - 4(5y - 5/2))$$

$$= 2(-10y + 5) = 10(1 - 2y)$$

$$\begin{cases} y_2 = .65 < 0 & \text{local max's} \\ y_3 = 1.90 < 0 & ? \end{cases}$$

$$f_{xx}f_{yy} - f_{xy}^2 \begin{cases} < 0 \text{ for } y_2 & \text{saddle} \\ > 0 \text{ for } y_3 & \text{local max} \end{cases}$$

[see Maple]

2 local max's,
2 saddle points