

[4.6b] gradient

The gradient vector field (vector-valued function of position) is the natural way to bundle together all the partial derivatives of a function into a "total" derivative.

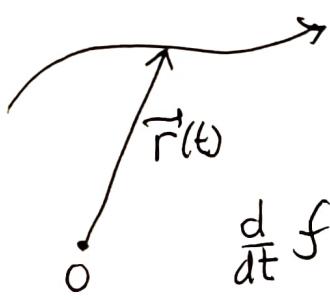
$$\vec{\nabla} f(x_1, \dots, x_n) = \left\langle \frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n) \right\rangle$$

We introduced it to define the directional derivative but the chain rule derivation applies to any parametrized curve

CHAIN RULE

The chain rule tells us how to evaluate the derivative of a function along a parametrized curve.

To be concrete consider 3-d:



$$\begin{aligned} \frac{d}{dt} f(\vec{r}(t)) &= \frac{\partial f}{\partial x}(\vec{r}(t)) \frac{dx(t)}{dt} + \frac{\partial f}{\partial y}(\vec{r}(t)) \frac{dy(t)}{dt} + \frac{\partial f}{\partial z}(\vec{r}(t)) \frac{dz(t)}{dt} \\ &= \left\langle \frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}(\vec{r}(t)), \frac{\partial f}{\partial y}(\vec{r}(t)), \frac{\partial f}{\partial z}(\vec{r}(t)) \right\rangle \\ &= \vec{r}'(t) \cdot (\vec{\nabla} f)(\vec{r}(t)) \end{aligned}$$

dot product of tangent vector with the gradient

$$\begin{aligned} &\underbrace{|\vec{r}'(t)|}_{\frac{ds(t)}{dt}} \underbrace{\hat{T}(t)}_{(\hat{D}\hat{T}(t)f)(\vec{r}(t))} \\ &= \frac{ds(t)}{dt} \underbrace{\frac{df(\vec{r}(t))}{ds}}_{\text{arc length derivative of } f} \\ &= \frac{ds(t)}{dt} \end{aligned}$$

converts to parameter derivative

If we consider a straight line

$$\begin{aligned} \vec{r} &= \vec{r}_0 + t \hat{u} \quad (\text{arc length!}) \\ \frac{d\vec{r}}{dt} &= \hat{u} \rightarrow \left| \frac{d\vec{r}}{dt} \right| = 1 = \frac{ds}{dt} \end{aligned}$$

$$\begin{aligned} \text{so we get } \frac{d}{dt} f(\vec{r}(t)) &= D \hat{u} f(\vec{r}(t)) \quad \text{previous result.} \\ &= \hat{u} \cdot \vec{\nabla} f(\vec{r}(t)) \end{aligned}$$

[4.6b]

gradient

(2)

[contours and contour surfaces]

= "level curves", "level surfaces"
(2d) (3d)

$n > 3$: level hypersurfaces

Suppose our parametrized curve traces out a path within the set of all points where the function has the same value:

$$f(\vec{r}(t)) = f(\vec{r}(0)) \quad \leftarrow \text{eqn of contour set thru } \vec{r}(0)$$

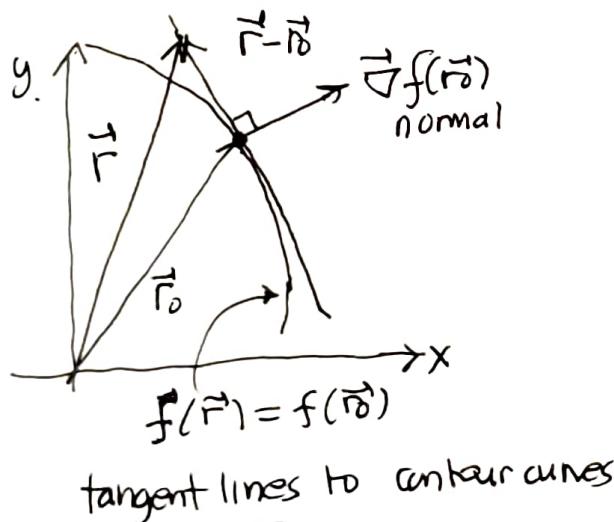
all points with same value as

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt} \underbrace{f(\vec{r}(0))}_{\text{constant}} = 0$$

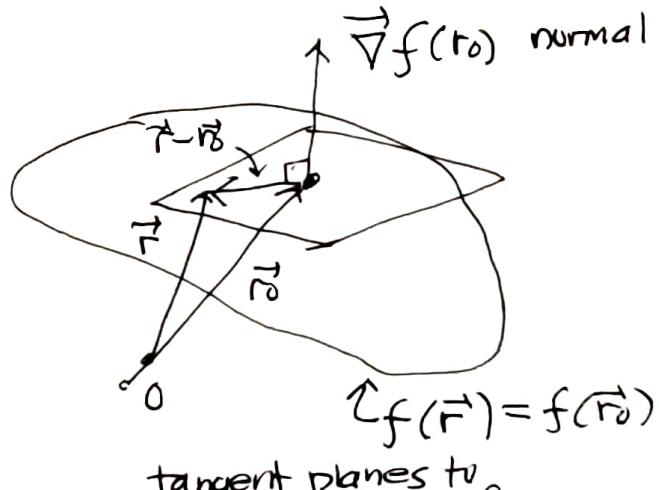
$$= \boxed{\vec{r}'(t) \cdot \nabla f(\vec{r}(t)) = 0}$$

all such tangent vectors are orthogonal to the
the tangent line / plane / hyperplane to the
level set has the eqn:

$$\boxed{\nabla f(\vec{r}(0)) \cdot (\vec{r} - \vec{r}(0)) = 0}$$



2d

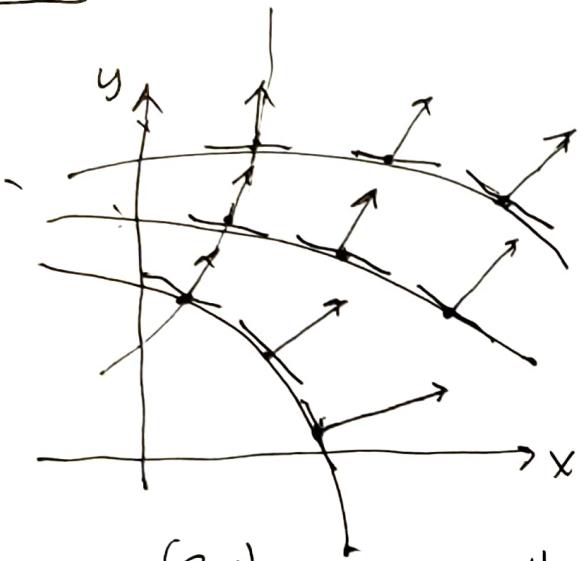


3d

14.6b

gradient

(3)



(2d)

the gradient vector field
is always orthogonal
to the contour curves

they point in the direction
the function is increasing
most rapidly

the most efficient path to climb
up the graph in 3d is to always
pick the steepest direction
(maximum rate of change)
like climbing a hill.

"path of steepest ascent"

Following the arrows is the most efficient way
to increase the value of the function.

(True in any dimension.)

14.6b) gradient

(4)

Example $f(x,y) = 16 - 4x^2 - y^2$ at $P(1,2)$

$$\vec{\nabla} f(x,y) = \langle -8x, -2y \rangle$$

$$f(1,2) = 16 - 4 - 4 = 8$$

$$\vec{\nabla} f(1,2) = \langle -8(1), -2(2) \rangle = \langle -8, -4 \rangle = 4 \langle -2, -1 \rangle$$

$$|\vec{\nabla} f(1,2)| = 4\sqrt{5}$$

$$\hat{\nabla} f(1,2) = -\frac{\langle 2, 1 \rangle}{\sqrt{5}}$$

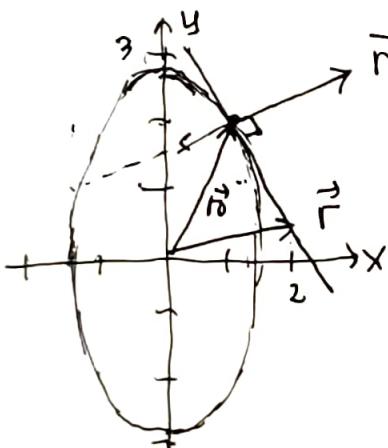
contours! $f(x,y) = 16 - 4x^2 - y^2 = k \leq 16$

$$4x^2 + y^2 = 16 - k \geq 0$$

$$\frac{x^2}{(\sqrt{16-k}/2)^2} + \frac{y^2}{(\sqrt{16-k})^2} = 1 \quad \text{ellipses semi-axes } a \neq b$$

contour thru point $P(1,2) : 16 - 4x^2 - y^2 = 8$

$$4x^2 + y^2 = 8 \rightarrow \frac{x^2}{2} + \frac{y^2}{8} = 1 \rightarrow a = \sqrt{2}, b = 2\sqrt{2} \approx 2.82 \approx 1.41 \quad \text{twice } a$$



$$\vec{n} = \langle 2, 1 \rangle \text{ or } \vec{\nabla} f(1,2)$$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\langle 2, 1 \rangle \cdot \langle x-1, y-1 \rangle = 0$$

$$2(x-1) + 1(y-1) = 0$$

$$2x + y = 2(1) + (1) = 5$$

$$\begin{array}{l} y = 5 - 2x \\ \downarrow \\ 0 \end{array}$$

$$y = 5$$

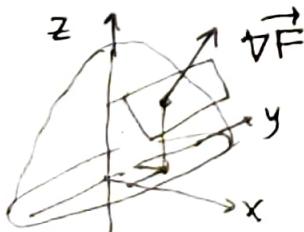
} intercepts

Graph in 3-d $z = f(x,y) = 16 - 4x^2 - y^2 \rightarrow z = \underbrace{4x^2 + y^2 + z - 16}_{z = f(x,y)} \equiv F(x,y,z)$

$$\vec{\nabla} F(x,y,z) = \langle 8x, 2y, 1 \rangle \quad \text{upward}$$

$$\vec{\nabla} F(1,2,8) = \langle 8, 4, 1 \rangle \quad \text{normal to tangent plane}$$

$$8(x-1) + 2(y-2) + 1(z-8) = 0 \rightarrow z = \underbrace{8 - 8(x-1) - 2(y-2)}_{\text{linear approximation}} = L(x,y)$$



[4.6b] gradient

(5)

textbook example

3-d temperature distribution peaked at the origin (units: °F)

$$T(x,y,z) = \frac{80}{1+x^2+y^2+3z^2} \quad \xleftarrow{\text{max: } T(0,0,0)=80}$$

causes decrease asymmetrically moving away from the origin

At $(x,y,z) = (1,1,2)$ what is the direction and maximum rate of change of T ?

$$T(1,1,2) = \frac{80}{1+1+2+3.4} = \frac{80}{7.4} = \frac{10}{2} = 5 \quad (\text{°F})$$

"isothermal" surface thru this point:

$$T(x,y,z) = \frac{80}{1+x^2+y^2+3z^2} = 5 \rightarrow 1+x^2+y^2+3z^2 = \frac{80}{5} = 16$$

$$x^2+y^2+3z^2 = 15$$

$$\frac{x^2}{15} + \frac{y^2}{15/2} + \frac{z^2}{15/3} = 1 \quad \text{ellipsoid}$$

$$\text{semiaxes: } a = \sqrt{15} > b = \sqrt{15/2} > c = \sqrt{15/3}$$

$$\begin{aligned} \frac{\partial T}{\partial x}(xyz) &= \frac{\partial}{\partial x} \left(\frac{80}{1+x^2+y^2+3z^2} \right)^{-1} \\ &= 80(-1) \left(1+x^2+y^2+3z^2 \right)^{-2} (0+2x) \\ &\quad \hookrightarrow 4y \rightarrow 6z \end{aligned}$$

$$\vec{\nabla} T(x,y,z) = \frac{-80}{(1+x^2+y^2+3z^2)^2} \underbrace{\langle 2x, 4y, 6z \rangle}_{\text{even}}$$

$$\vec{\nabla} T(1,1,2) = -\frac{80}{16^2} \underbrace{2 \langle 1, 2, 6 \rangle}_{\frac{5 \cdot 2}{16} = \frac{5}{8}} = -\frac{5}{8} \langle 1, 2, 6 \rangle$$

points backwards
"towards" origin general
direction in first octant.

$$|\vec{\nabla} T(1,1,2)| = \frac{5}{8} \sqrt{1+4+36} = \frac{5}{8} \sqrt{41}$$

$$\approx 4.002 \quad \left. \begin{array}{l} \text{interpretation: if we move } \Delta s = 0.1 \\ \text{along } \vec{\nabla} T(1,1,2), \Delta T \approx 4(.1) = 0.4 \\ \text{so temperature increases to} \\ \text{about } 5.4^\circ \text{ from } 5^\circ. \end{array} \right\}$$

$$\hat{\vec{\nabla}} T(1,1,2) = -\frac{\langle 1, 2, 6 \rangle}{\sqrt{41}}$$

14.6b

gradient

(6)

tangent plane: $\vec{n} = \langle 1, 2, 6 \rangle \propto \nabla T(1, 1, 2)$

$$\begin{aligned} 0 &= \vec{n} \cdot (\vec{r} - \vec{r}_0) = \langle 1, 2, 6 \rangle \cdot \langle x-1, y-1, z-2 \rangle \\ &= 1(x-1) + 2(y-1) + 6(z-2) \\ x+2y+6z &= 1+2+12 = 15 \end{aligned}$$

normal line: $\vec{r} = \vec{r}_0 + t\vec{n} = \langle 1, 1, 2 \rangle + t \langle 1, 2, 6 \rangle$

$$\langle x, y, z \rangle = \langle 1+t, 1+2t, 2+6t \rangle$$

but if we use $\hat{\nabla T}(1, 1, 2)$ we get the arclength parametrized curve pointing in the direction of fastest increase

$$\vec{r} = \langle 1, 1, 2 \rangle + s \frac{\langle 1, 2, 6 \rangle}{\sqrt{41}}$$

$$\langle x, y, z \rangle = \left\langle 1 - \frac{s}{\sqrt{41}}, 1 - \frac{2s}{\sqrt{41}}, 2 - \frac{6s}{\sqrt{41}} \right\rangle$$

$s=0, 1$; evaluate T at new point:

$T \approx 5.42$ numerical value of exact temperature

$T_{\text{linear}} = 5.40$ close to exact value.

14.6b

gradient

⑦

math geek fun textbook - 14.6.61

consider the level surfaces

$$F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} \equiv \sqrt{c}, c > 0$$

Show that the sum of the three axis intercepts of any tangent plane has the same value. Find it.

SOLUTIONFix a point (x_0, y_0, z_0) to evaluate the tangent plane.

$$\vec{\nabla} F(x, y, z) = \left\langle \frac{1}{2}x^{-1/2}, \frac{1}{2}y^{-1/2}, \frac{1}{2}z^{-1/2} \right\rangle$$

$$\vec{\nabla} F(x_0, y_0, z_0) = \left\langle \frac{1}{2\sqrt{x_0}}, \frac{1}{2\sqrt{y_0}}, \frac{1}{2\sqrt{z_0}} \right\rangle$$

$$\text{choose } \vec{n}(\vec{r}) = \left\langle \frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}} \right\rangle \text{ simpler}$$

tangent plane:

$$0 = \vec{n}(r) \cdot (\vec{r} - \vec{r}_0) = \left\langle \frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}} \right\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$= \frac{x - x_0}{\sqrt{x_0}} + \frac{y - y_0}{\sqrt{y_0}} + \frac{z - z_0}{\sqrt{z_0}} = \underbrace{\frac{x}{\sqrt{x_0}} - \frac{\sqrt{x_0}}{\sqrt{x_0}}}_{= -F(x_0, y_0, z_0)} + \underbrace{\frac{y}{\sqrt{y_0}} - \frac{\sqrt{y_0}}{\sqrt{y_0}}}_{= -F(x_0, y_0, z_0)} + \underbrace{\frac{z}{\sqrt{z_0}} - \frac{\sqrt{z_0}}{\sqrt{z_0}}}_{= -F(x_0, y_0, z_0)} = -F(x_0, y_0, z_0) = -\sqrt{c}$$

$$\Rightarrow \boxed{\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{c}}$$

$$\underline{\text{Intercepts}} \quad y = z = 0 \rightarrow \frac{x}{\sqrt{x_0}} = \sqrt{c} \rightarrow x = \sqrt{x_0} \sqrt{c}$$

$$x = y = 0 \quad \therefore \text{diff to } - \quad y = \sqrt{y_0} \sqrt{c}$$

$$x = y = 0 \quad \therefore \text{diff to } - \quad z = \sqrt{z_0} \sqrt{c}$$

$$\text{sum: } \sqrt{x_0} \sqrt{c} + \sqrt{y_0} \sqrt{c} + \sqrt{z_0} \sqrt{c}$$

$$\text{sum of intercepts: } \underbrace{(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})}_{\sqrt{c}} \sqrt{c} = c !$$