

13.2b derivatives and vector operations / definite integrals (1)

Derivative rules for vector-valued functions mirror the rules for scalar functions.

Symbolically, erasing arrows over vector symbols, erasing dot and cross product symbols, reduces vector rules to the corresponding

scalar rules:

sum + : $\frac{d}{dt}(f(t) + g(t)) = \frac{df}{dt}(t) + \frac{dg}{dt}(t)$ (generalize to any number of terms)

constant factor $c \times$: $\frac{d}{dt}(c f(t)) = c \frac{df}{dt}(t)$ combine \downarrow

$$\frac{d}{dt}(c_1 f_1(t) + c_2 f_2(t) + \dots) = c_1 \frac{df_1}{dt}(t) + c_2 \frac{df_2}{dt}(t) + \dots$$

"linear combination rule"

product \times : $\frac{d}{dt}(f(t)g(t)) = \frac{df}{dt}(t)g(t) + f(t)\frac{dg}{dt}(t)$ (keep factor order)

iterate: $\frac{d}{dt}(f(t)g(t)h(t)) = \frac{df}{dt}(t)g(t)h(t) + f(t)\frac{dg}{dt}(t)h(t) + f(t)g(t)\frac{dh}{dt}(t)$

quotient \div : $\frac{d}{dt}\left(\frac{N(t)}{D(t)}\right) = \frac{D(t)\frac{dN}{dt}(t) - N(t)\frac{dD}{dt}(t)}{D(t)^2}$

chain rule "o": $\frac{d}{dt}f(g(t)) = \frac{df}{dt}(g(t))\frac{dg}{dt}(t)$
 \swarrow
 $(f \circ g)(t)$

since order of factors matters with the cross product, if order of factors in expressions involving them is maintained, the vector rules look exactly like the scalar rules.

13.2b derivatives and vectorops / def integrals

(2)

vector derivative rules

Expressing them in component form they easily follow from the corresponding ~~problems~~ rules for the scalar component functions.

sum + : $\frac{d}{dt} (\vec{F}(t) + \vec{G}(t)) = \frac{d\vec{F}}{dt}(t) + \frac{d\vec{G}}{dt}(t)$

constant factor * : $\frac{d}{dt} (c\vec{F}(t)) = c \frac{d\vec{F}}{dt}(t)$, $\frac{d}{dt} (f(t)\vec{C}) = \frac{df(t)}{dt} \vec{C}$

$\frac{d}{dt} (c_1 \vec{F}_1(t) + c_2 \vec{F}_2(t) + \dots)$
 $= c_1 \frac{d\vec{F}_1}{dt}(t) + c_2 \frac{d\vec{F}_2}{dt}(t) + \dots$

2 factor products * : $\frac{d}{dt} (f(t)\vec{F}(t)) = \frac{df(t)}{dt} \vec{F}(t) + f(t) \frac{d\vec{F}}{dt}(t)$

$\frac{d}{dt} (\vec{F}(t) \cdot \vec{G}(t)) = \frac{d\vec{F}}{dt}(t) \cdot \vec{G}(t) + \vec{F}(t) \cdot \frac{d\vec{G}}{dt}(t)$

$\frac{d}{dt} (\vec{F}(t) \times \vec{G}(t)) = \frac{d\vec{F}}{dt}(t) \times \vec{G}(t) + \vec{F}(t) \times \frac{d\vec{G}}{dt}(t)$

3 factor products * : $\frac{d}{dt} (H(t) \cdot (\vec{F}(t) \times \vec{G}(t)))$
 $= \frac{dH}{dt}(t) \cdot (\vec{F}(t) \times \vec{G}(t)) + H(t) \cdot (\frac{d\vec{F}}{dt}(t) \times \vec{G}(t)) + H(t) \cdot (\vec{F}(t) \times \frac{d\vec{G}}{dt}(t))$

$\frac{d}{dt} (H(t) \times (\vec{F}(t) \times \vec{G}(t))) = \frac{dH}{dt}(t) \times (\vec{F}(t) \times \vec{G}(t)) + \dots$ similar

chain rule: o : $\frac{d}{dt} (\vec{F}(f(t))) = \frac{d\vec{F}}{dt}(f(t)) \frac{df(t)}{dt}$

useful for

change of parametrization of a parametrized curve

$\frac{d}{dt} \vec{r}(t(s)) = \vec{r}'(t(s)) \frac{dt(s)}{ds}$

related rate of change

change from t to $t = t(s)$
 trace out same path at different rate

13.2b derivatives and vector ops / def integrals

(3)

quotient rule \div :
$$\frac{d}{dt} \left(\frac{\vec{F}(t)}{f(t)} \right) = \frac{f(t) \frac{d\vec{F}(t)}{dt} - \vec{F}(t) \frac{df(t)}{dt}}{f(t)^2}$$

why useful? $\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ makes evaluating derivative easier.

example twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

$|\vec{r}'(t)| = \sqrt{1+4t^2+9t^4} \rightarrow \hat{T}(t) = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1+4t^2+9t^4}}$

now differentiate:

$$\hat{T}'(t) = \frac{\sqrt{1+4t^2+9t^4} \frac{d}{dt} \langle 1, 2t, 3t^2 \rangle - \langle 1, 2t, 3t^2 \rangle \frac{d}{dt} \sqrt{1+4t^2+9t^4}}{(\sqrt{1+4t^2+9t^4})^2}$$

$$= \frac{\sqrt{1+4t^2+9t^4} \langle 0, 2, 6t \rangle - \langle 1, 2t, 3t^2 \rangle \frac{1}{2} (1+4t^2+9t^4)^{-1/2} (0+8t+36t^3)}{(1+4t^2+9t^4)}$$
 combine fractions!

$$= \frac{(1+4t^2+9t^4) \langle 0, 2, 6t \rangle - (4t+18t^3) \langle 1, 2t, 3t^2 \rangle}{(1+4t^2+9t^4)^{3/2}}$$

$$= \frac{\langle 0 - 4t - 18t^3, 2 + 8t^2 + 18t^4 - 8t^2 - 36t^4, 6t + 24t^3 + 54t^5 - 12t^3 - 54t^5 \rangle}{(1+4t^2+9t^4)^{3/2}}$$

$$= \frac{\langle -4t - 18t^3, 2 - 18t^4, 6t + 12t^3 \rangle}{(1+4t^2+9t^4)^{3/2}} = \frac{2 \langle -t(t+9t^2), 1 - 9t^4, 6t(1+2t^2) \rangle}{(1+4t^2+9t^4)^{3/2}}$$

$|\hat{T}'(t)| = ?$

use Maple!

WHY?
13.3

example parametrized curve: $\vec{r} = \vec{r}(t)$
 tangent vector: $\vec{r}'(t)$

suppose the tangent vector length is constant:

$$|\vec{r}'(t)| = v_0 \text{ (constant)}$$

Then we can apply a vector derivative rule to draw a useful conclusion.

$$0 = \frac{d}{dt} |\vec{r}'(t)|^2 \quad \text{length squared also constant}$$

$$= \frac{d}{dt} (\vec{r}'(t) \cdot \vec{r}'(t))$$

$$= \frac{d\vec{r}'(t)}{dt} \cdot \vec{r}'(t) + \vec{r}'(t) \cdot \frac{d\vec{r}'(t)}{dt}$$

$$= \vec{r}''(t) \cdot \vec{r}'(t) + \vec{r}'(t) \cdot \vec{r}''(t)$$

symbol definition

$$= 2 \vec{r}'(t) \cdot \vec{r}''(t)$$

order doesn't matter

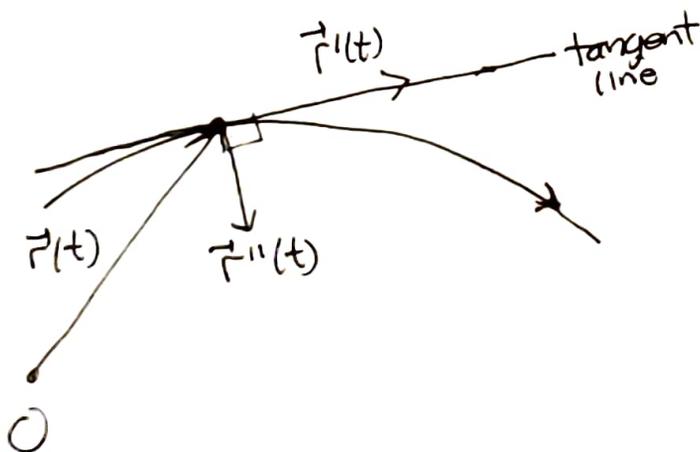
$$\vec{r}'(t) \cdot \vec{r}''(t) = 0$$

$$\vec{v}(t) \cdot \vec{a}(t) = 0 \quad \text{physics terminology}$$

The first and second derivatives are orthogonal

"The acceleration is orthogonal to the velocity

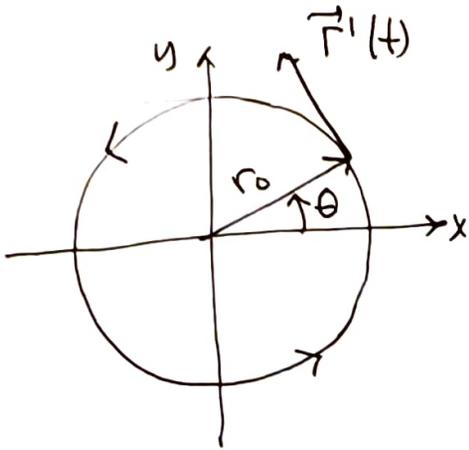
if the speed is constant."



13.2b) derivatives and vector ops / def. integrals

(5)

Example: circular motion at frequency ω : $\theta = \omega t$ \leftarrow sec
rad



$$\vec{r}(t) = \langle r_0 \cos \omega t, r_0 \sin \omega t \rangle$$

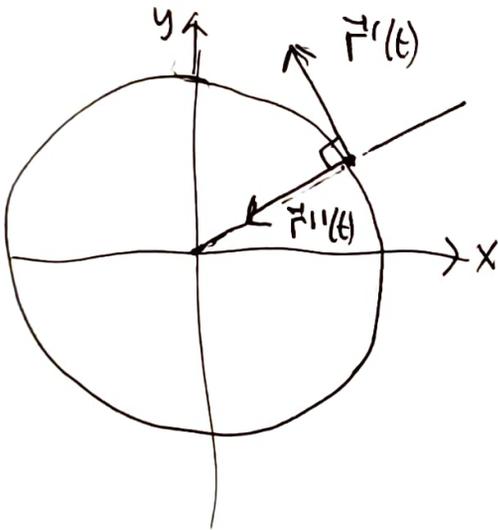
$$\vec{r}'(t) = \langle -\omega r_0 \sin \omega t, \omega r_0 \cos \omega t \rangle$$

$$\vec{r}''(t) = \langle -\omega^2 r_0 \cos \omega t, -\omega^2 r_0 \sin \omega t \rangle$$

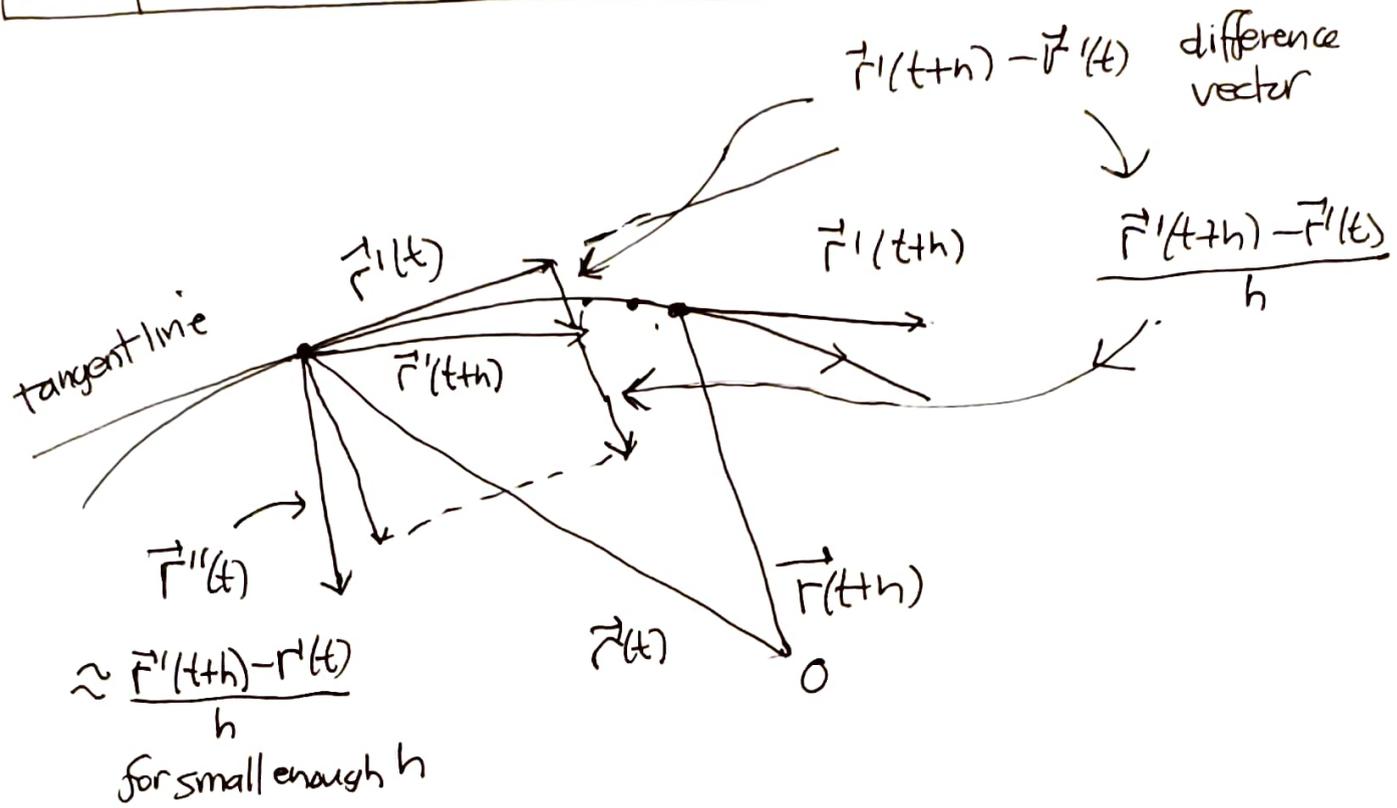
$$= -\omega^2 \langle r_0 \cos \omega t, r_0 \sin \omega t \rangle$$

$$= -\omega^2 \vec{r}(t)$$

↑
points radially inward
orthogonal to tangential direction



Interpretation of second derivative



To compare tangent vectors at successive points,
 must have same initial points
 same applies to comparing rescaled difference vector
 with second derivative.

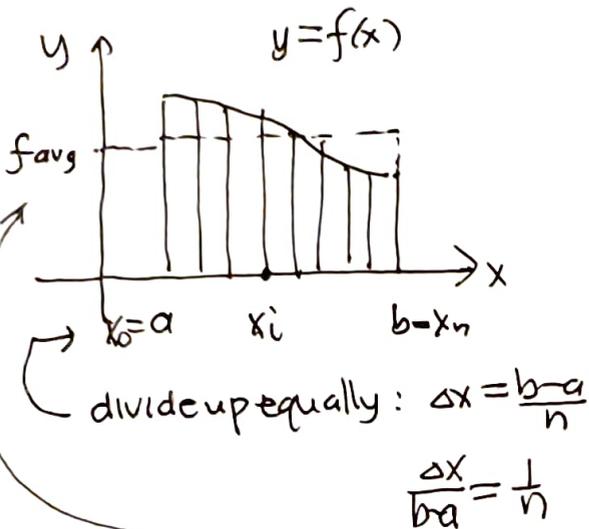
$\vec{a}(t) = \vec{r}''(t)$ lies on the same side of the tangent line
 in which the curve is bending (proportional)
 when $\vec{r}''(t)$ and $\vec{r}'(t) = \vec{v}(t)$ are not ~~collinear~~ parallel
 they determine a plane through the tip of $\vec{r}'(t)$ containing
 the tangent line, $\vec{r}'(t)$ and $\vec{r}''(t)$
 "the velocity-acceleration" plane or
 the plane of the "instantaneous motion"
 (later)

13.2b) derivatives and vector ops / def integration

(7)

Can we visualize vector integration? (definite integral!)

calc 2 scalar case



$$f_{avg} = \frac{\int_a^b f(x) dx}{b-a} \quad \text{average value}$$

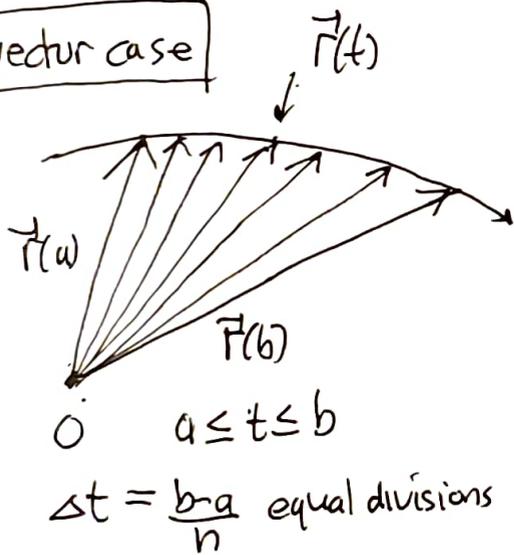
$$= \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n f(x_i) \Delta x}{b-a}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{\sum_{i=1}^n f(x_i)}{n} \right)$$

avg of sampled values

so $\int_a^b f(x) dx = (b-a) f_{avg}$
 rectangle, same area as under curve

vector case



parametrized curve segment: $\vec{r}(t) = \vec{F}(t)$

$$\vec{F}_{avg} = \frac{1}{b-a} \int_a^b \vec{F}(t) dt$$

$$= \frac{1}{b-a} \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \vec{F}(t_i) \Delta t \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{\sum_{i=1}^n \vec{F}(t_i)}{n} \right)$$

average of sampled vector values

so \vec{F}_{avg} "averages out" all the vectors along this curve.

[see Maple]

BUT This kind of vector integral not so useful!