

14.4b

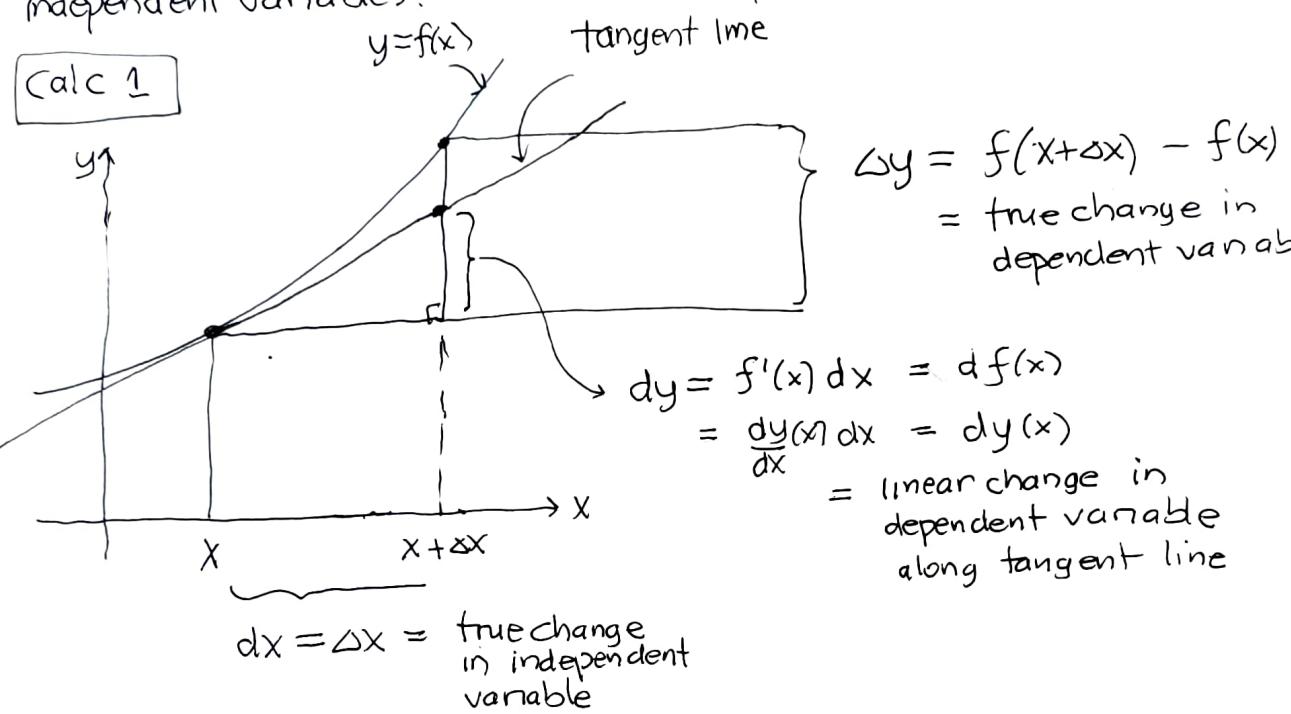
differentials

①

When we want to approximate (or "estimate") a small change in a function and don't need the actual new value itself, we can calculate the linear approximation change in the function value and adapt our notation to fit this new perspective.

This defines the [differential] of a function.

The differential is a new function not only of the original independent variables but also of their increments, away from the values of those independent variables.



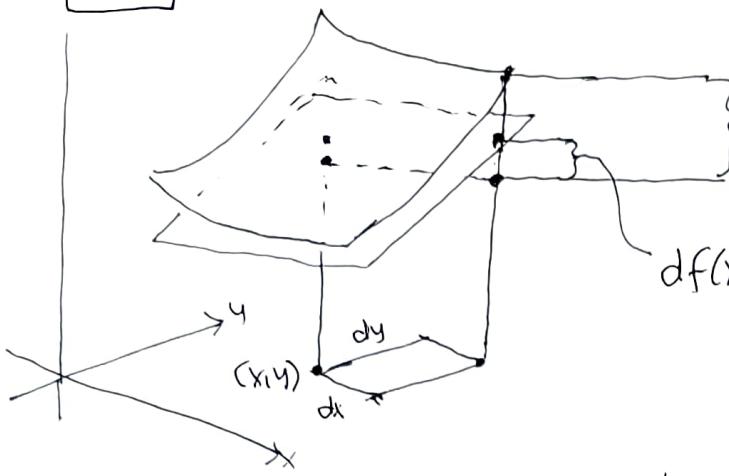
The differential of a function is just the product of its derivative and the differential of the independent variable.

14.4b

differentials

(2)

n=2



$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y) \\ &= \text{true increment in dependent variable}\end{aligned}$$

$$\begin{aligned}df(x, y) &= f_x(x, y) dx + f_y(x, y) dy \\ &= \text{linear change in dependent variable along tangent plane.}\end{aligned}$$

$$\begin{aligned}dx &= \Delta x \\ dy &= \Delta y\end{aligned}\left.\right\} \text{true increments in independent variables}$$

The differentials are just new notation for the linear increments of all the variables along the tangent plane from their values at the point on the graph where the linear approximation is constructed:

$$f(x, y) \rightarrow z = L(x, y) = \underbrace{f(x_0, y_0)}_{z_0} + \underbrace{f_x(x_0, y_0)}_{\Delta x} \frac{(x-x_0)}{\Delta x} + \underbrace{f_y(x_0, y_0)}_{\Delta y} \frac{(y-y_0)}{\Delta y}$$

$$\Delta z = z - z_0 = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y$$

now we can switch notation and let (x_1, y_1) be the point where we construct the linear approximation

$$dz = \boxed{f_x(x_1, y_1) dx + f_y(x_1, y_1) dy = df(x_1, y_1)}$$

$$\text{or } \boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy} \quad \text{if we suppress function notation}$$

The differential approximation is: $\Delta f \approx df$

n=3

$$df(x_1, y_1, z_1) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

n>3

The multivariable differential is just the sum of the independent variable differentials of the function.

14.4b differentials

(3)

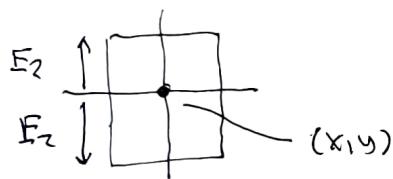
error propagation

When using differentials, often "precision" is not as important as obtaining a rough estimate of the change in a function.

A useful application is in estimating the error in the computed value of a function when the input variables are only known to lie within an interval of values with a certain precision about some central value. We then need to estimate the error range in the set of computed values of the function. This is called "error propagation" because the error in the input variables translates directly into error in the value of the function of those variables. The error "propagates" from input to output. We usually only need a linear estimate for this effect.

$n=2$

$$f(x_1, y) \rightarrow \text{true error : } |\delta x| \leq E_1, |\delta y| \leq E_2$$
$$|f(x_1 + \delta x, y + \delta y) - f(x_1, y)| \leq E_3 ?$$



linear error good enough :

$$|df(x_1, y)| \leq E_3 ?$$

$$\underbrace{\left| \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right|}_{\text{triangle inequality:}} \leq E_3 ?$$

$$\leq \left| \frac{\partial f}{\partial x} dx \right| + \left| \frac{\partial f}{\partial y} dy \right|$$

triangle inequality:

$$|A+B| \leq |A| + |B|$$

If of opposite sign

cancellation makes sum smaller

14.4b) differentials

(4)

Example Estimate the error in the computed volume of a cone if the dimensions are only known to within $\pm 0.1 \text{ cm}$.



$$r \rightarrow 10 \text{ cm} \pm 0.1 \text{ cm}$$

$\underbrace{r}_{\text{r}}$ $\underbrace{dr}_{\text{dr}}$

$$h \rightarrow 25 \text{ cm} \pm 0.1 \text{ cm}$$

$\underbrace{h}_{\text{h}}$ $\underbrace{dh}_{\text{dh}}$

$$V = \frac{\pi}{3} r^2 h \quad \begin{aligned} \frac{\partial V}{\partial r} &= \frac{2\pi}{3} r h \\ \frac{\partial V}{\partial h} &= \frac{\pi}{3} r^2 \end{aligned}$$

$$\begin{aligned} dV &= \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh \\ &= \frac{2\pi}{3} r h dr + \frac{\pi}{3} r^2 dh \end{aligned}$$

$$\begin{aligned} dV \Big|_{\substack{r=10 \\ h=25}} &= \frac{2\pi}{3} (10)(25) dr + \frac{\pi}{3} (10)^2 dh \\ &= \frac{\pi}{3} (500 dr + 100 dh) \\ &= 100\pi \left(5 dr + dh \right) \end{aligned}$$

use triangle inequality

↑ ↓
changing r changes V
5 times as fast as changing h

$$\begin{aligned} |dV| &\leq \frac{100\pi}{3} (5 |\pm 0.1| + 1 |\pm 0.1|) && \text{"absolute error"} \\ &= \frac{100\pi}{3} (.6) = \boxed{20\pi \text{ cm}^3} && \approx \boxed{62.8 \text{ cm}^3} \end{aligned}$$

$$V(10, 25) = \frac{\pi}{3} (10)^2 (25) = \frac{2500\pi}{3} \text{ cm}^3 \approx 2618.0 \text{ cm}^3 \quad \leftarrow$$

Is this big, small?
must compare
with V to know

$$\left| \frac{dV}{V} \right| \leq \frac{20\pi}{2500\pi/3} = \frac{6}{250} = \frac{3}{125} = .024 \approx 2.4\%$$

If approximation makes sense

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okay, small in comparison

fractional error

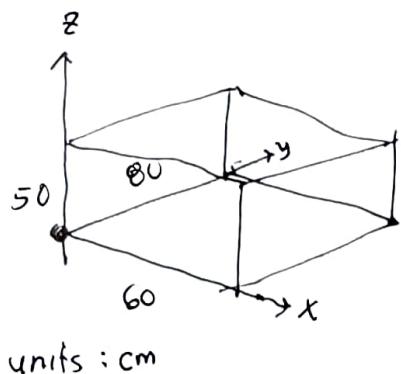
How good really is this linear approximation?

$$\begin{aligned} \text{high: } V(10.1, 25.1) - V(10, 25) &\approx 63.3 = 62.8 + 0.5 && \} \text{ pretty close if} \\ \text{low: } V(9.9, 24.9) - V(10, 25) &\approx -62.4 = -(62.8 - 0.4) && \text{we are just} \\ &&& \text{looking for a} \\ &&& \text{rough estimate.} \end{aligned}$$

14.9b) differentials

(5)

example



What is the approximate change in the box volume if we increase all its dimensions by 1%?

$$(x_1, y_1, z_1) = (60, 80, 50)$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = .01$$

$$V = 60 \cdot 80 \cdot 50 \\ = 240,000 \text{ cm}^3$$

= .24 m<sup>3</sup> more appropriate unit!

$$V = xyz$$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \\ = yz \underbrace{dx}_{.01x} + xz \underbrace{dy}_{.01y} + xy \underbrace{dz}_{.01z}$$

$$= \frac{xyz}{V} (.01 + .01 + .01) = .03V$$

3 contributions  
from 3 dimensions  
add

Volume increases  
by 3%.

$$dV = .03(24) = .0072 \text{ m}^3$$

Compare with true change:

$$\Delta V =$$

$$.60(1.01) \cdot .80(1.01) \cdot .50(1.01) - (.6)(.8)(.5) \\ = (.60)(.80)(.50)[(1.01)^3 - 1] \approx 0.007272 \text{ m}^3$$

pretty good

aside:  $(1.01)^3 = \underbrace{1 + 3(0.01)}_{1.03} + \underbrace{3(0.01)^2}_{.000301} + \underbrace{(0.01)^3}_{.000001}$  binomial expansion

linear estimate of increase      higher order corrections