

13.3b

curvature etc

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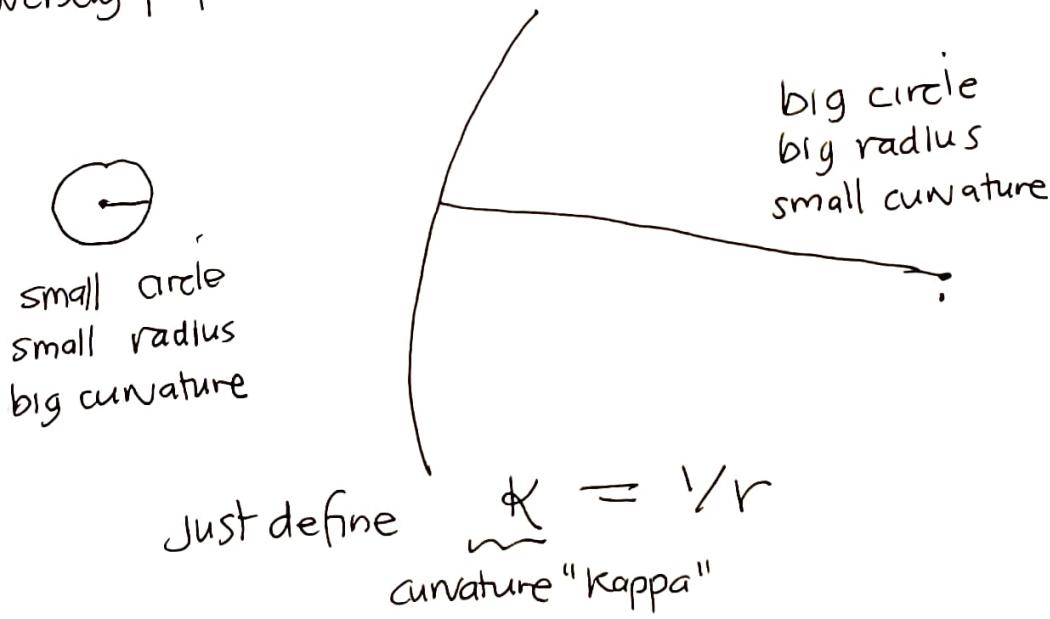
plane curves in  $\mathbb{R}^2$  "curve" and one can quantify how much by comparing with a circle.

space curves in  $\mathbb{R}^3$  curve in the same way with the same measure of curvature BUT they can also twist in space simultaneously. This twisting is what happens when the instantaneous plane of the motion containing  $\vec{r}'$  and  $\vec{r}''$  changes its orientation, i.e., its normal direction rotates.

The study of how space curves "curve" and "twist" utilizes the various derivatives along the curve, but one needs a parametrization to define derivatives and rates of change. The geometry of a curve is independent of how we move along it, so we need to remove the effects of "nonuniform" motion along it — we need to trace it out "uniformly" without "speeding up" or "slowing down".

This is accomplished by referring derivatives to an arclength parametrization which corresponds to tracing out the curve at "unit speed." The catch is that very few curves can be parametrized by their arclength so we simply use the chain rule with related rates of change to evaluate arclength derivatives!

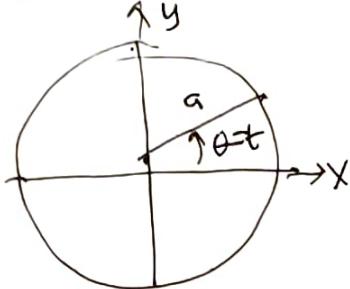
Any measure of the curvature of a circle should be inversely proportional to its radius:



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Calculate:



$$\vec{r} = \langle a \cos t, a \sin t \rangle \quad \text{but } s = a\theta = at$$

$$= \langle a \cos \frac{s}{a}, a \sin \frac{s}{a} \rangle \quad \text{arclength parametrization}$$

$$\frac{d\vec{r}}{ds} = \langle a(-\frac{1}{a} \sin \frac{s}{a}), a(\frac{1}{a}) \cos \frac{s}{a} \rangle$$

$$= \langle -\sin \frac{s}{a}, \cos \frac{s}{a} \rangle = \hat{T}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\hat{T}}{ds} = \langle -\frac{1}{a} \cos \frac{s}{a}, -\frac{1}{a} \sin \frac{s}{a} \rangle = -\frac{1}{a} \underbrace{\langle \cos \frac{s}{a}, \sin \frac{s}{a} \rangle}_{\hat{T}}$$

$$\left| \frac{d^2\vec{r}}{ds^2} \right| =$$

$$K \equiv \left| \frac{d\hat{T}}{ds} \right| = \frac{1}{a}$$

We use this for any curve to define curvature!

Curvature is the magnitude of the second arclength derivative of position.

Direction of change:  $\hat{N} = \frac{\hat{T}'}{\|\hat{T}'\|}$  = normal vector to curve.

$$\frac{d\hat{T}}{ds} = K \hat{N}$$

We will use this for space curves too.

$$\vec{r}(t)$$

**CHAIN RULE**

For any parametrized curve  $\vec{r}(t)$  the arclength rate of change of any quantity  $Q(t)$  along the curve is:

$$\frac{dQ}{ds} = \frac{dQ/dt}{ds/dt} = \frac{Q'}{s'}$$

divide by  $s' = \|\vec{r}'\|$ . to get arclength rate of change

**product rule**

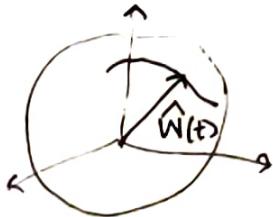
For any vector along the curve, its geometric decomposition is a product  $\vec{W} = \|\vec{W}\| \hat{W}$ .

The product rule then shows how changing length and direction contribute to the changing vector.

For a unit vector

$$0 = \frac{d}{dt} \|\hat{W}\|^2 = \frac{d}{dt} (\hat{W} \cdot \hat{W}) = 2 \hat{W}' \cdot \hat{W}$$

so its change is orthogonal to its direction.



The tip of a unit vector with initial pt at the origin for comparison can only rotate, moving around the unit sphere, so its derivative must be tangential to the sphere

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Now we are ready to analyze the geometry of a curve through its first and second derivatives.

start:  $\vec{r} = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

"differentiate"

$$\begin{array}{c} D \\ \downarrow \\ \vec{r}' = \frac{d\vec{r}}{dt} = \langle \dot{x}, \dot{y}, \dot{z} \rangle \end{array}$$

$| \vec{r}' | = s'$

$\hat{T} = \frac{\vec{r}'}{| \vec{r}' |} = \frac{\vec{r}'}{s'}$

decompose

$\vec{r}' = s' \hat{T}$

$$\vec{r}'' = \frac{d}{dt} (s' \hat{T}) = s'' \hat{T} + s' \left( \frac{d\hat{T}}{dt} \right)$$

we want  $\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt} / \frac{dt}{ds}$

$$= \frac{1}{s'} \hat{T}'$$

↓ get rid of term along  $\vec{r}'$

$$\begin{aligned} \vec{r}' \times \vec{r}'' &= (s' \hat{T}) \times (s'' \hat{T}' + s' \hat{T}'') \\ &= (s')^2 \underbrace{\hat{T} \times \hat{T}'}_{\text{orthogonal}} \end{aligned}$$

$$\underbrace{|\hat{T}|}_{1} |\hat{T}'| \underbrace{\sin \pi/2}_{1}$$

$$\begin{aligned} |\vec{r}' \times \vec{r}''| &= (s')^2 \underbrace{|\hat{T} \times \hat{T}'|}_{\substack{\text{unit normal} \\ |\hat{T}'|=s'|\frac{d\hat{T}}{ds}|}} \\ &= (s')^3 \left| \frac{d\hat{T}}{ds} \right| \end{aligned}$$

$$\boxed{\frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \left| \frac{d\hat{T}}{ds} \right| = k}$$

unit normal:

$$\frac{d\hat{T}}{ds} = k \hat{N}, \quad \hat{N} = \frac{\hat{T}'}{|\hat{T}'|} \quad \left. \begin{array}{l} \text{just} \\ \text{normalize } \hat{T}' \end{array} \right\}$$

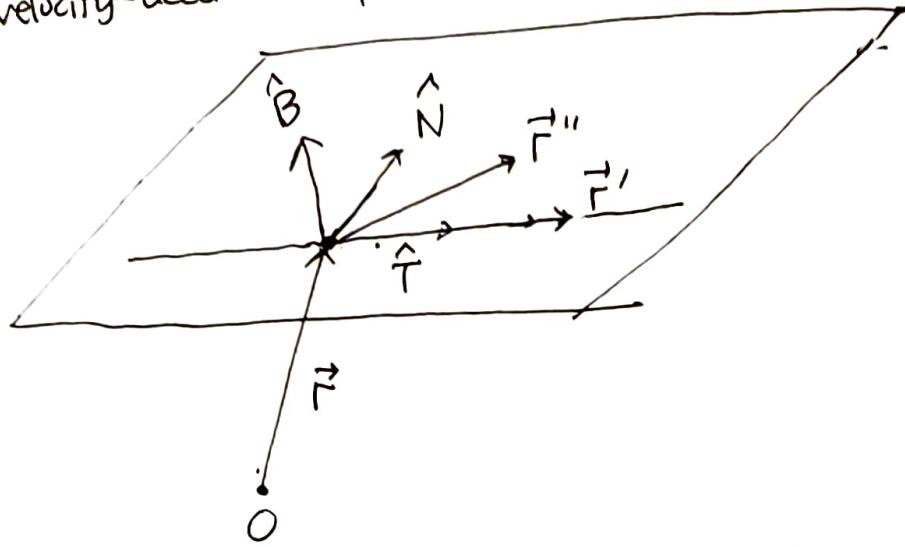
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$\hat{T}$ - $\hat{N}$ - $\hat{B}$  orthogonal triad of unit vectors!

velocity-acceleration plane:



The unit normal  $\hat{N}$  is orthogonal to the unit tangent  $\hat{T}$  within this plane and either cross product pair gives a normal to the plane.

Define the unit binormal:

$$\hat{B} = \hat{T} \times \hat{N} = \frac{\vec{r}' \times \vec{r}''}{\|\vec{r}' \times \vec{r}''\|} \quad \left. \begin{array}{l} \text{easiest formula to} \\ \text{evaluate, just} \\ \text{normalize } \vec{r}' \times \vec{r}'' \end{array} \right\}$$

$\hat{T}, \hat{N}, \hat{B}$  have the same cross-product algebra as  $\hat{i}, \hat{j}, \hat{k}$ :  
 mutually  
orthogonal  
unit vectors

$$\hat{N} = \hat{B} \times \hat{T}$$

easier to calculate than  
 $\hat{T}'$

If a curve remains in any such given velocity-acceleration plane, then  $\hat{B}$  does not change and the curve is a [plane curve].

The "tilting" of the velocity-acceleration plane along the curve causes the "twisting" of that space curve so that it cannot remain in the same plane.

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Helix example

2-parameter family!

$$\vec{r} = \langle a \cos t, a \sin t, bt \rangle$$

$$\vec{r}' = \langle -a \sin t, a \cos t, b \rangle$$

$$|\vec{r}'| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2} = s'$$

$$\hat{T} = \frac{1}{\sqrt{a^2 + b^2}} \langle -a \sin t, a \cos t, b \rangle$$

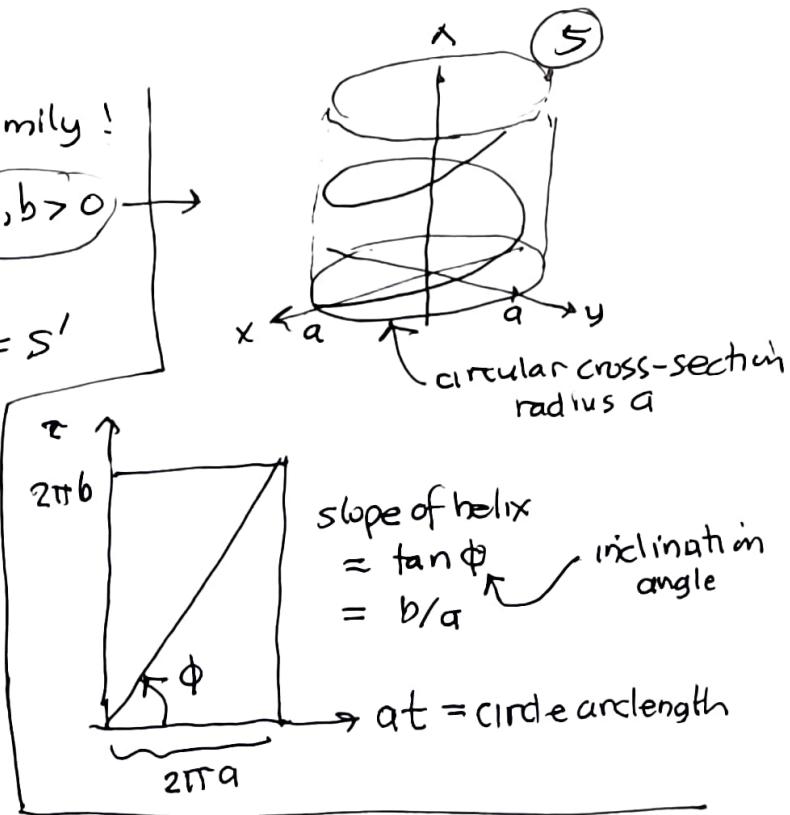
$$\hat{T}' = \frac{1}{\sqrt{a^2 + b^2}} \langle -a \cos t, -a \sin t, 0 \rangle$$

$$= \frac{-a}{\sqrt{a^2 + b^2}} \langle \cos t, \sin t, 0 \rangle$$

radially out from  $\hat{T}$   
inward!  
unit vector

$$|\hat{T}'| = \frac{a}{\sqrt{a^2 + b^2}}$$

$$K = \frac{|\hat{T}'|}{s'} = \frac{a}{\frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2}}} = \frac{a}{a^2 + b^2}$$



$$\hat{T}' = - \langle \cos t, \sin t, 0 \rangle = \hat{N}$$

horizontal unit normal

or:  $\vec{r}' \times \vec{r}'' = a \langle b \sin t, -b \cos t, a \rangle$

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{a \sqrt{b^2 \sin^2 t + b^2 \cos^2 t + a^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}$$

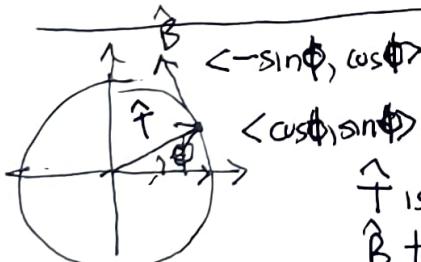
$\hat{T}'$  is a difficult quotient rule in general

$$\hat{B} = \frac{\vec{r}' \times \vec{r}''}{|\vec{r}' \times \vec{r}''|} = \frac{a \langle b \sin t, -b \cos t, a \rangle}{a \sqrt{b^2 + a^2}} = \frac{\langle b \sin t, -b \cos t, a \rangle}{\sqrt{a^2 + b^2}}$$

↓ or

$$\hat{B} = \hat{T} \times \hat{N} \quad \text{Maple}$$

same result but by hand much more tedious to calculate  $\hat{N}$  first



$\hat{T}$  is tilted up by  $\phi$   
 $\hat{B}$  tilted back by  $\phi$

$$= \frac{b}{\sqrt{a^2 + b^2}} \langle \sin t, -\cos t, 0 \rangle \leftarrow (\text{hor.})$$

$$+ \frac{0}{\sqrt{a^2 + b^2}} \langle 0, 0, 1 \rangle \leftarrow (\text{vert.})$$



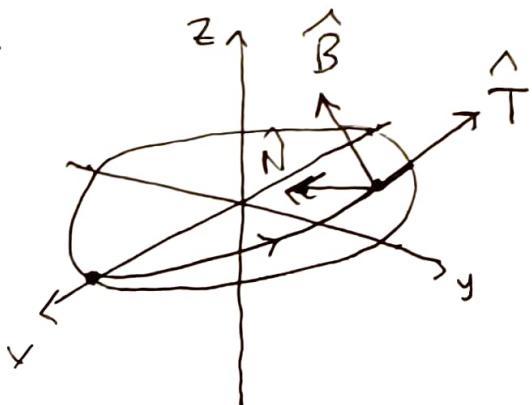
$$\cos \phi = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin \phi = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\text{but } \hat{T} = \cos \phi \langle -\sin \phi, \cos \phi, 0 \rangle + \sin \phi \langle 0, 0, 1 \rangle$$

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visualize:



$\hat{T}$  tilts up  
 $\hat{B}$  tilts back to remain orthogonal

$\hat{N}$  points towards  $z$ -axis

$\hat{T}$  and  $\hat{B}$  are tangent to the cylinder but rotated about  $\hat{N}$  by the tilt angle  $\phi$  of the helix, as it climbs up the cylinder.

interpret and visualize curvature:

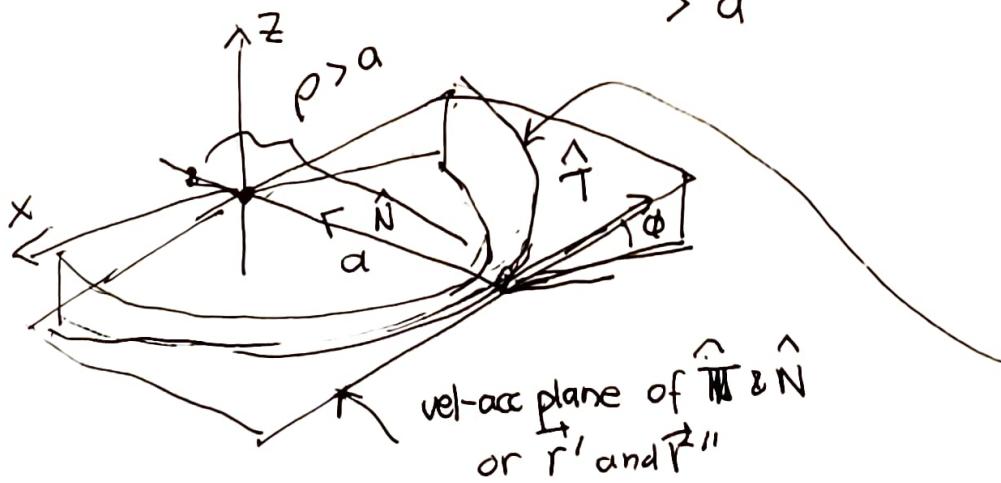
Define radius of curvature  $\rho \equiv 1/k$

$$\text{helix: } \frac{1}{\rho} = \frac{a^2+b^2}{a} = \sqrt{a^2+b^2} \left( \frac{\sqrt{a^2+b^2}}{a} \right)$$

$= \frac{\sqrt{a^2+b^2}}{\cos \phi}$  } gives scale of radius  
 ← tilt factor:



dividing by  $\cos \phi$   
 "stretches" lengths by tilt factor



"osculating circle"  
 in this plane  
 with center a distance  $\rho$  along  $\hat{N}$  from point on curve

Think of a "Slinky" spring.

As you pull it upward the opening coil is increasingly less curved than the circular cross-section

13.3b) curvature etc

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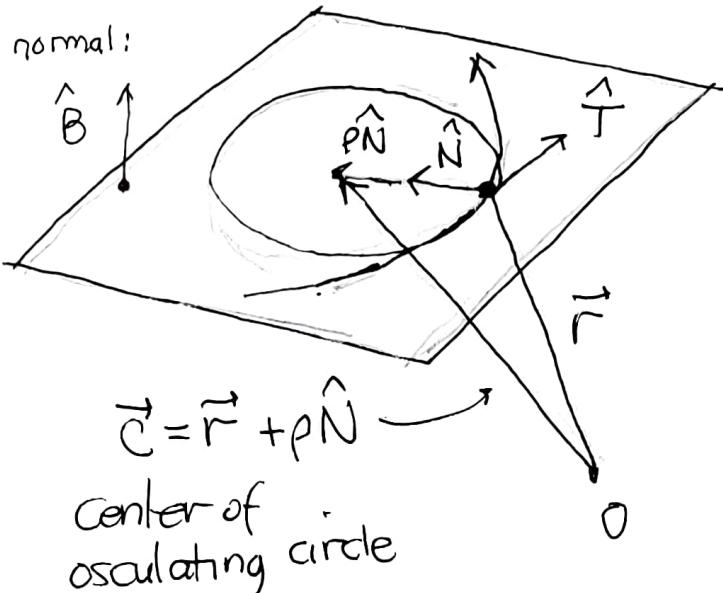
## osculating circle geometry

$\hat{T}$ - $\hat{N}$  plane = velocity-acceleration plane

= "osculating plane"

mathematical name

"kissing" (latin root)



Center of  
osculating circle

more distance  $p$   
along unit normal  $\hat{N}$   
from point of tangency

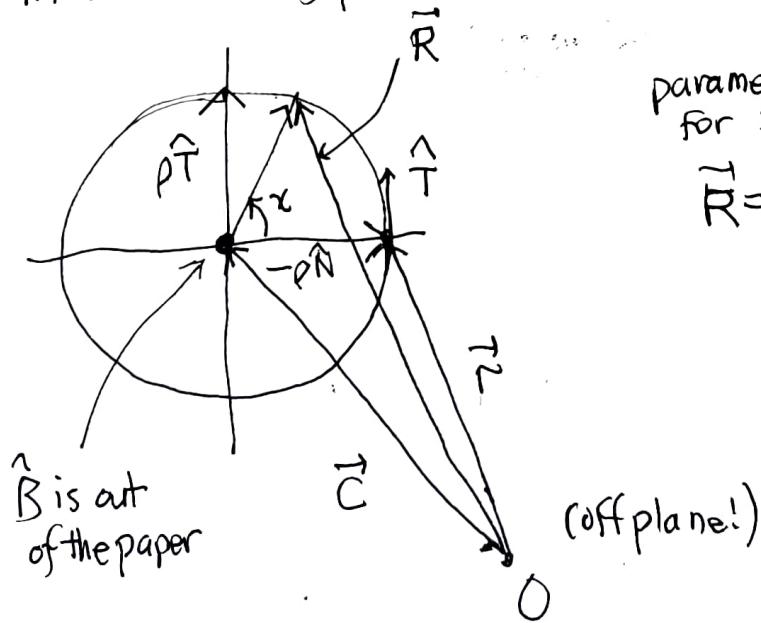
zoom in to point  
of tangency  
(tip of  $\vec{r}$ )

and curve  
merges with  
osculating circle

which is a quadratic  
approximation  
to the curve.

continue zoom in  
& both straighten,  
out to tangent line.  
(linear approximation)

in the osculating plane:



parametrization for osculating circle  
for fixed  $t$  on curve:

$$\vec{R} = \vec{C} + \underbrace{(-p\vec{N}) \cos x + (p\vec{N}) \sin x}_{\text{analogous to}}$$

$\langle r_0 \cos \theta, r_0 \sin \theta \rangle$

$$= r_0 \cos \theta \hat{i} + r_0 \sin \theta \hat{j}$$

Note:  $\vec{R} = \vec{R}(t, x)$

$x = 0 \leftrightarrow$  point of tangency

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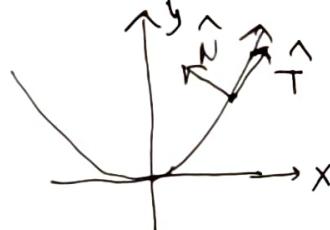
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okay, you don't have to evaluate all this stuff but now you understand how Maple can illustrate all of this curve geometry for you.

pretend in  $\mathbb{R}^3$   
in  $xy$  plane  
 $z=0$

quadratic approximation

plane curve  $y = x^2 \rightarrow x = t$   
 $y = t^2$



$$\vec{r} = \langle t, t^2, 0 \rangle$$

$$\vec{r}' = \langle 1, 2t, 0 \rangle$$

$$|\vec{r}'| = \sqrt{1+4t^2}$$

$$\hat{T} = \frac{\langle 1, 2t, 0 \rangle}{\sqrt{1+4t^2}}$$

$$\vec{r}'' = \langle 0, 2, 0 \rangle$$

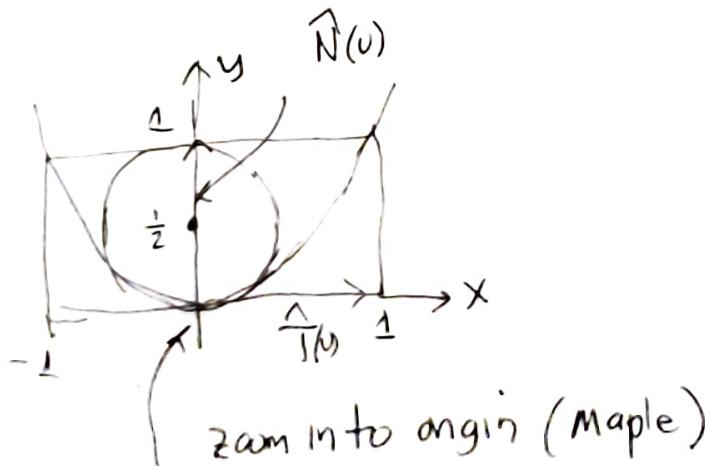
$$\vec{B} = \vec{r}' \times \vec{r}'' = \langle 1, 2t, 0 \rangle \times \langle 0, 2, 0 \rangle = \underset{\text{Maple}}{2} \langle 0, 0, 1 \rangle \quad \begin{array}{l} \text{(avoids} \\ \text{quotient} \\ \text{rule)} \end{array}$$

$$\hat{B} = \langle 0, 0, 1 \rangle = \hat{k} \quad (\text{binormal to plane } z=0)$$

$$\hat{N} = \hat{B} \times \hat{T} = \langle 0, 0, 1 \rangle \times \frac{\langle 1, 2t, 0 \rangle}{\sqrt{1+4t^2}} = \frac{\langle -2t, 1, 0 \rangle}{\sqrt{1+4t^2}}$$

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{2}{(1+4t^2)^{3/2}} \rightarrow \rho = \frac{(1+4t^2)^{3/2}}{2} \geq \frac{1}{2}$$

$\rho(0)$   
minimum  
radius



$$\vec{r}(0) = \langle 0, 0, 0 \rangle$$

$$\hat{N}(0) = \langle 0, 1, 0 \rangle$$

$$\hat{T}(0) = \langle 1, 0, 0 \rangle$$