

14.4[a] tangent planes, linear approximation and differentiability ①

$$z = f(x, y) \quad \left. \begin{array}{l} f_x(x, y) \\ f_y(x, y) \end{array} \right\} \text{ both exist at } (x_0, y_0):$$

The two partial derivatives DON'T TELL US what happens in all the directions in between the coordinate axes, where things can go wrong.

This is new for $n > 1$ independent variables

Example [see Maple!]

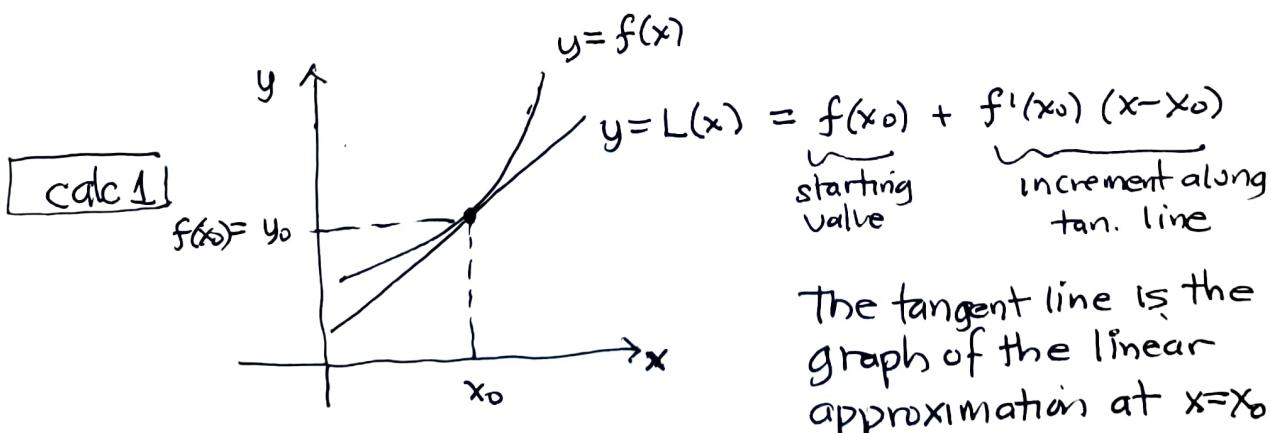
$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

identically 0
on axes
so both
partial derivatives
are 0 at origin

BUT does not zoom in to any tangent plane

problem: both partial derivatives are discontinuous at origin

It turns out the all partial derivatives must not only exist but also be continuous at a point for a limiting tangent plane to exist. This is how differentiability is defined for $n > 1$.

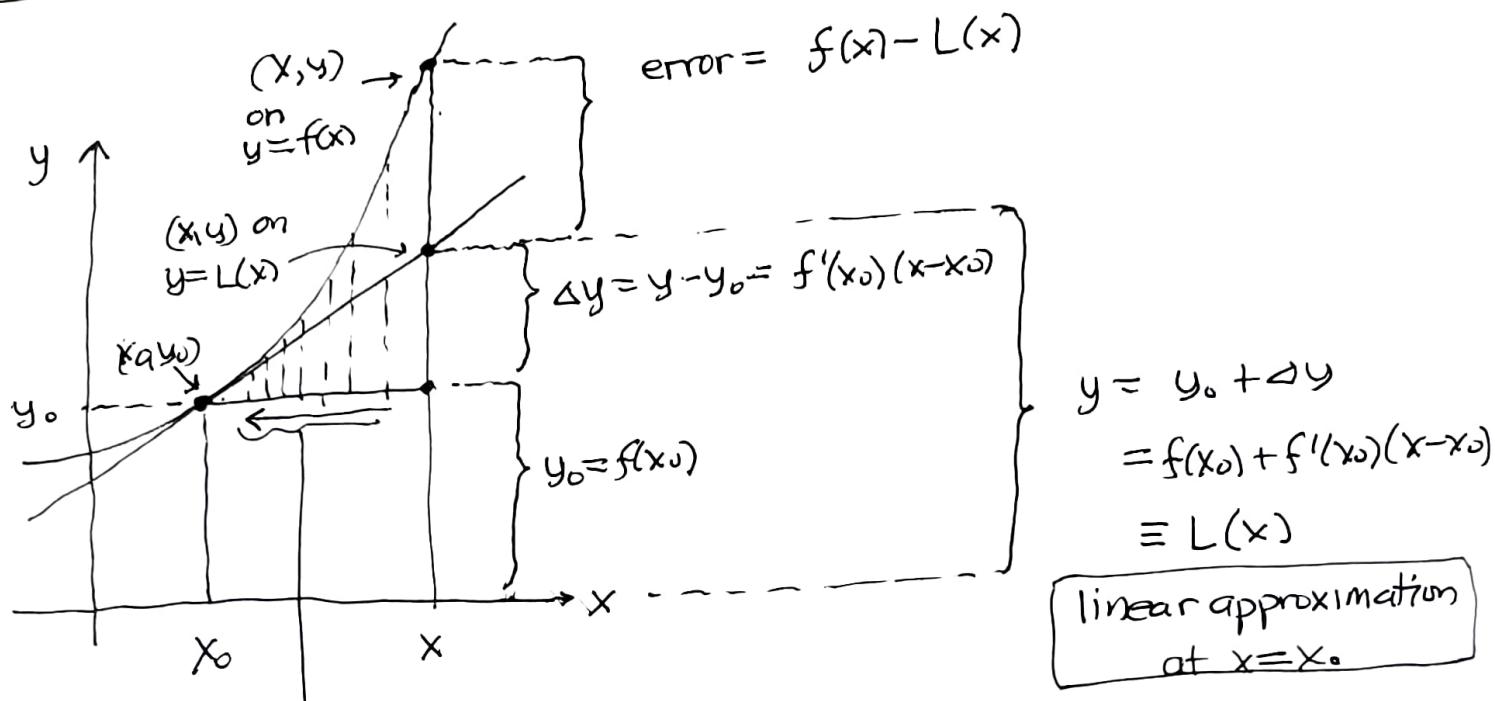


The tangent line is the graph of the linear approximation at $x = x_0$

14.4a

tangent planes, linear approximation and differentiability ②

Calc 1: how does the linear approximation work]



as $x \rightarrow x_0$ and you zoom into $x = x_0$, the error squeezes to zero FASTER than Δy so that the curved graph squeezes to the tangent line and approximates it better and better :

$$\lim_{\Delta x \rightarrow 0} \frac{\text{error}}{\Delta y} = 0 \quad \begin{matrix} \text{error goes to zero} \\ \underline{\text{faster than } \Delta y} \end{matrix}$$

Remark: Taylor series:

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{L(x)} + \frac{1}{2} f''(x_0)(x-x_0)^2 + \dots$$

$$\text{error} = f(x) - L(x) = (x-x_0)^2 \left(\frac{1}{2} f''(x_0) + \dots \right)$$

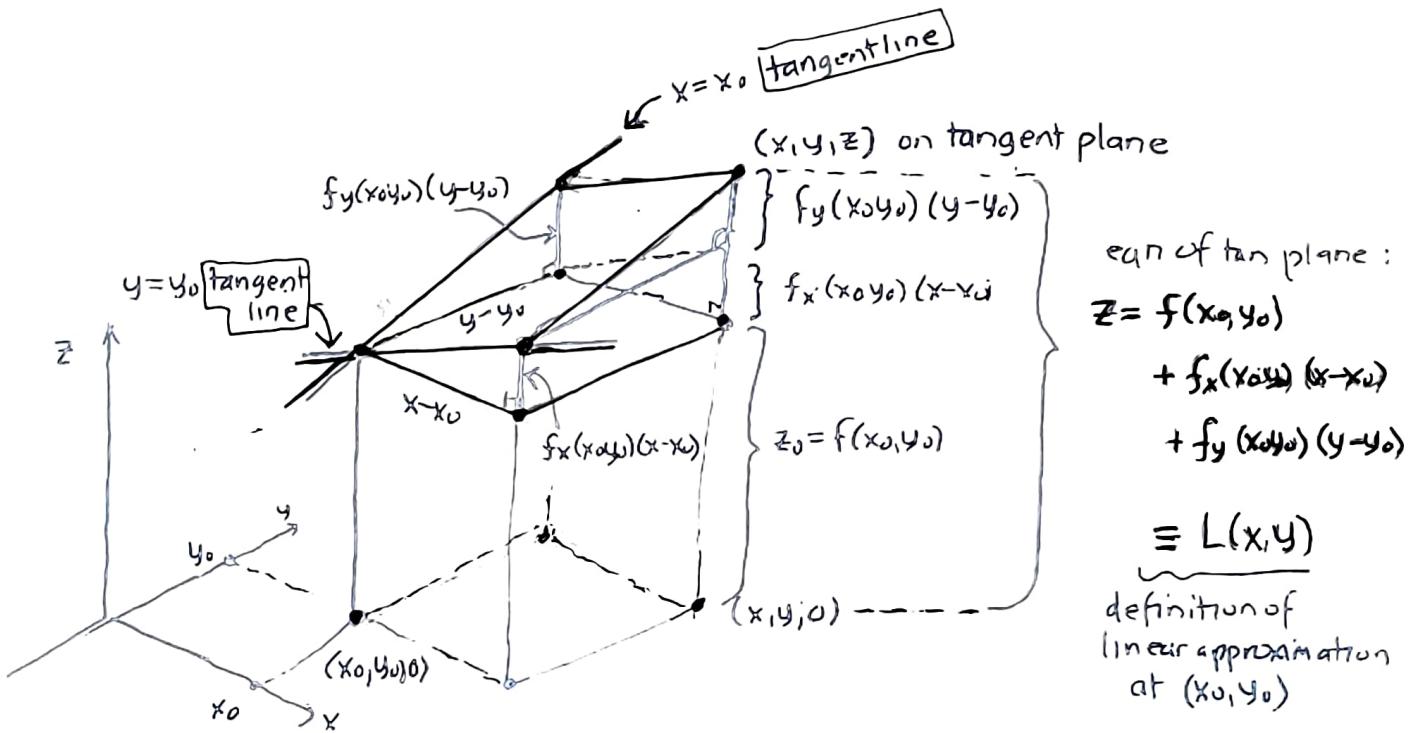
$$\frac{\text{error}}{x-x_0} = \underbrace{(x-x_0)}_{\rightarrow 0} \left(\frac{1}{2} f''(x_0) + \dots \right) \xrightarrow{\rightarrow 0} 0 \quad \text{as } x \rightarrow x_0$$

higher order terms go to zero faster than any given order

14.4a tangent planes, linear approximation and differentiability

(3)

$n=2$ increments Δx & Δy away from (x_0, y_0) determine two tangent line segments which complete to a parallelogram belonging to the tangent plane



tangent plane = graph of linear approximation

$$z = L(x, y)$$

$$= f(x_0, y_0) + \underbrace{f_x(x_0, y_0)}_{\text{all must be evaluated to constants at } (x_0, y_0)}(x - x_0) + \underbrace{f_y(x_0, y_0)}_{\text{all must be evaluated to constants at } (x_0, y_0)}(y - y_0)$$

all must be evaluated to constants at (x_0, y_0)

standard eqn for tangent plane:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

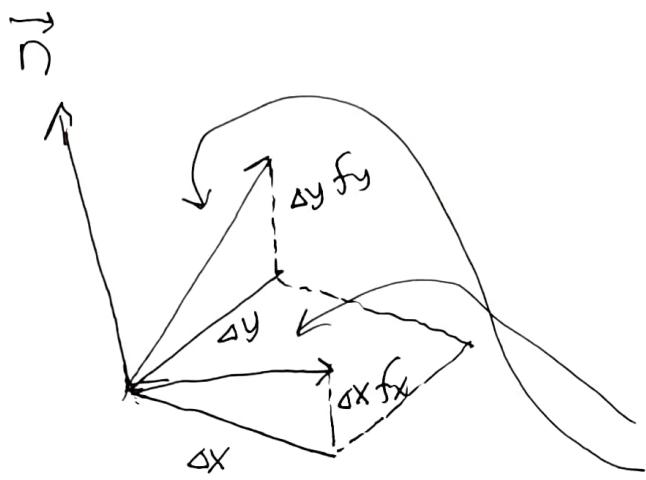
$$- f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + \frac{1}{2}(z - z_0) = 0$$

$$\vec{n} = \langle -f_x, -f_y, \frac{1}{2} \rangle \text{ at } (x_0, y_0) \quad \text{normal (upward)}$$

$$-\vec{n} = \langle f_x, f_y, -\frac{1}{2} \rangle \quad \text{downward normal}$$

[4.4g] tangent planes, linear approximation and differentiability (4)

[upward normal from vector geometry]



graph: $z = f(x, y)$

$$\vec{r} = \langle x, y, f(x, y) \rangle$$

vector-valued
function

$$\vec{r}_x = \langle 1, 0, f_x \rangle$$

$$\vec{r}_y = \langle 0, 1, f_y \rangle$$

$$\vec{r}_x \Delta x = \langle \Delta x, 0, f_x \Delta x \rangle$$

$$\vec{r}_y \Delta y = \langle 0, \Delta y, f_y \Delta y \rangle$$

$$\vec{n} = \vec{r}_x \times \vec{r}_y = \dots = \underbrace{\langle -f_x, -f_y, 1 \rangle}$$

normal must tilt back compared
to directions where function is
increasing

$$\text{at } (x_0, y_0): \quad -f_x(x_0, y_0)(x-x_0) - f_y(x_0, y_0)(y-y_0) + 1(z-z_0) = 0$$

$$\begin{aligned} \downarrow \\ z &= f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \\ &\equiv L(x, y) \end{aligned}$$

$|$
 $f(x_0, y_0)$

$n=3$

$$\begin{aligned} L(x, y, z) &= f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x-x_0) \\ &\quad + f_y(x_0, y_0, z_0)(y-y_0) \\ &\quad + f_z(x_0, y_0, z_0)(z-z_0) \end{aligned}$$

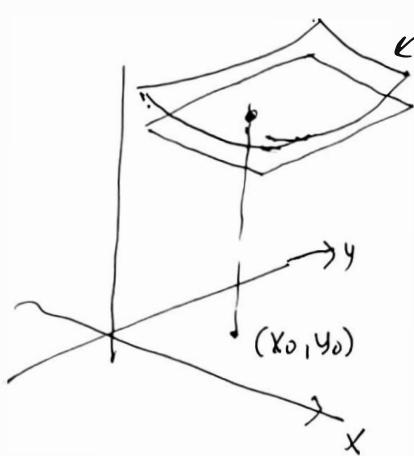
$n>3$ similar

\uparrow
add independent 1-d increments
for all independent variables

4.4a) tangent plane, linear approximation & differentiability

(5)

[differentiability] requires function graph squeeze to tangent plane



want this to flatten out to tangent plane as we zoom in

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

$$= f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1(x_0 + \Delta x, y_0 + \Delta y) \Delta x$$

linear approx. to change

+ $\epsilon_2(x_0 + \Delta x, y_0 + \Delta y) \Delta y$

(both go to 0 as we zoom in)

fractional error

$$= \frac{\epsilon_1(x_0 + \Delta x, y_0 + \Delta y) \Delta x + \epsilon_2(x_0 + \Delta x, y_0 + \Delta y) \Delta y}{f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y} \rightarrow 0$$

in order to squeeze to tangent plane as $(x_0, y_0) \rightarrow (0, 0)$

In particular

$$\text{if } \Delta y = 0, \text{ fractional error} = \frac{\epsilon_1(x_0 + \Delta x, y_0 + \Delta y)}{f_x(x_0, y_0)} \rightarrow 0$$

forces $\lim_{\Delta x \rightarrow 0} \epsilon_1 = 0$

$$\text{if } \Delta x = 0, \text{ fractional error} = \frac{\epsilon_2(x_0 + \Delta x, y_0 + \Delta y)}{f_y(x_0, y_0)} \rightarrow 0$$

forces $\lim_{\Delta y \rightarrow 0} \epsilon_2 = 0$

As 2-d limits, differentiability

$$\text{requires } \lim_{(x_1, y_1) \rightarrow (x_0, y_0)} \epsilon_1(x_1, y_1) = 0 = \lim_{(x_1, y_1) \rightarrow (x_0, y_0)} \epsilon_2(x_1, y_1)$$

Both fractional error functions must go to zero

It turns out that this is equivalent to the continuity of both partial derivatives at (x_0, y_0) .

$f(x, y)$ is differentiable at (x_0, y_0) if all partial derivatives exist & are continuous there

14.4a) tangent plane, linear approximation & differentiability (6)

Example

$$f(x,y) = x^2 + 4y^2$$

$$f(1,1) = 1 + 4 = 5 \rightarrow (1,1,5) \text{ on graph}$$

$$f_x(x,y) = 2x$$

$$f_x(1,1) = 2$$

$$f_y(x,y) = 8y$$

$$f_y(1,1) = 8$$

↓

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$= 5 + 2(x-1) + 8(y-1)$$

useful without multiplying out

$$\left. \begin{aligned} &= 5 + 2x - 2 + 8y - 8 = -5 + 2x + 8y = z \\ &\quad \downarrow \\ &2x + 8y - z = 5 \end{aligned} \right\} \quad \vec{n} = \langle 2, 8, -1 \rangle$$

small increments hidden

using the linear approximation

$$f(0.9, 1.1) = (.9)^2 + 4(1.1)^2 = 5.65 \text{ exactly}$$

$\approx 1 \approx 1$

$$\approx L(0.9, 1.1) = 5 + 2(\underbrace{0.9-1}_{-0.1}) + 8(\underbrace{1.1-1}_{+0.1})$$

$$= 5 - 0.2 + 0.8$$

$$= 5.6$$

small changes
to 5

$$\text{error } f(0.9, 1.1) - L(0.9, 1.1) = 5.65 - 5.6 = 0.05 > 0 \text{ approx below}$$

fractional
error

$$\frac{f(0.9, 1.1) - L(0.9, 1.1)}{f(0.9, 1.1)} = \frac{0.05}{5.65} = 0.008 \approx 1\%$$

(compared to
correct value)

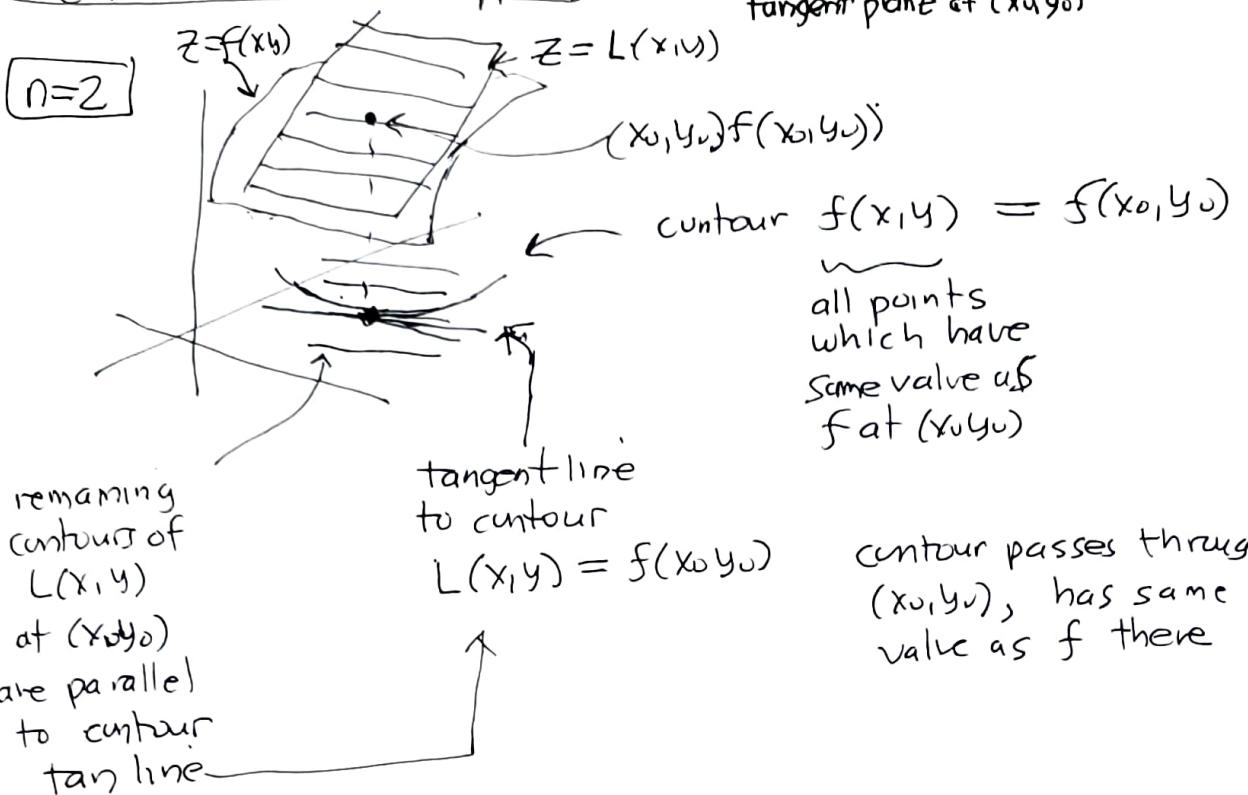
so the linear approx
is about 1% to low

similar for $n > 2$

[14.4a] tangent plane, linear approximation and differentiability (7)

contours and linear approx]

$n=2$



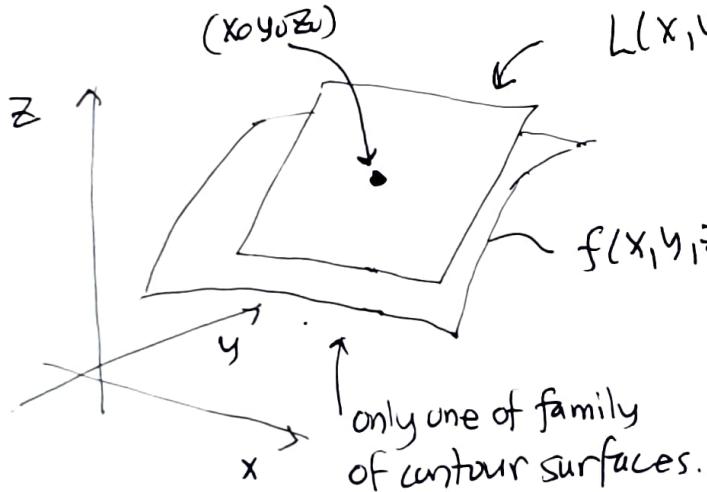
tangent plane at (x_0, y_0)

all points which have same value as f at (x_0, y_0)

contour passes through (x_0, y_0) , has same value as f there

$n=3$

we only have the contour surface point of view to visualize



$L(x,y,z) = f(x_0, y_0, z_0)$ ← contour surface of linear approx at (x_0, y_0, z_0)

is tangent plane to:

$f(x,y,z) = f(x_0, y_0, z_0)$ ← contour surface through (x_0, y_0, z_0)

remaining contour surfaces of $L(x,y,z)$ are parallel to this one and equally spaced

14.4 a)

tangent plane, linear approximation and differentiability (8)
Example

Approximate

$$\sqrt{(6.02)^2 + (6.99)^2 + (5.97)^2}$$

$$f(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$$

$$\text{choose } (x_0, y_0, z_0) = (6, 7, 6)$$

$$\text{because } f(6, 7, 6) = \sqrt{36 + 49 + 36} = \sqrt{121} = \sqrt{11^2} = 11!$$

easy
exact.

$$(dx, dy, dz) = (6.02 - 6, 6.99 - 7, 5.97 - 6)$$

$$= (0.02, -0.01, -0.03) \text{ very small}$$

$$f_x = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} (2x) = \frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$

$$f_y = \dots = \frac{y}{(x^2 + y^2 + z^2)^{1/2}}$$

$$f_z = \dots = \frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$(f_x, f_y, f_z)|_{(6, 7, 6)} = \left(\frac{6}{11}, \frac{7}{11}, \frac{6}{11}\right)$$

$$L(x, y, z) = 11 + \frac{6}{11}(x - 6) + \frac{7}{11}(y - 7) + \frac{6}{11}(z - 6)$$

$$L(6.02, 6.99, 5.97) = 11 + \underbrace{\frac{6}{11}(6.02 - 6)}_{0.02} + \underbrace{\frac{7}{11}(6.99 - 7)}_{-0.01} + \underbrace{\frac{6}{11}(5.97 - 6)}_{-0.03}$$

$$\underbrace{\frac{1}{11}(6(0.02) + 7(0.01) - 6(0.03))}_{.12 - .07 - .18} = -0.13$$

$$= 11 - \frac{0.13}{11}$$

$$\approx 11 - \underbrace{0.018}_{\substack{\text{very small} \\ \text{correction}}} \approx 10.982 \rightarrow .006 \text{ too low}$$

exact value: $f(6.02, 6.99, 5.97) \approx 10.988$ fractional error: $\frac{.006}{10.988} \approx .0005 \approx .05\%$ very small indeed