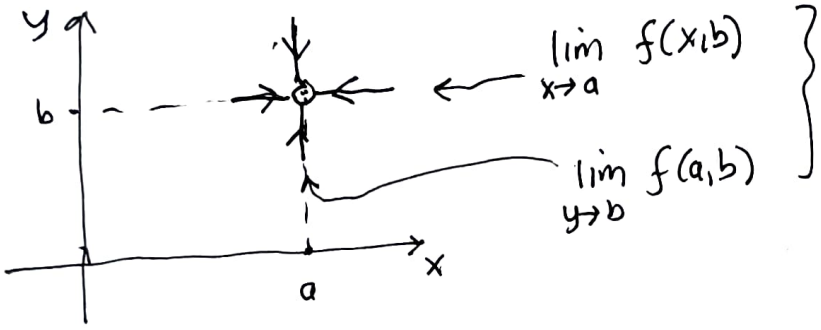
multivariable limits

$\lim_{x \to a} f(x,y)$, $\lim_{y \to b} f(x,y)$ make perfect sense!

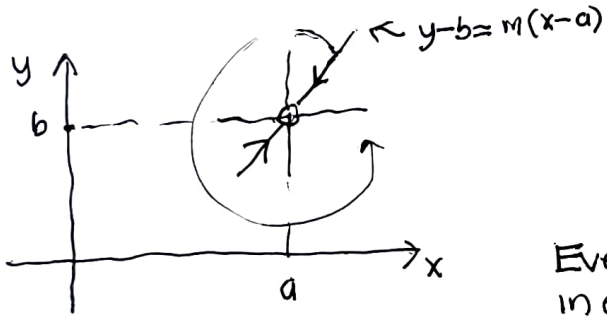
With any multivariable function we can easily take limits with respect to each independent variable just like in calc 1, holding all other independent variables fixed during the calculation of course.

But it turns out that these 1-dimensional limits are **insufficient** to capture the multivariable behavior near a point.

For $n=2$ we would like to define such a multivariable limit. The limit should not depend on how we approach the limiting point.



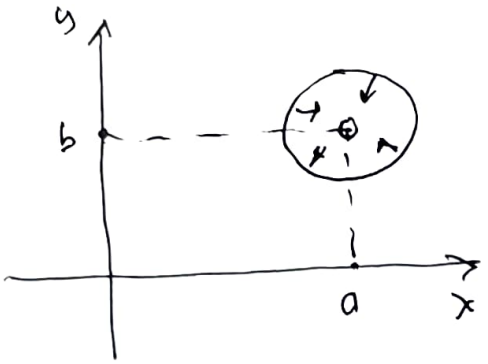
only test limiting value at (a,b) along axis directions but NOT along the other infinite number of directions of approach to (a,b).



We can fix that by taking the limit along **all** the directions in between the axes (all m values).

Even if we get the same 1-d limit in every direction at (a,b) THIS IS STILL NOT ENOUGH to guarantee that all nearby values of the function actually get close to that common limit!

14.2 multivariable limits



The idea is simple.
As we squeeze a circle (or box) towards a point, all values assumed by the function within it should also squeeze towards some limiting value in order to call that the limit.

This is easily seen in a plot of the graph.

we will examine 4 examples, each a bit more complicated than the previous case, to explore the problems which arise.

And not go further.

We just need to be aware of the complications to be careful later in defining differentiability.

All have only a problem at the origin where division by zero occurs.

① $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

② $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

③ $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

④ $f(x,y) = \dfrac{xy^2}{x^{(2)}+y^{(4)}}$

only example where net power in each term of a sum is not the same

Example 1

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} = \frac{\sin u}{u}, \quad u = x^2+y^2$$

really a 1-d function that is the problem, independent of direction (rotationally symmetric)

calculate 1-d limits along axes:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{\sin(x^2+0^2)}{x^2+0^2} = \lim_{x\to 0} \frac{\sin x^2}{x^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

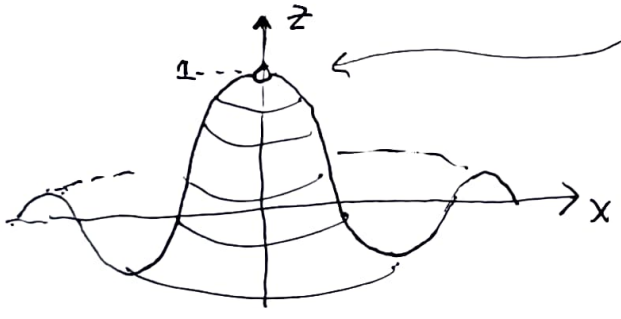
fundamental trig limit needed for trig derivatives!

obviously same for:

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{\sin(0^2+y^2)}{0^2+y^2} = \lim_{y\to 0} \frac{\sin y^2}{y^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1$$

indeed true for every direction.

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$



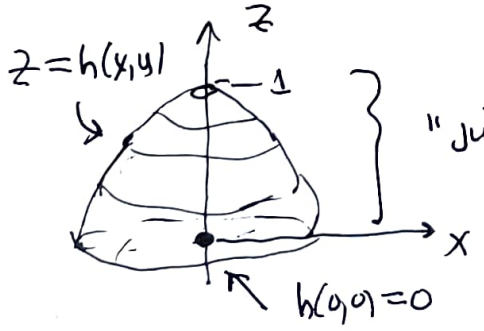
hole in graph "removable" discontinuity

We can fill it in with the limiting value to make a continuous function

$$g(x,y) = \begin{cases} f(x,y), & (x,y)\neq(0,0) \\ 1, & (x,y)=(0,0) \end{cases}$$

but

$$h(x,y) = \begin{cases} f(x,y), & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$



"jump" discontinuity

Maple may or not show signs of this in 3d plot.

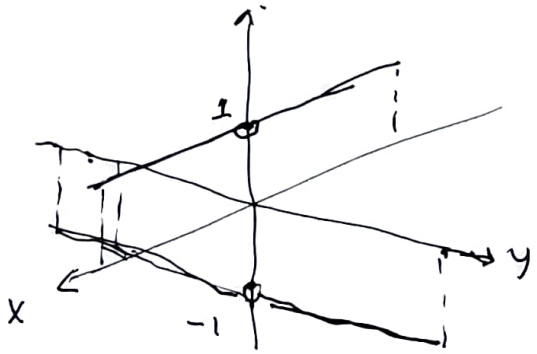
(a single point can be missed)

Example 2 $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \dfrac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} \dfrac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$

$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \dfrac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} \dfrac{-y^2}{y^2} = \lim_{y\to 0} (-1) = -1$

both constant along axes but dont agree!



No common limit can exist!

$\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E. as $x \to 0$

no need to go further.

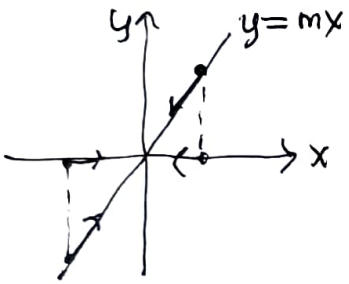
Example 3 $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{3x^2(0)}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{3\cdot 0^2(y)}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

both constant along axes and agree!

what about other directions?



$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{3x^2(mx)}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{x^3(3m)}{x^2(1+m^2)} = \lim_{x\to 0} x\left(\frac{3m}{1+m^2}\right) = 0$$

agrees!

Indeed $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ here

(see Maple later to confirm)

Example 4: $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ ← note unequal powers!

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x0^2}{x^2+0^4} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0y^2}{0^2+y^4} = \lim_{y\to 0} \frac{0}{y^4} = \lim_{y\to 0} 0 = 0$$

constant along axes and agree!

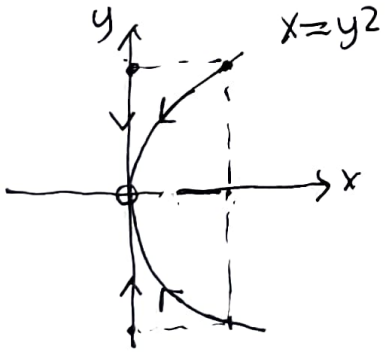
next:

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{x\to 0} \frac{x^3m^2}{x^2(1+m^4x^2)}$$

$$= \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0$$

again.

BUT suppose we sneak up on the origin along a parabola.



$$\lim_{y\to 0} f(y^2,y) = \lim_{y\to 0} \frac{(y^2)(y^2)}{(y^2)^2+y^4}$$

$$= \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \underbrace{\frac{1}{2}}_{\text{constant value along parabola (contour curve)!}} = \frac{1}{2} \neq 0$$

$f(x,y) = \frac{1}{2}$

In fact: $x = my^2$

$$\lim_{y\to 0} f(my^2,y) = \lim_{y\to 0} \frac{(my^2)(y^2)}{(my^2)^2+y^4}$$

$$= \lim_{y\to 0} \frac{y^4 m}{y^4(m^2+1)} = \lim_{y\to 0} \underbrace{\left(\frac{m}{m^2+1}\right)}_{\text{constant along parabola}}$$

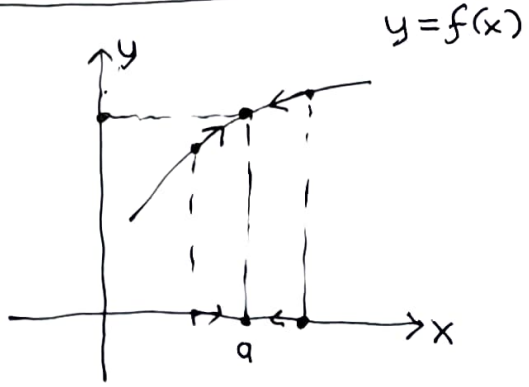
contour curves:

$f(x,y) = \underbrace{\dfrac{m}{m^2+1}}_{\text{takes all values between } -\frac{1}{2} \text{ and } \frac{1}{2}!}$

[see Maple now]

[contours all "pileup" at the origin!!]

Continuity?



calc 1:

f is continuous at $x=a$ if 3 conditions are met:

$$\underbrace{\lim_{x \to a} f(x)}_{\text{② its limit there exists (is finite!)}} = \underbrace{f(a)}_{\text{① f is defined at } x=a}$$

③ they agree

The same definition holds for multivariable functions.

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

② ③ ①

We only need to check limits at obvious "bad points" of a function, otherwise functions defined by formulas are continuous everywhere else.

Examples 1–4 are all continuous everywhere away from the origin.

We don't really need $n=3$ examples. Section 14.2 only serves to explain why differentiability is more tricky for $n>1$.