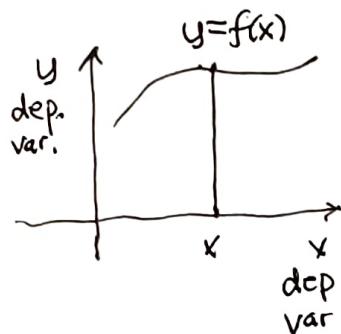


## 14.1 functions of $n > 1$ independent variables

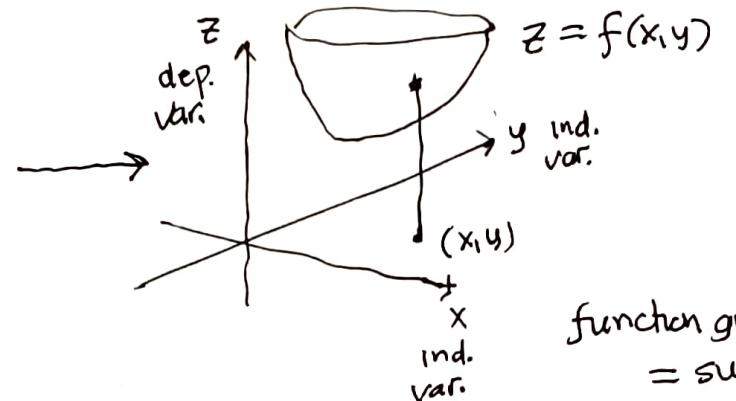
①

In chapter 13 we increased the number of dependent variables to  $n \geq 1$ , keeping 1 independent variable. Now we keep 1 dependent variable and increase the number of independent variables to  $n > 1$ .



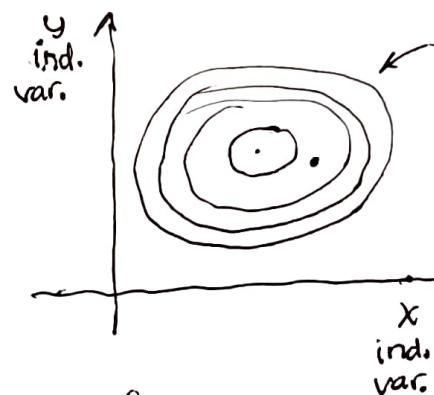
function graph = curve

$n=1$



function graph  
= surface

$n=2$



$f(x,y) = k_i$   
equally spaced values  
"contour curves"  
called "level curves"

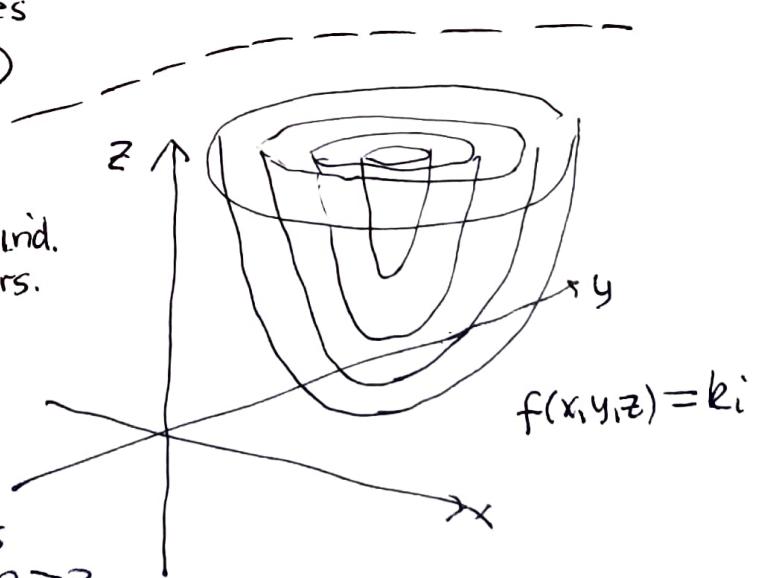
We have a second way  
of visualizing functions  
entirely in the space of  
independent variables.

"Level curves" are the intersections of  
the graph with a family of equally  
spaced horizontal (=level) planes  
 $z = k_i$ , ( $\Delta k = k_i - k_{i-1} = \text{fixed}$ )

$n=3$  graph in  $\mathbb{R}^4$

cannot be visualized  
but we can visualize  
"level surfaces" = contoursurfaces

We will explore graphs & contours  
for  $n=2$  and contoursurfaces  
for  $n=3$  to get intuition about  $n > 3$ .

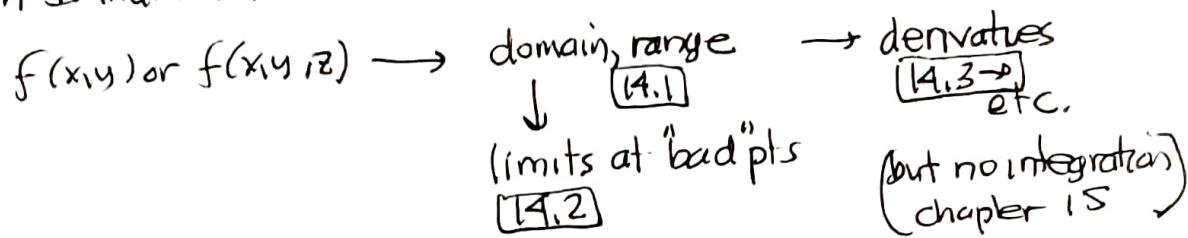
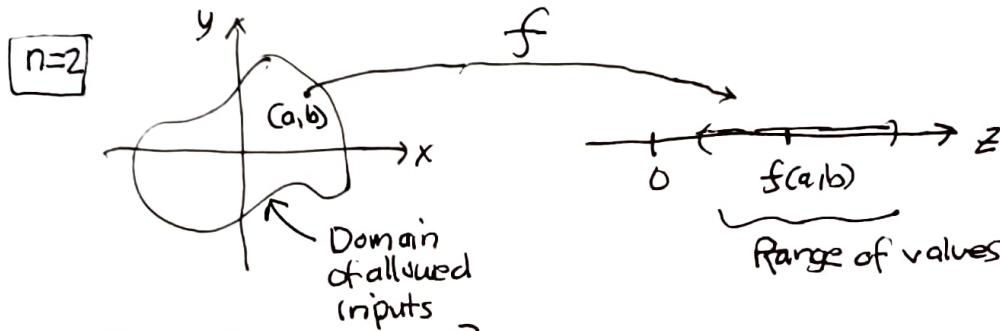


14.1

functions of  $n > 1$  independent variables

2

We follow the same path as with 1 ind. var:

Functions defined by formulas

[where formula produces a unique real number]

Troubles with formula that can arise:

exclude points from domain where this happens

- division by zero
- even root negative number (complex)
- $\ln(u)$ :  $u=0$  or  $u < 0$  ( $-\infty$ ) (complex)
- etc.

[limits] are only needed to explore behaviors near "bad points", otherwise just plug in  $(x_1, y)$  values to get function values.

14.1

functions of  $n > 1$  independent variables

(3)

Example

$$f(x_1, y) = x^2 + y^2$$

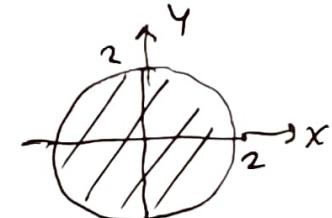
Domain:  $\mathbb{R}^2$  (no restrictions!)  
 Range:  $[0, \infty)$  nonnegative  
 no need for limits anywhere

Example

$$f(x_1, y) = \sqrt{4 - x^2 - y^2} \geq 0$$

$$\begin{aligned} 4 - x^2 - y^2 &\geq 0 \\ 4 &\geq x^2 + y^2 \\ x^2 + y^2 &\leq 4 \end{aligned}$$

Domain:



interior of circle  
of radius 2 &  
boundary

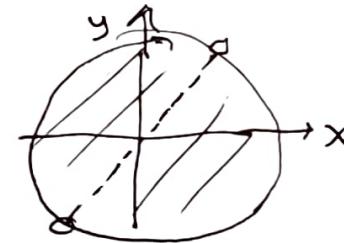
Range:  $[0, \infty)$  all nonnegative  
 $\uparrow$        $\downarrow$  at  $(0,0)$   
 on circle

can consider limits on circular boundary  
 = edge of domain (goes complex)

Example

$$f(x_1, y) = \sqrt{\frac{4 - x^2 - y^2}{x - y}} \quad \left. \begin{array}{l} \text{Domain:} \\ \text{exclude also} \\ y = x \end{array} \right\}$$

Domain:  
 exclude  
 also  
 $y = x$



Range:  $\mathbb{R}$  (why?)

Example

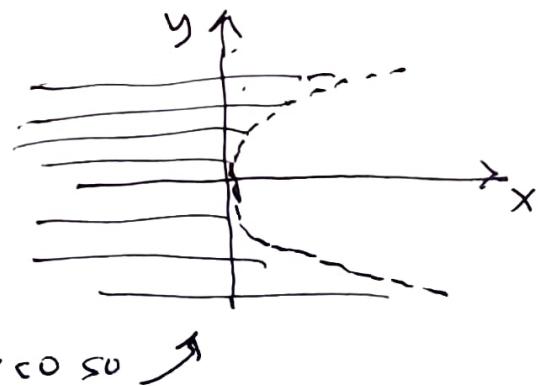
$$f(x_1, y) = x \ln(y^2 - x)$$

$$\underbrace{y^2 - x}_{> 0}$$

$$y^2 > x$$

$$\text{Domain: } x < y^2$$

↑ includes all  $x < 0$  so



Range:  $\ln$  produces all real values from all  
 nonnegative inputs so:  $\mathbb{R}$

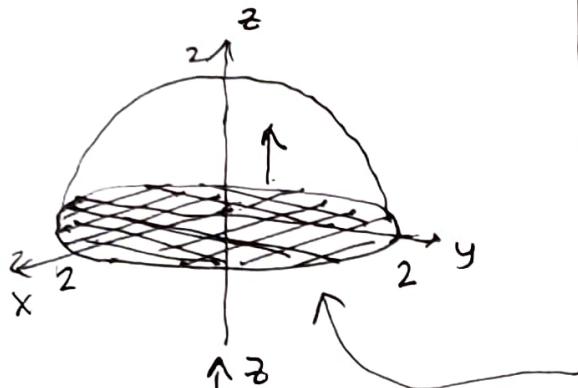
Ranges are not usually easy to describe, not so important.

# 14.1 functions of $n > 1$ independent variables

(4)

$n=2$  graphs:  $z = f(x, y)$  surface in space

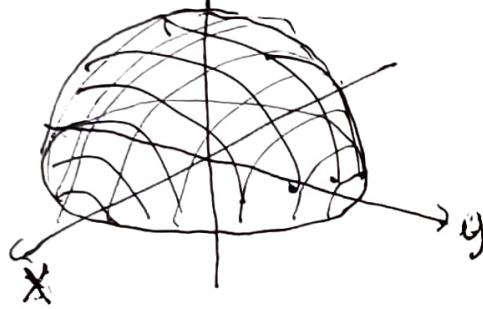
Example  $z = \sqrt{4 - x^2 - y^2} \geq 0 \rightarrow z^2 = 4 - x^2 - y^2 \rightarrow x^2 + y^2 + z^2 = 4$   
 sphere of radius 2 at origin but  
 only upper hemisphere.



visualize 3d surface on 2-d paper/screen  
 using coordinate grid projected vertically  
 onto surface

x-y grid (equally spaced divisions)

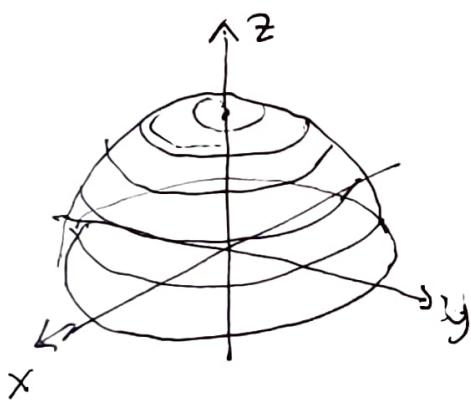
Maple: "Surface-WireFrame"



OR

use equally spaced horizontal  
 plane cross-sections

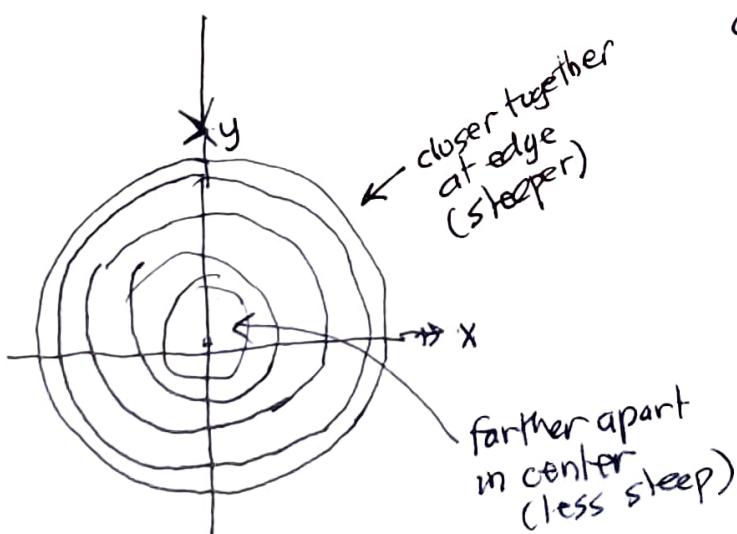
Maple: "Surface-contour"



OR

project these contours onto  
 the x-y plane

get "level curves" or  
 "contour curves"  
 visualization of function graph



BUT we are  
 not  
 artists  
 we need technology!

# 14.1 functions of $n > 1$ independent variables

(5)

Example  $f(x, y) = 6 - 3x - 2y$

$$z = 6 - 3x - 2y \rightarrow 3x + 2y + z = 6 \quad \text{standard form eqn}$$

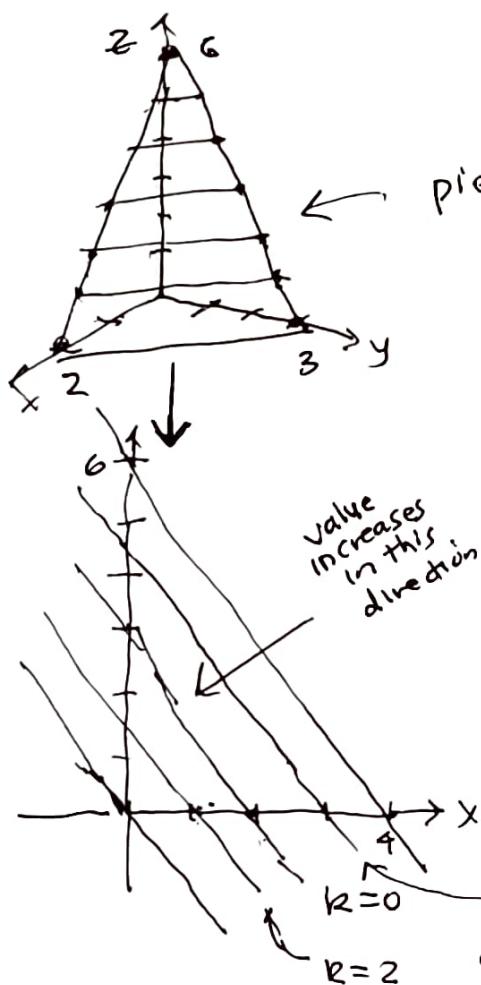
↓  
plane

Find axis intercepts:  $y=z=0 \rightarrow x = 6/3 = 2$

$$x=z=0 \rightarrow y = 6/2 = 3$$

$$x=y=0 \rightarrow z = 6/1 = 6$$

3 pts determine a plane!

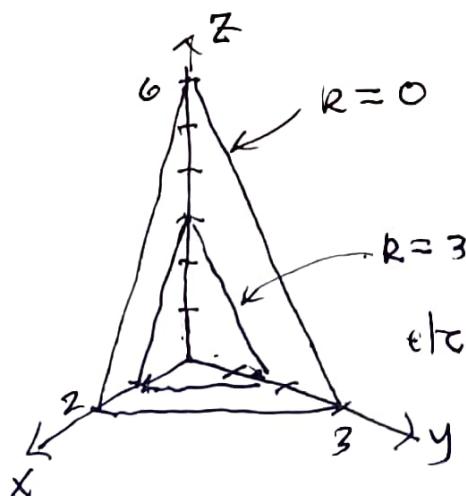


$$6 - 3x - 2y = k$$

$$\downarrow y = \frac{6-3x-k}{2} = 3 - \frac{3}{2}x - \frac{k}{2}$$

equally spaced parallel lines  
level curves / contours

Example  $n = 3$   $f(x, y, z) = 6 - 3x - 2y - z = k$  ( $= 0$  previous graph!)



get family of equally spaced parallel planes for level surfaces / contour surfaces

value increases towards origin

# 14.1 functions of $n > 1$ independent variables

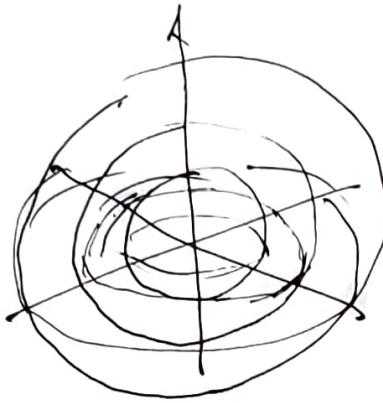
(6)

Example

$n=3$

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2} = R \geq 0$$

= distance from origin  
spheres concentric about origin  
(equally spaced radii)



? I repeat we are not  
Artists!  
not even bob  
we need Maple.

But before the show and tell ...

14.1

functions of  $n > 1$  independent variables

(7)

functions defined by tabular data

calc 1:

$x$	$x_1$	...	$x_{i-1}$	$x_i$	$x_{i+1}$	...	$x_n$
$y = f(x)$	$y_1$	...	$y_{i-1}$	$y_i$	$y_{i+1}$	...	$y_n$

$\frac{(\frac{\Delta y}{\Delta x})_{\text{left}} + (\frac{\Delta y}{\Delta x})_{\text{right}}}{2}$

1-d array of values

average nearest neighbor avg rates of change to get tabular "derivative" at  $x = x_i$

calc 3:

$x_1$	...	$x_{i-1}$	$x_i$	$x_{i+1}$	...
$y_1$					
$\vdots$					
$y_{j-1}$					
$y_j$					
$y_{j+1}$					
$\vdots$					

$Z_{i,j} = f(x_i, y_j)$

repeat calc 1 "x" calculations horizontally

repeat calc 1 "y" calculations vertically

only nearest neighbors in same row or column

needed for x or y tabular derivatives (Later) at each data point

textbook: wind chill tabular data function

$$W = W(V, T)$$

↑                      ↑                      ↑  
 perceived      wind      actual  
 temperature    speed    temperature

A.1

# functions of $n > 1$ independent variables

8

Functions defined by fitting data

textbook example      ~~Cost~~ Douglas production function

$$P(L, K) = b \underbrace{L^\alpha}_{\substack{\text{production} \\ \text{labour}}} \underbrace{K^{1-\alpha}}_{\text{capital}}$$

scales property:

$$\begin{aligned} P(2L, 2K) &= b(2L)^\alpha(2K)^{1-\alpha} \\ &= b \underbrace{2^\alpha L^\alpha}_{= 2} 2^{1-\alpha} K^{1-\alpha} \\ &= 2 P(L, K) \end{aligned}$$

doubling inputs should double output  
(true for any multiple!)

obtained by cleverly fitting  
annual data 1999–2000?

year	P	L	K
1999	*	x	*
2000	x	x	x
2001	x	x	x
⋮			

data fit yields  
fixed parameter  
values.

} so only some pairs of  $(L, K)$   
have values  $P$   
(not like a data table)

$$P = 1.01 L^{.75} K^{.25}$$

↑  
can then use continuous function  
on all nonnegative values  
of  $(L, K)$ .