

13.2a) Calc ops on vector-valued functions

(1)

We apply calc 1 operations simultaneously to all components of a vector-valued function.

Example. $\vec{F}(t) = \langle t, t^2, t^3 \rangle$

$$\lim_{t \rightarrow 2} \vec{F}(t) = \lim_{t \rightarrow 2} \langle t, t^2, t^3 \rangle = \left\langle \lim_{t \rightarrow 2} t, \lim_{t \rightarrow 2} t^2, \lim_{t \rightarrow 2} t^3 \right\rangle \\ = \langle 2, 2^2, 2^3 \rangle$$

$$\frac{d}{dt} \vec{F}(t) = \vec{F}'(t) = \frac{d}{dt} \langle t, t^2, t^3 \rangle = \left\langle \frac{d}{dt} t, \frac{d}{dt} t^2, \frac{d}{dt} t^3 \right\rangle \\ = \langle 1, 2t, 3t^2 \rangle$$

$$\int \vec{F}(t) dt = \int \langle t, t^2, t^3 \rangle dt = \left\langle \int t dt, \int t^2 dt, \int t^3 dt \right\rangle \\ = \left\langle \frac{t^2}{2} + C_1, \frac{t^3}{3} + C_2, \frac{t^4}{4} + C_3 \right\rangle$$

$$\int_0^1 \vec{F}(t) dt = \int_0^1 \langle t, t^2, t^3 \rangle dt = \left\langle \int_0^1 t dt, \int_0^1 t^2 dt, \int_0^1 t^3 dt \right\rangle \\ = \left\langle \frac{t^2}{2} \Big|_0^1, \frac{t^3}{3} \Big|_0^1, \frac{t^4}{4} \Big|_0^1 \right\rangle = \left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right\rangle$$

higher derivatives:

$$\vec{F}''(t) = \frac{d}{dt} \vec{F}'(t) = \frac{d}{dt} \langle 1, 2t, 3t^2 \rangle = \dots = \langle 0, 2, 6t \rangle \\ = 2 \langle 0, 1, 3t \rangle$$

B3.2a

calc ops on vector-valued functions

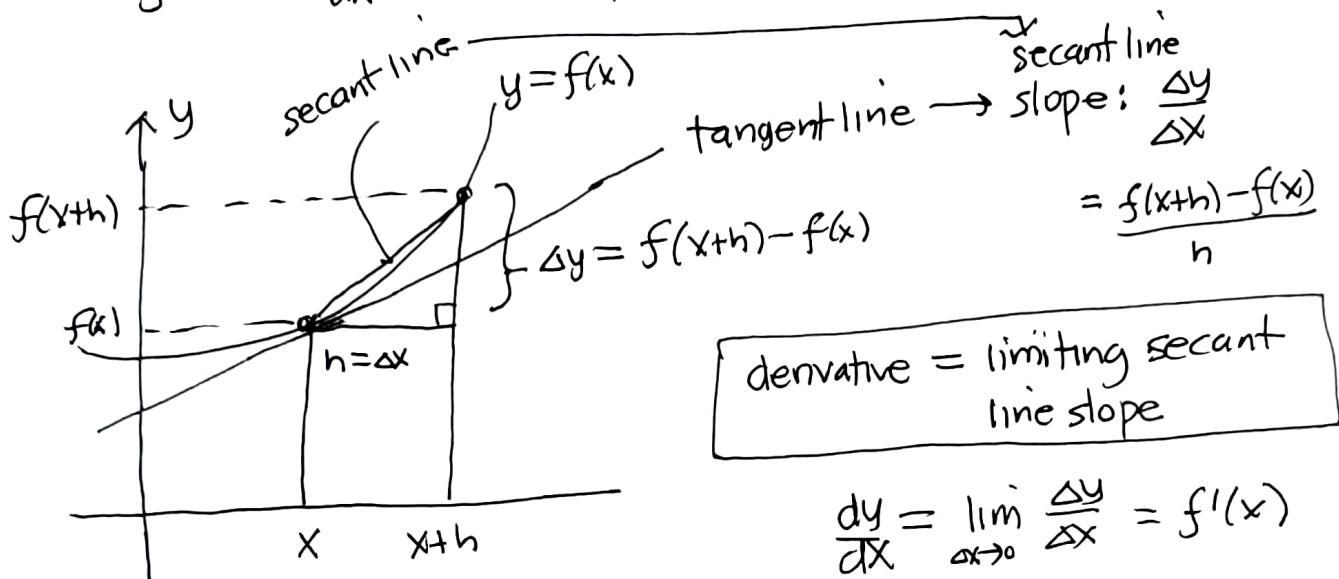
(2)

Derivatives

we have a component definition of the derivative $\vec{F}'(t)$
 but what is the geometrical meaning? What would the
 equivalent geometrical definition be?

Calc 1 g

$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$



derivative = limiting secant line slope

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(x)$$

defines tangent line:

At $x=a$:

$$\underbrace{y - f(a)}_{\text{rise}} = \underbrace{f'(a)}_{\text{slope}} \underbrace{(x-a)}_{\text{run}}$$

$$y = f(a) + f'(a)(x-a) \quad \text{eqn of tan line}$$

The tangent line is a concrete visualization of the slope, which is just a number.

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The limit definition of the vector derivative gives a geometric interpretation

The limit definition is:

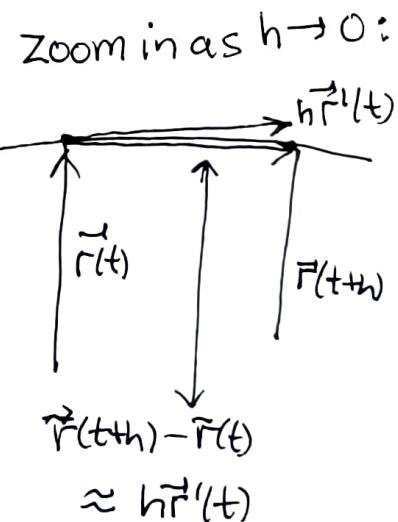
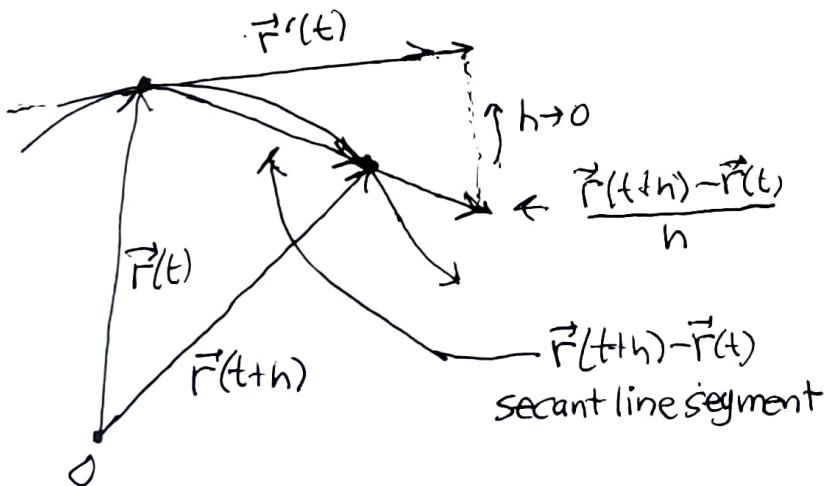
$$\frac{d}{dt} \vec{F}(t) = \lim_{h \rightarrow 0} \underbrace{\frac{\vec{F}(t+h) - \vec{F}(t)}{h}}_{\langle \vec{F}_1(t+h), \vec{F}_2(t+h), \vec{F}_3(t+h) \rangle - \langle \vec{F}_1(t), \vec{F}_2(t), \vec{F}_3(t) \rangle} \quad \text{"geometric definition"}$$

$$= \lim_{h \rightarrow 0} \frac{\langle \vec{F}_1(t+h) - \vec{F}_1(t), \vec{F}_2(t+h) - \vec{F}_2(t), \vec{F}_3(t+h) - \vec{F}_3(t) \rangle}{h}$$

$$= \left\langle \lim_{h \rightarrow 0} \frac{\vec{F}_1(t+h) - \vec{F}_1(t)}{h}, \lim_{h \rightarrow 0} \frac{\vec{F}_2(t+h) - \vec{F}_2(t)}{h}, \lim_{h \rightarrow 0} \frac{\vec{F}_3(t+h) - \vec{F}_3(t)}{h} \right\rangle$$

$$= \left\langle \vec{F}'_1(t), \vec{F}'_2(t), \vec{F}'_3(t) \right\rangle = \vec{F}'(t) \quad \text{component definition}$$

The visualization of this is best in the context of a parametrized curve.



since $\frac{\vec{r}(t+h) - \vec{r}(t)}{h} \approx \vec{r}'(t)$
for small enough h

This is just the linear approximation

$$\vec{r}(t+h) \approx \vec{r}(t) + h \vec{r}'(t)$$

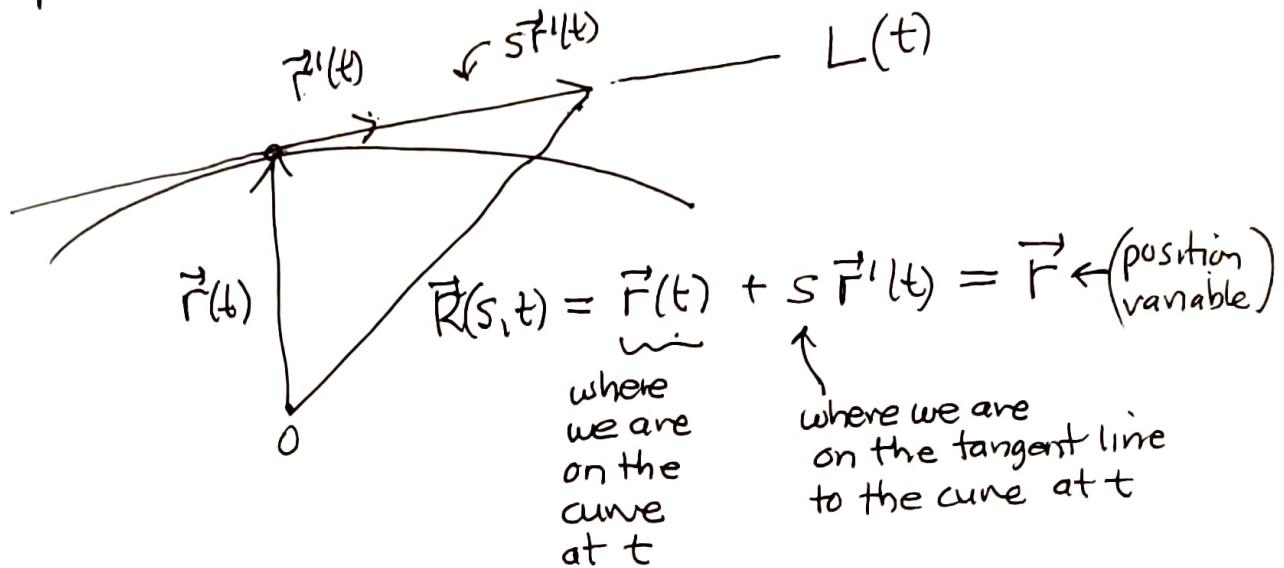
where $h = \Delta t$

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④

The tangent vector attached to the position vector and the tangent line which contains it are the visual representation of the value of the derivative.

example: twisted cubic

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

$$\vec{r}'(1) = \langle 1, 1, 1 \rangle$$

$$\vec{r}'(1) = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{r}'(1) = \langle 1, 2, 3 \rangle \rightarrow |\vec{r}'(1)| = \sqrt{1+4+9} = \sqrt{14}$$

$$\text{tangent line: } \vec{r} = \vec{r}(1) + s \vec{r}'(1)$$

$$\begin{aligned} &= \langle 1, 1, 1 \rangle + s \langle 1, 2, 3 \rangle \\ &= \langle 1+s, 1+2s, 1+3s \rangle \end{aligned}$$

$$\text{or } x = 1+s, y = 1+2s, z = 1+3s$$

The tangent vector has geometrical properties:

unit tangent:

$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

is defined when $\vec{r}'(t) \neq \vec{0}$.

$$\text{so } \vec{r}'(t) = |\vec{r}'(t)| \hat{T}(t)$$

length direction

$$\text{example: twisted cubic: } |\vec{r}'(t)| = \sqrt{1+4t^2+9t^4}, \hat{T}(t) = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1+4t^2+9t^4}}$$

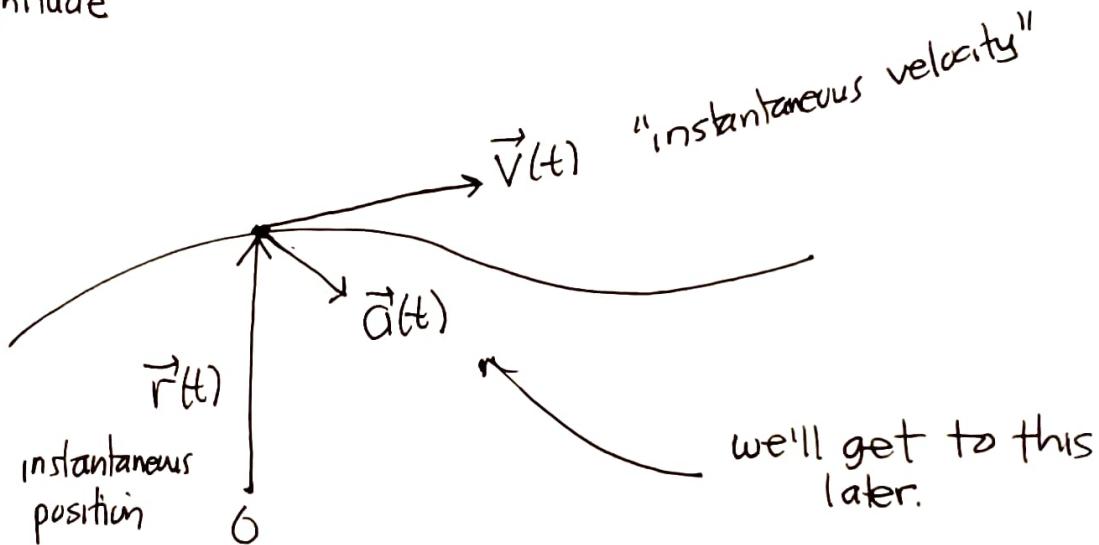
$$\hat{T}(1) = \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}}$$

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physics terminology for motion in space

position	$\vec{r}(t)$
velocity	$\vec{r}'(t) = \vec{v}(t)$
speed	$ \vec{r}'(t) = v(t)$
direction (of motion)	$\hat{T}(t) = \hat{v}(t)$
acceleration	$\vec{r}''(t) = \vec{v}'(t) = \vec{a}(t)$
acceleration magnitude	$ \vec{r}''(t) = a(t)$



example if we know $\vec{V}(t)$ for all t , and the initial position $\vec{R}(0)$ we can integrate to get $\vec{R}(t)$:

$$\begin{aligned}
 \vec{r}'(t) &= \langle \sin t, \cos t, 2t \rangle \longrightarrow \vec{r}(t) = \int \frac{d\vec{r}(t)}{dt} dt = \int \langle \sin t, \cos t, 2t \rangle dt \\
 \vec{r}(0) &= \langle 1, 1, 2 \rangle \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad = \langle -\cos t + c_1, \sin t + c_2, t^2 + c_3 \rangle \\
 &\quad \longrightarrow \vec{r}(0) = \langle c_1 - 1, c_2 + 0, c_3 + 0 \rangle = \langle 1, 1, 2 \rangle \\
 c_1 - 1 &= 1, \quad c_2 = 1, \quad c_3 = 2 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{ backsub:} \\
 \downarrow c_1 &= 2 \\
 \vec{r}(t) &= \langle 2 - \cos t, \sin t, t^2 + 2 \rangle
 \end{aligned}$$

B.2a

calc qps on vector-valued functions

⑥

vector ops?

example: $\vec{r}(t) = \langle 2 - \cos t, 1 + \sin t, 2 + t^2 \rangle$

what angle does $\vec{r}'(\pi/2)$ make with $\vec{r}(\pi/2)$?

what angle does $\vec{r}''(\pi/2)$ make with $\vec{r}'(\pi/2)$?

$$\vec{r}'(t) = \langle \sin t, \cos t, 2t \rangle$$

$$\vec{r}''(t) = \langle \cos t, -\sin t, 2 \rangle$$

$$\vec{r}'(\pi/2) = \langle 1, 0, \pi \rangle$$

$$\vec{r}''(\pi/2) = \langle 0, -1, 2 \rangle$$

$$\vec{r}(\pi/2) = \langle 2, 2, 2 + \pi^2/4 \rangle$$

$$\hat{\vec{r}}(\pi/2) = \frac{\langle 2, 2, 2 + \pi^2/4 \rangle}{\sqrt{4 + 4 + (2 + \pi^2/4)}}$$

$$\hat{\vec{r}''}(\pi/2) = \frac{\langle 0, -1, 2 \rangle}{\sqrt{1 + 4}}$$

$$\hat{\vec{r}''}(\pi/2) = \hat{\vec{r}}(\pi/2) = \frac{\langle 1, 0, \pi \rangle}{\sqrt{1 + \pi^2}}$$

$$\theta = \arccos(\hat{a} \cdot \hat{b})$$

$$* \frac{180^\circ}{\pi} \approx \text{interpret}$$

angle 23.1° angle 31.5°

Just put formulas into Maple!