

# 14.5a chain rule

The chain rule governs derivatives of composed functions ①

Calc 1  
(n=1)

intermediate variable

dep. var.      final ind. var.

$$y = f(u) \text{ but } u = g(x)$$

$$= f(g(x)) \equiv (f \circ g)(x)$$

The two functions are linked together by this chain of relationships, hence the name "chain rule!"

$$\frac{d}{dx} f(g(x)) = \underbrace{f'(g(x))}_{\text{outer der.}} \underbrace{g'(x)}_{\text{inner der.}}$$

function notation

or:

$$y = y(u(x))$$

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{\text{outer der.}} \underbrace{\frac{du}{dx}}_{\text{inner der.}}$$

→ related rate of change of u wrt x corrects the outer function's derivative wrt u.

dependent variable notation

example  $F(x) = (1+x^2)^{1/2} = \underbrace{\text{sqrt}}_f(\underbrace{1+x^2}_{g(x)})$        $\begin{cases} f(x) = x^{1/2} \\ g(x) = 1+x^2 \end{cases}$

$$F'(x) = \frac{d}{dx} (1+x^2)^{1/2} = \underbrace{\frac{1}{2}(1+x^2)^{-1/2}}_{\text{outer}} \underbrace{\frac{d}{dx} (1+x^2)}_{\text{inner}}$$

$$= \frac{1}{2} (1+x^2)^{-1/2} (0+2x) = \frac{x}{(1+x^2)^{1/2}}$$

n > 1

$$\frac{dy}{dx} = \frac{dy(u(x))}{du} \frac{du}{dx}$$

→ replace by sum of 1-d chain rule terms for each ind. var.

14.5a chain rule

(2)

n=2

$$z = f(x, y) \quad \begin{matrix} x = g(t) \\ y = h(t) \end{matrix} = f(g(t), h(t))$$

$\frac{dz}{dt} =$  f has  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  ??  
 how do we express this in terms of them?  $\frac{d}{dt} f(g(t), h(t))$

result  
function notation

$$\frac{d}{dt} f(g(t), h(t)) = \frac{\partial f}{\partial x}(g(t), h(t)) g'(t) + \frac{\partial f}{\partial y}(g(t), h(t)) h'(t)$$

$\downarrow$   
 if only were a function of x ↑  
↑  
 if only were a function of y

Just add the chain rule terms for each independent variable

dep. var. notation:

$$\frac{d}{dt} z(x(t), y(t)) = \frac{\partial z}{\partial x}(x(t), y(t)) \frac{dx(t)}{dt} + \frac{\partial z}{\partial y}(x(t), y(t)) \frac{dy(t)}{dt}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \left( = \frac{\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}{dt} \right)$$

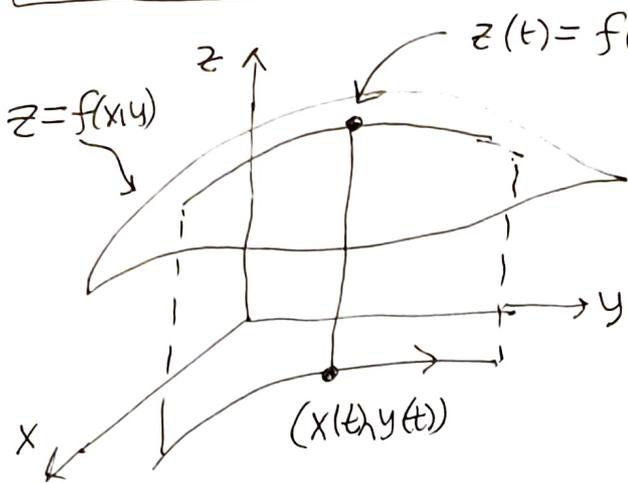
$$\left( = \frac{dz(x, y)}{dt} \leftarrow \text{differential} \right)$$

Why? see next page.

14.5a chain rule

(3)

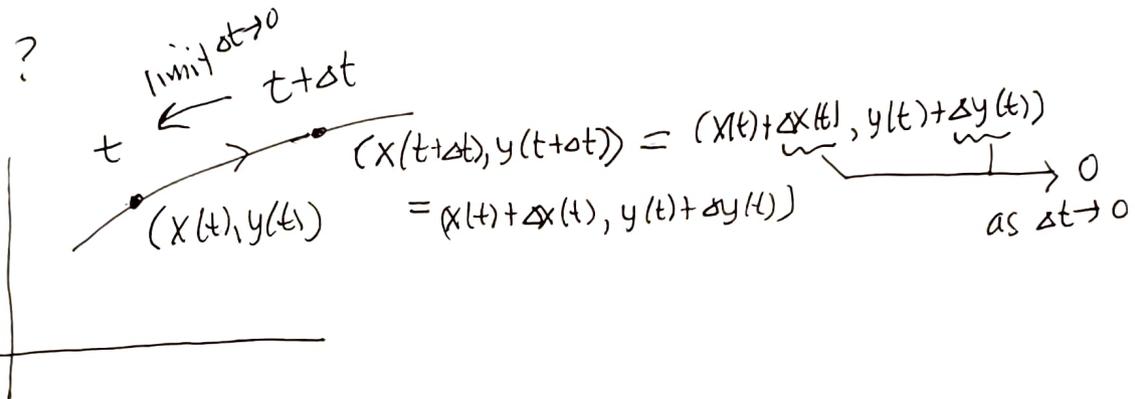
interpretation: derivative of a function along a curve



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

sum of 1d chain rules for each ind. variable

why?



linear approximation to change in f:  $\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$

exact change:

$$\Delta z = \underbrace{\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y}_{\text{linear change}} + \underbrace{\epsilon_1 \Delta x + \epsilon_2 \Delta y}_{\text{error}}$$

so: 
$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

$$\frac{dz}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\partial f}{\partial x} \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{\partial f}{\partial y} \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} + \lim_{\Delta t \rightarrow 0} \epsilon_1 \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \rightarrow 0} \epsilon_2 \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t}$$

$$= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

if f is differentiable there

14.5a

4

poor example

$$z = x^2 - y^2 \left\{ \begin{array}{l} \frac{\partial z}{\partial x} = 2x \\ \frac{\partial z}{\partial y} = -2y \end{array} \right. \left. \begin{array}{l} \frac{\partial z}{\partial x} = 10 \\ \frac{\partial z}{\partial y} = -6 \end{array} \right\} \leftarrow z = 5^2 - 3^2 = 25 - 9 = 16$$

$$x = t^2 + 1 \quad \frac{dx}{dt} = 2t \quad \left. \begin{array}{l} x = 2^2 + 1 = 5 \\ y = 2^2 - 1 = 3 \end{array} \right\} t = 2 \rightarrow \frac{dx}{dt} = 2(2) = 4$$

$$y = t^2 - 1 \quad \frac{dy}{dt} = 2t \quad \left. \begin{array}{l} y = 2^2 - 1 = 3 \end{array} \right\} \frac{dy}{dt} = 2(2) = 4$$

The chain rule tells us how to combine all these derivatives:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= (2x)(2t) + (-2y)(2t) = 4t(x-y)$$

mixed formula  
(not very elegant)

but values combine easily

$$\left. \frac{dz}{dt} \right|_{t=2} = 10(4) + (-6)(4) = 40 - 24 = 16$$

better example

$$PV = 8.31 T$$

kPascals liters      °K

Find  $\frac{dP}{dt}$  when  $\left\{ \begin{array}{l} T = 300^\circ\text{K} \\ V = 100\text{L} \end{array} \right. \quad \left. \begin{array}{l} \frac{dT}{dt} = 0.1^\circ\text{K/s} \\ \frac{dV}{dt} = 0.2\text{L/s} \end{array} \right\} \leftarrow \begin{array}{l} \text{only given rates} \\ \text{of change at a point -} \\ \text{not formula to} \\ \text{differentiate.} \end{array}$

$$P = 8.31 \frac{T}{V} = 8.31 T V^{-1} \left\{ \begin{array}{l} \frac{\partial P}{\partial T} = \frac{8.31}{V} \\ \frac{\partial P}{\partial V} = -8.31 T V^{-2} \end{array} \right.$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T} \frac{dT}{dt} + \frac{\partial P}{\partial V} \frac{dV}{dt} = \frac{8.31}{V} \frac{dT}{dt} - 8.31 \frac{T}{V^2} \frac{dV}{dt}$$

$$\left. \frac{dP}{dt} \right|_{t_0} = \left( \frac{8.31}{V} \frac{dT}{dt} \right) \Big|_{t_0} - \left( 8.31 \frac{T}{V^2} \frac{dV}{dt} \right) \Big|_{t_0}$$

$$= \left( \frac{8.31}{100} \right) (0.1) - 8.31 \frac{(300)}{100^2} (0.2) = \frac{8.31}{100} (1 - 6)(0.1) = -\frac{5(8.31)}{10^3}$$

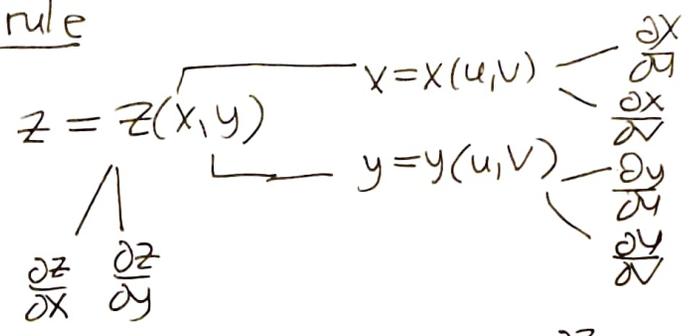
$$= .06831 - 6(.06831) = -0.04155 \approx \boxed{-0.042 \frac{\text{kPascal}}{\text{s}}}$$

increase in V decreases P 6 times faster during same time interval than increase in T

14.5a chain rule

(5)

best example



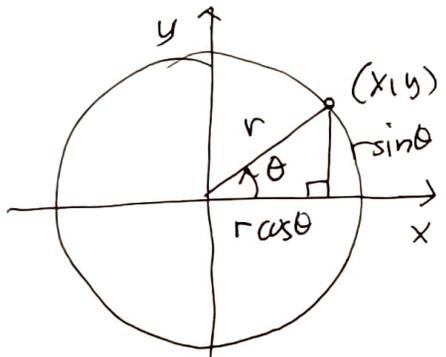
$z = z(x(u, v), y(u, v))$  calculate each one with the chain rule for differentiation wrt that variable

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

u partial derivative

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

v partial derivative



$$x = r \cos \theta \begin{cases} \frac{\partial x}{\partial r} = \cos \theta \\ \frac{\partial x}{\partial \theta} = -r \sin \theta \end{cases}$$

$$y = r \sin \theta \begin{cases} \frac{\partial y}{\partial r} = \sin \theta \\ \frac{\partial y}{\partial \theta} = r \cos \theta \end{cases}$$

coordinate transformation  
Cartesian  $\leftrightarrow$  polar  
in plane

how to convert derivatives from Cartesian to polar coords and vice versa:

$C = 0$  in quads 1, 4  
 $C = \pi$  in quad 2  
 $C = -\pi$  in quad 3

$$r = (x^2 + y^2)^{1/2} \begin{cases} \frac{\partial r}{\partial x} = \frac{1}{2} (x^2 + y^2)^{-1/2} (2x) \\ \frac{\partial r}{\partial y} = \frac{1}{2} (x^2 + y^2)^{-1/2} (2y) \end{cases}$$

$$\theta = \arctan \frac{y}{x} + "C" \begin{cases} \frac{\partial \theta}{\partial x} = \frac{1}{(1 + \frac{y^2}{x^2})} \left( \frac{-y}{x^2} \right) \\ \frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{1}{x} \right) \end{cases}$$

Starting with

$$f(x, y) = f(r \cos \theta, r \sin \theta) = F(r, \theta)$$

$$g(r, \theta) = g((x^2 + y^2)^{1/2}, \arctan \frac{y}{x} + c) = G(x, y)$$

we can re-express either function in terms of the other coords and so too for the corresponding derivatives

14.5a chain rule

(6)

chain rule calculations:

$$x = r \cos \theta \begin{cases} \frac{\partial x}{\partial r} = \cos \theta \\ \frac{\partial x}{\partial \theta} = -r \sin \theta \end{cases}$$

$$y = r \sin \theta \begin{cases} \frac{\partial y}{\partial r} = \sin \theta \\ \frac{\partial y}{\partial \theta} = r \cos \theta \end{cases}$$

1)  $z = f(x, y) = f(r \cos \theta, r \sin \theta)$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = z_x \cos \theta + z_y \sin \theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = z_x (-r \sin \theta) + z_y (r \cos \theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = (z_x \cos \theta + z_y \sin \theta)^2 + (-z_x \sin \theta + z_y \cos \theta)^2$$

$$= z_x^2 \cos^2 \theta + z_y^2 \sin^2 \theta + 2 z_x z_y \cos \theta \sin \theta$$

$$+ (z_x^2 \sin^2 \theta + z_y^2 \cos^2 \theta - 2 z_x z_y \sin \theta \cos \theta)$$

$$= z_x^2 + z_y^2 = \underbrace{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}_{\text{important in PDES}}$$

better for rotational symmetry problems

$$r = \sqrt{x^2 + y^2} \begin{cases} \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \\ \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \end{cases}$$

$$\theta = \arctan \frac{y}{x} + c \begin{cases} \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} \\ \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} \end{cases}$$

2)

$$z = g(r, \theta) = g(\sqrt{x^2 + y^2}, \arctan \frac{y}{x} + c) = G(x, y)$$

$$\frac{\partial z}{\partial x} = z_r \frac{\partial r}{\partial x} + z_\theta \frac{\partial \theta}{\partial x} = \frac{x}{r} z_r + \left(\frac{-y}{r^2}\right) z_\theta$$

$$\frac{\partial z}{\partial y} = z_r \frac{\partial r}{\partial y} + z_\theta \frac{\partial \theta}{\partial y} = \frac{y}{r} z_r + \left(\frac{x}{r^2}\right) z_\theta$$

where  $r = \sqrt{x^2 + y^2}$   
just shorthand

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{x}{r}\right)^2 z_r^2 + \left(\frac{y}{r}\right)^2 z_\theta^2 - \frac{2xy}{r^2} z_r z_\theta$$

$$+ \left(\frac{y}{r}\right)^2 z_r^2 + \left(\frac{x}{r}\right)^2 z_\theta^2 + \frac{2xy}{r^2} z_r z_\theta$$

$$= \left(\frac{x^2 + y^2}{r^2}\right) z_r^2 + \left(\frac{y^2 + x^2}{r^2}\right) z_\theta^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$$

$r \partial \theta \rightarrow$  arc length increment }  $(r, \theta)$  "orthogonal coords", Pythagorean Thm holds for arc length increments

(later)

14.5a chain rule

7

Iterate for higher derivatives.

$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2} \rightarrow$  convert

Find  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r}$

(pythag thm) product rule term

↓  
n=3 spherical coords  
(x, y, z) → (ρ, φ, θ)

$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \rightarrow$  spherical coord

3-d Laplacian

Key to describing electromagnetic waves  
(all modern communication)

key to Schrodinger's eq for ground states  
of atoms underlying all of chemistry

(organic chemistry etc)

partial differential equations!

Chapter 15 will discuss spherical coordinates