

14.3a partial derivatives (first order)

Calc 1 notation for explicit differentiation

$\frac{d}{dx} [f(x) = x^2]$ function definition

"take derivative wrt x of expression to the immediate right"

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{d}{dx} (x^2) = 2x \\ &= \frac{df}{dx}(x) \\ &= f'(x) \end{aligned}$$

limiting ratio

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

useful symbolism

OR explicit functional relationship:

$$\frac{d}{dx} [y = x^2] \rightarrow \frac{dy}{dx} = \frac{d}{dx} (x^2) = 2x$$

n=2

$f(x,y) = x^2 + 4y^2$ at (1,1) where $f(1,1) = 5$

we can take both calc 1 derivatives of this function but what is our mental picture?

graph it:

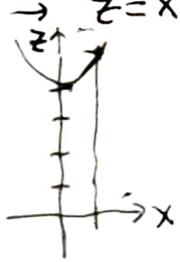
$z = x^2 + 4y^2 = k \geq 0$
 contour surfaces
 horizontal plane cross-sections

$$\frac{x^2}{(\sqrt{k})^2} + \frac{y^2}{(\sqrt{k}/2)^2} = 1$$

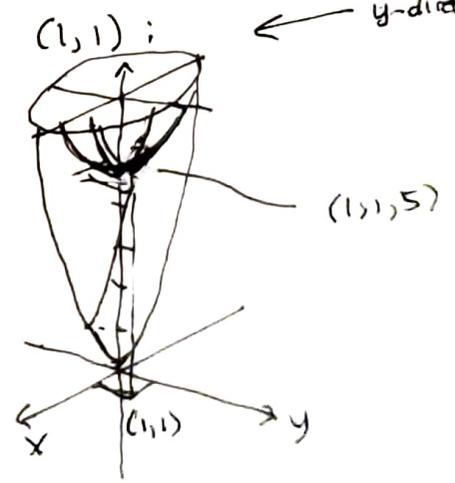
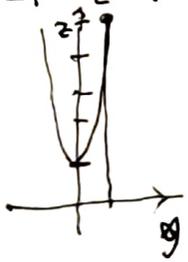
ellipses with semi-axes \sqrt{k} , $\sqrt{k}/2$
 half y-direction

(part to coord plane) vertical cross-sections through (1,1):

$y=1 \rightarrow z = x^2 + 4$



$x=1 \rightarrow z = 1 + 4y^2$



[See Maple now]

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(2)

In the two $x=1$ and $y=1$ vertical coord plane cross-sections, these are just calc 1 functions.

$$y=1: z = f(x, 1) = x^2 + 4 \rightarrow \frac{dz}{dx} = 2x + 0 = 2x$$

$$x=1: z = f(1, y) = 1 + 4y^2 \rightarrow \frac{dz}{dy} = 0 + 4(2y) = 8y$$

$$\text{at } (1, 1): \left. \frac{dz}{dx} \right|_{x=1} = 2 \quad \text{slope in } y=1 \text{ cross-section}$$

$$\left. \frac{dz}{dy} \right|_{y=1} = 8 \quad \text{slope in } x=1 \text{ cross-section}$$

"partial" notation recognizes that each calc 1 derivative only gives partial information about the derivatives. The "total" derivative is better represented as a vector quantity of such calc 1 derivatives.

$$\frac{\partial f}{\partial x}(x, y) = \frac{d}{dx} f(x, y) = 2x$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{d}{dy} f(x, y) = 8y$$

↑
the operations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ require understanding which are the full set of independent variables AND that all other variables are held fixed during the derivative operation.

sometimes read as "partial x of f" etc
or "partial f partial x"

ALSO

subscript notation

$$f_x(x, y) \equiv \frac{\partial f}{\partial x}(x, y)$$

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y)$$

sometimes more convenient.

like more compact prime notation but also has to identify which variable is acting.

4.3 a partial derivatives (first order)

3

Example $f(x,y) = x^3 + 4x^2y^3 - 2y^2 + 1$ $f(1,1) = 1 + 4 - 2 + 1 = 4$

when you differentiate you have to restrain yourself from applying power rules to the other variable!

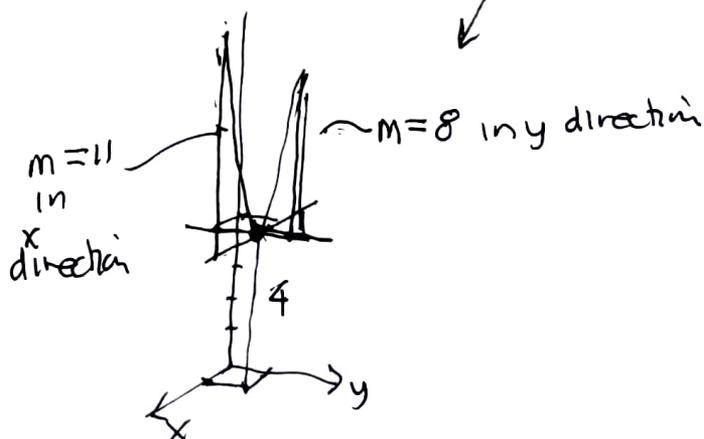
$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x} (x^3 + 4x^2y^3 - 2y^2 + 1) \\ &= \frac{\partial}{\partial x} (x^3) + \frac{\partial}{\partial x} (4x^2y^3) - \frac{\partial}{\partial x} (2y^2) + \frac{\partial}{\partial x} (1) \\ &= 3x^2 + 4y^3 \frac{\partial}{\partial x} (x^2) \\ &= 3x^2 + 8xy^3\end{aligned}$$

remember:
 $\frac{\partial}{\partial x} g(y) = 0$
 $\frac{\partial}{\partial y} h(x) = 0$

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y} (x^3 + 4x^2y^3 - 2y^2 + 1) \\ &= \frac{\partial}{\partial y} (x^3) + \frac{\partial}{\partial y} (4x^2y^3) - \frac{\partial}{\partial y} (2y^2) + \frac{\partial}{\partial y} (1) \\ &= 0 + 4x^2 \frac{\partial}{\partial y} (y^3) - 2 \frac{\partial}{\partial y} (y^2) + 0 \\ &= 12x^2y^2 - 4y\end{aligned}$$

$$\frac{\partial f}{\partial x}(1,1) = 3 + 8 = 11 \rightarrow$$

$$\frac{\partial f}{\partial y}(1,1) = 12 - 4 = 8 \rightarrow$$



14.3a partial derivatives (first order)

4

Example $f(x,y) = \sin\left(\frac{x}{1+y}\right) = \sin\left(x(1+y)^{-1}\right)$

linear function of x. easy!

not a quotient function in either variable - don't use quotient rule

as function of y it is a constant times the -1 power function: chain rule

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x} \sin\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \frac{\partial}{\partial x} \left(\frac{x}{1+y}\right) = \frac{1}{1+y} \cos\left(\frac{x}{1+y}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y} \sin\left(x(1+y)^{-1}\right) = \cos\left(\frac{x}{1+y}\right) \frac{\partial}{\partial y} \left(x(1+y)^{-1}\right) = \frac{-x}{(1+y)^2} \cos\left(\frac{x}{1+y}\right)$$

$x(-1)(1+y)^{-2}(0+1)$

Example

$f(x,y) = x^y$ $y \leftarrow$ power function of $x \rightarrow$ power rule

exp function of y
base x not e
(rewrite) $\rightarrow (e^{\ln x})^y = e^{y \ln x}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x} x^y = y x^{y-1}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y} e^{y \ln x} = \frac{e^{y \ln x}}{x^y} \frac{\partial}{\partial y} (y \ln x) = (\ln x) x^y$$

[compare: $\frac{\partial}{\partial x} e^{y \ln x} = e^{y \ln x} \frac{\partial}{\partial x} (y \ln x) = \frac{y}{x} x^y = y x^{y-1}$]

14.3a

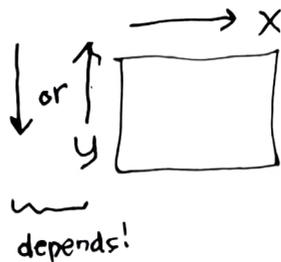
partial derivatives (first order)

5

The horizontal rows and vertical columns of 2-d tabular data are each 1-d tabular data where we can easily evaluate the 1-d tabular derivatives.

$$z = f(x, y)$$

↑ ↑
 label label
 cols rows



Example

Wind chill

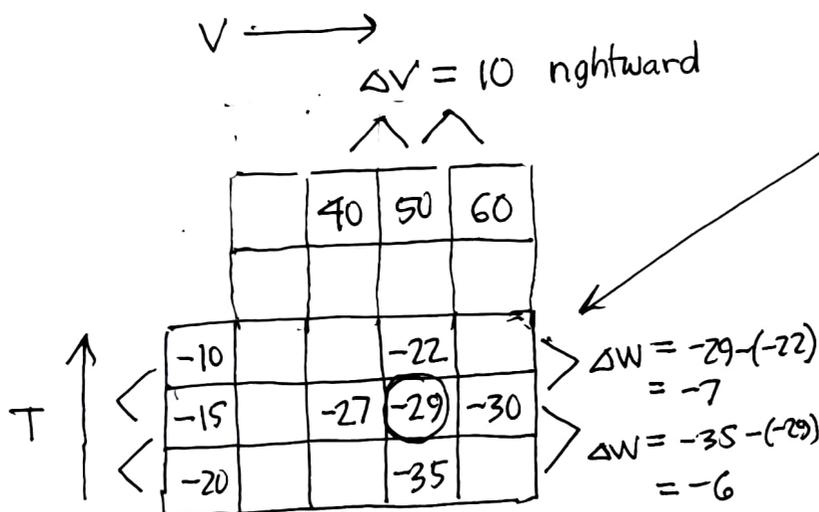
$$W = W(V, T)$$

↙ ↘
 perceived temperature of
 temperature speed mph

central data point

$$W(50, -15) = -29$$

for derivative calculations



$$\frac{\Delta W}{\Delta T} = \frac{-7}{-5} = \frac{7}{5}$$

$$\frac{\Delta W}{\Delta T} = \frac{-6}{-5} = \frac{6}{5}$$

$$\text{avg: } \frac{1}{2} \left(\frac{7+6}{5} \right) = \frac{13}{10} = 1.3$$

$$\frac{\partial W}{\partial T}(50, -15) = 1.3 \frac{^{\circ}\text{F}}{^{\circ}\text{F}}$$

decrease T by 1°
decreases W by 1.3°!
makes sense!

linear approximation!
(for small changes)

$$\frac{\partial W}{\partial V}(50, -15) = -0.15 \frac{^{\circ}\text{F}}{\text{mph}}$$

increase V by Δmph
decreases W by -0.15 °F