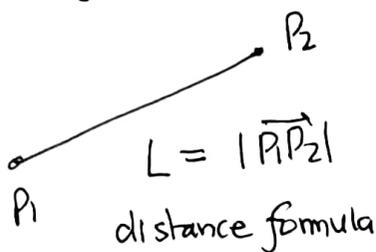


### 13.3a arclength

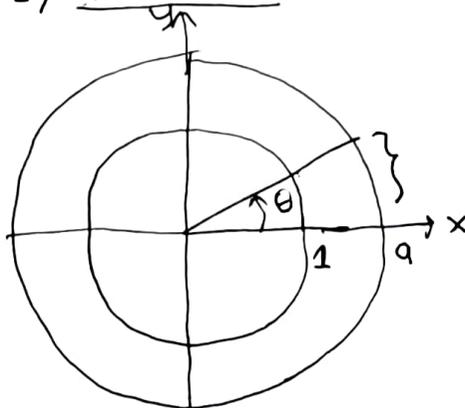
①

pre calc we only know 2 things about arclength:

1) straight lines!

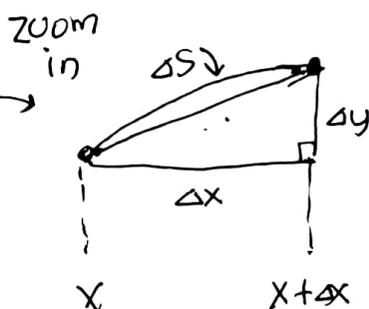
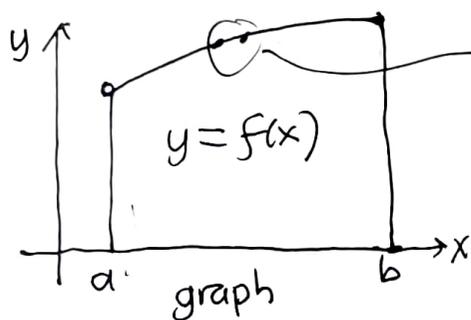


2) circular arcs:



arc of circle  
 $S = a\theta$   
 "arc" length = angle  $\times$  radius  
 radians = arclength on unit circle.

### calc 2 plane curves



secant line length approximates true "arc"

$$\Delta S \approx \sqrt{\Delta x^2 + \Delta y^2}$$

↓ limit

$$ds = \sqrt{dx^2 + dy^2}$$

differential of arclength

For a graph:

$$y = f(x), \frac{dy}{dx} = f'(x), dy = f'(x)dx \rightarrow ds = \sqrt{dx^2 + f'(x)^2 dx^2} = \sqrt{1 + f'(x)^2} dx \quad (dx > 0)$$

$$L = \int_a^b ds(x) = \int_a^b \sqrt{1 + f'(x)^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

For a parametrized curve:

$$\left. \begin{array}{l} x = x(t), y = y(t) \\ dx = x'(t)dt, dy = y'(t)dt \end{array} \right\} ds = \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2} = \sqrt{x'^2 + y'^2} dt \quad (dt > 0)$$

$$L = \int_a^b ds(t) = \int_a^b \sqrt{x'^2 + y'^2} dt$$

↓ space curves

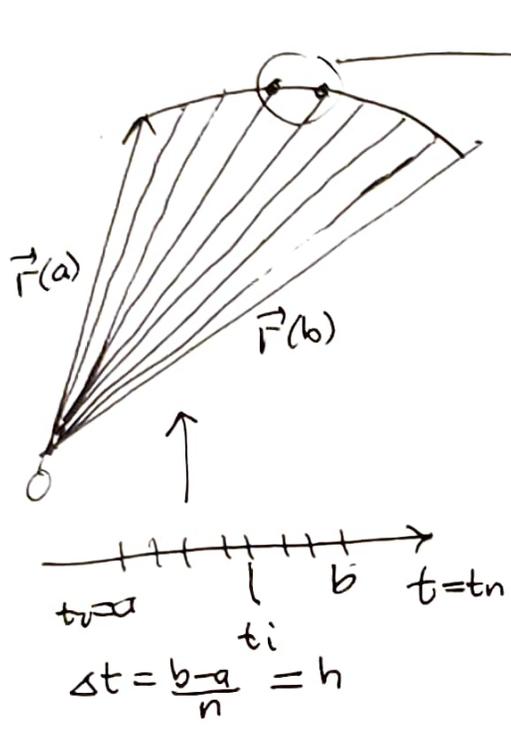
$$L = \int_a^b \sqrt{x'^2 + y'^2 + z'^2} dt$$

Let's revisit this.

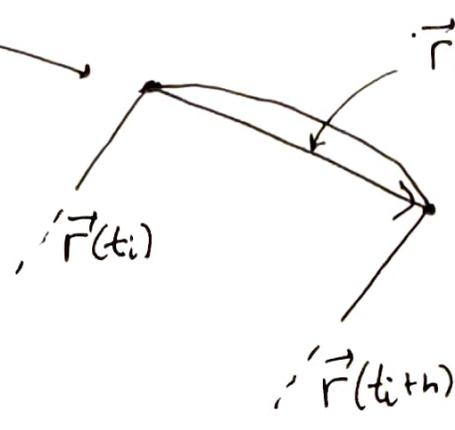
13.3a arclength

(2)

space curve arclength



zoom in



$\vec{r}(t_i+h) - \vec{r}(t_i) \approx h \vec{r}'(t_i)$   
 since  $\vec{r}'(t_i) \approx \frac{\vec{r}(t_i+h) - \vec{r}(t_i)}{h}$   
 for small enough  $h$  if  $\vec{r}'(t_i)$  exists

$$\Delta L_i \approx |\vec{r}(t_i+h) - \vec{r}(t_i)| \approx |\vec{r}'(t_i)| h = |\vec{r}'(t_i)| \Delta t$$

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta L_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n |\vec{r}'(t_i)| \Delta t = \int_a^b |\vec{r}'(t)| dt$$

$ds$  differential of arclength

arclength = integral of length of tangent vector

in physics jargon:  $\vec{r}'(t) = \vec{v}(t)$   
 $|\vec{r}'(t)| = |\vec{v}(t)| = v(t) = \text{speed}$

integral of speed = distance traveled

this generalizes the precalc fact

$d = vt$  when constant speed  
 units: length  $\leftarrow \left\{ \begin{array}{l} \text{length} \\ \text{time} \end{array} \right.$  time

explicitly:  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$   
 $|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

13.3a arclength

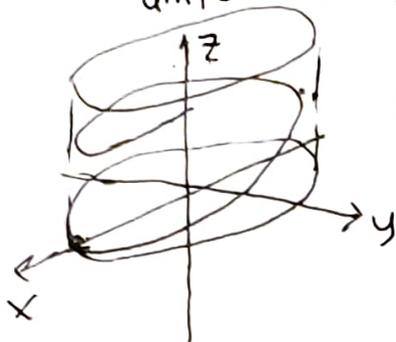
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example: helix

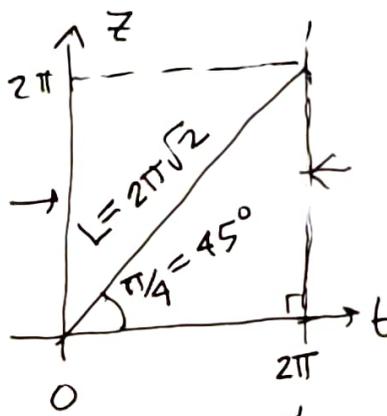
$$\vec{r} = \langle \cos t, \sin t, t \rangle = \langle x, y, z \rangle$$

lies within cylinder  $x^2 + y^2 = 1$

unit circle in x-y plane



$\theta = t$   
arc length on circle



roll this into a cylinder identifying vertical edges

tilt angle of  $\vec{r}$  wrt horizontal is  $45^\circ$

calculate

$$\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

clearly length is  $(2\pi)\sqrt{2}$

$$L = \int_0^{2\pi} \sqrt{2} dt = \sqrt{2} t \Big|_0^{2\pi} = 2\pi\sqrt{2} \checkmark$$

(one loop)

integrand constant so integral is trivial

BUT only in special cases does the integral allow an exact result in terms of our familiar special functions

All textbook exercises rely on a perfect square to eliminate the square root or a special factorization allowing a simple u-substitution.

13.3a arclength

4

perfect square example

$$\vec{r} = \langle \sqrt{2}t, e^t, e^{-t} \rangle$$

$$\vec{r}' = \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$|\vec{r}'| = \sqrt{2 + (e^t)^2 + (e^{-t})^2} = \sqrt{\underbrace{(e^t + e^{-t})^2}_{\text{perfect square!}}} = e^t + e^{-t} \quad \text{radical gone!}$$

$$L = \int_0^1 |\vec{r}'(t)| dt = \int_0^1 e^t + e^{-t} dt = (e^t - e^{-t}) \Big|_0^1 \\ = e^1 - e^{-1} = e - e^{-1} \quad \text{easy!}$$

perfect square example

$$\vec{r} = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$$

rescaled twisted cubic  
very special choice

$$\vec{r}' = \langle 2, 2t, t^2 \rangle$$

$$|\vec{r}'|^2 = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

$$L = \int_0^1 t^2 + 2 dt = \left. \frac{t^3}{3} + 2t \right|_0^1 = \frac{1}{3} + 2 = \frac{7}{3} \quad \text{easy!}$$

without these coefficients (modulo an overall constant),  
there is no familiar antiderivative so we (not Maple!)  
are forced to do a numerical approximation.

$$\vec{r} = \langle t, t^2, t^3 \rangle \rightarrow \vec{r}' = \langle 1, 2t, 3t^2 \rangle$$

$$L = \int_0^1 \sqrt{1 + 4t^2 + 9t^4} dt = \text{exact formula in terms of special functions}$$

≈

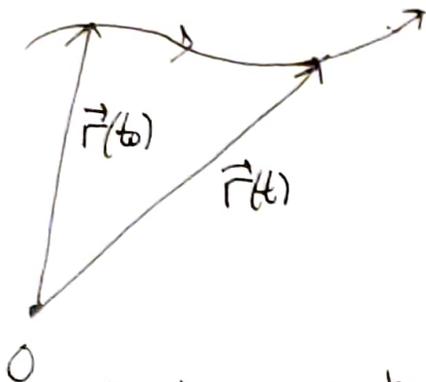
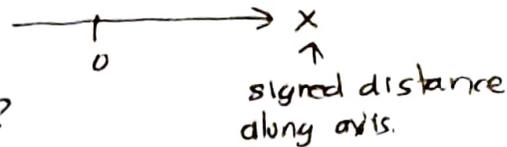
# 13.3a arclength

(5)

## arclength function

fix a reference point on a curve and measure the "signed" arclength along it

just like on a straight line axis:



confusing?

$$s = \int_{t_0}^t |\vec{r}'(t)| dt = \int_{t_0}^t |\vec{r}'(u)| du$$

dummy integration variable

$$\begin{cases} > 0 & \text{if } t > t_0 \\ = 0 & \text{if } t = t_0 \\ < 0 & \text{if } t < t_0 \end{cases}$$

Rarely can we actually evaluate this exactly and if so, we cannot often invert the relationship between  $s$  and  $t$ .

Helix  $\vec{r} = \langle \cos t, \sin t, t \rangle \rightarrow |\vec{r}'(t)| = \sqrt{2} \rightarrow s = \int_0^t \sqrt{2} du = \sqrt{2}u \Big|_0^t = \sqrt{2}t$

invert:  $t = s/\sqrt{2}$

## reparametrize this curve by this arclength

$$\vec{r} = \left\langle \cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle$$

$$\vec{r}' = \left\langle \frac{1}{\sqrt{2}}(-\sin \frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}} \langle -\sin \frac{s}{\sqrt{2}}, \cos \frac{s}{\sqrt{2}}, 1 \rangle$$

$$|\vec{r}'| = \frac{1}{\sqrt{2}} \sqrt{\sin^2 \frac{s}{\sqrt{2}} + \cos^2 \frac{s}{\sqrt{2}} + 1} = \frac{1}{\sqrt{2}} \sqrt{2} = 1 \text{ unit vector!}$$

why?

$$s = \int_{t_0}^t |\vec{r}'(u)| du \xrightarrow[\text{thm of calculus}]{\text{fund}} \frac{ds}{dt} = |\vec{r}'(t)|$$

chain rule:  $\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \frac{dt}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{r}'}{|\vec{r}'|} = \hat{T}$  unit tangent!

Tracing out an arclength parametrized curve corresponds to moving with unit speed, eliminating the effects of speeding up or slowing down.

# 13.3a arclength

(B)

more interesting example

$$\vec{r} = \langle \sqrt{2}t, e^t, e^{-t} \rangle$$

$$\vec{r}' = \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$|\vec{r}'| = \sqrt{2 + e^{2t} + e^{-2t}} = e^t + e^{-t}$$

$$s = \int_0^t e^u + e^{-u} du = e^u - e^{-u} \Big|_0^t = e^t - e^{-t} - (e^0 - e^0) = \underbrace{e^t - e^{-t}}_{2 \sinh t}$$

fact:  $\cosh x = \frac{e^x + e^{-x}}{2}$      $\frac{d}{dx} \cosh x = \sinh x$   
 $\sinh x = \frac{e^x - e^{-x}}{2}$      $\frac{d}{dx} \sinh x = \cosh x$

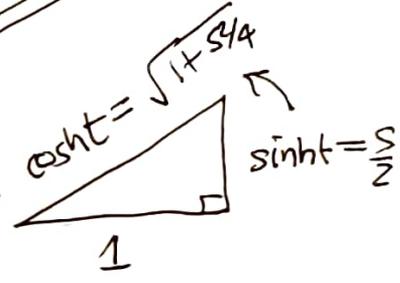
$s = 2 \sinh t \rightarrow t = \operatorname{arcsinh}(\frac{s}{2})$  inverted!

$$\vec{r} = \langle \sqrt{2} \operatorname{arcsinh}(\frac{s}{2}), e^{\operatorname{arcsinh} s/2}, e^{-\operatorname{arcsinh} s/2} \rangle$$

ignorable aside:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4} \left( \cancel{e^{2x}} + 2e^x e^{-x} + \cancel{e^{-2x}} - (\cancel{e^{2x}} - 2e^x e^{-x} + \cancel{e^{-2x}}) \right) = \frac{1}{4}(4) = 1! \end{aligned}$$

fundamental hyperbolic identity  
 analogous to  $\cos^2 x + \sin^2 x = 1$



$\cosh^2 t = 1 + \sinh^2 t$   
 Pythagoras!

$$\begin{aligned} \cosh x + \sinh x &= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x = \frac{s}{2} + \sqrt{1 + s^2/4} \\ \cosh x - \sinh x &= \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x} = -\frac{s}{2} + \sqrt{1 + s^2/4} \end{aligned}$$

$$\vec{r} = \langle \sqrt{2} \operatorname{arcsinh} \frac{s}{2}, \frac{s}{2} + \sqrt{1 + s^2/4}, -\frac{s}{2} + \sqrt{1 + s^2/4} \rangle$$

$\ln(\frac{s}{2} + \sqrt{1 + s^2/4})!$  but that is another story for those who love math!