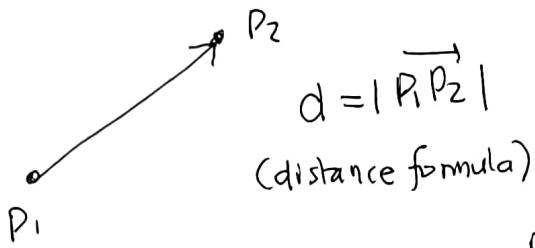
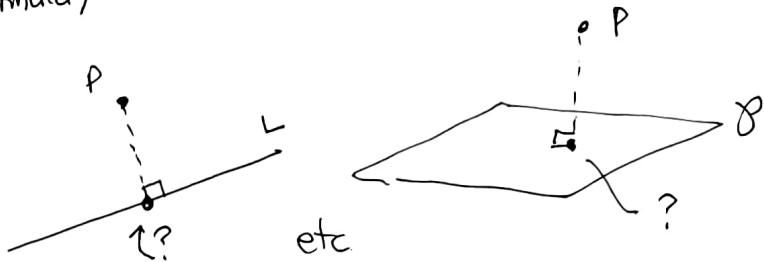


12.5b distances between pairs of points, lines, planes in space ①

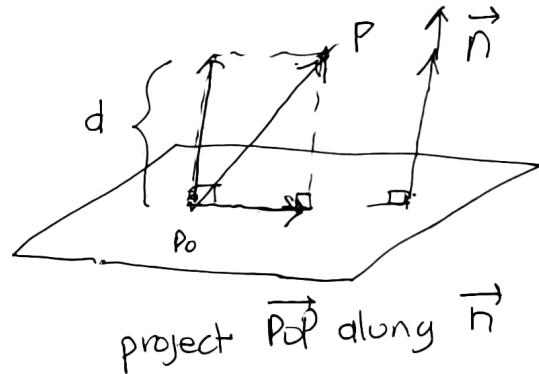
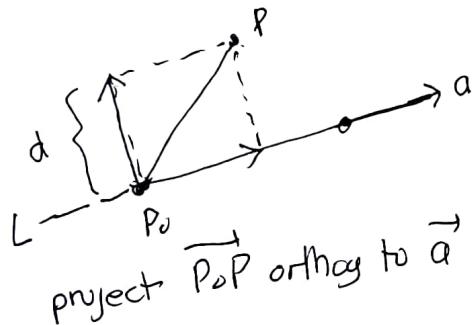
The distance between sets of points in space (points, lines, planes, curves, surfaces) is the length of the separation vector between their nearest points:



The problem is that we don't know in advance which are the nearest points:



But we can take any separation vector and project it orthogonal to a line or a plane, hence orthogonal to the direction of the line or along the normal to a plane to get the nearest point distance.



We can consider any combination of two objects taken from the set of points, lines and planes and determine their separation (at their closest points assuming they do not intersect).

These exercises are NOT terribly interesting in themselves, but provide valuable practice in applying the projection process in a variety of contexts.

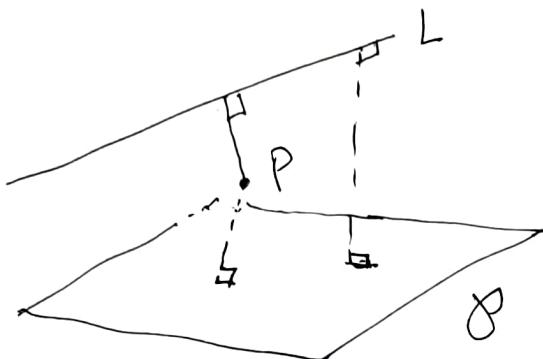
Plugging into some formula serves NO PURPOSE!

A basic understanding of the projection process is all we need, plus common sense in applying it in each of the following scenarios!

12.5b distances between pts, lines, planes in space

(2)

[pairings] of points, lines, planes

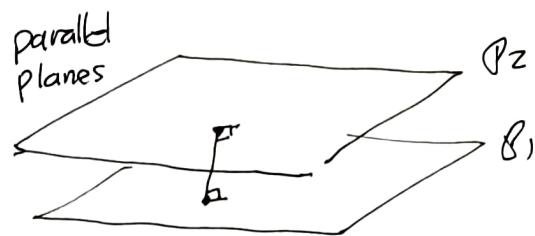


$$P - L$$

$$P - \gamma$$

$$L - \gamma \text{ (if parallel)}$$

$$\vec{a} \cdot \vec{n} = 0$$

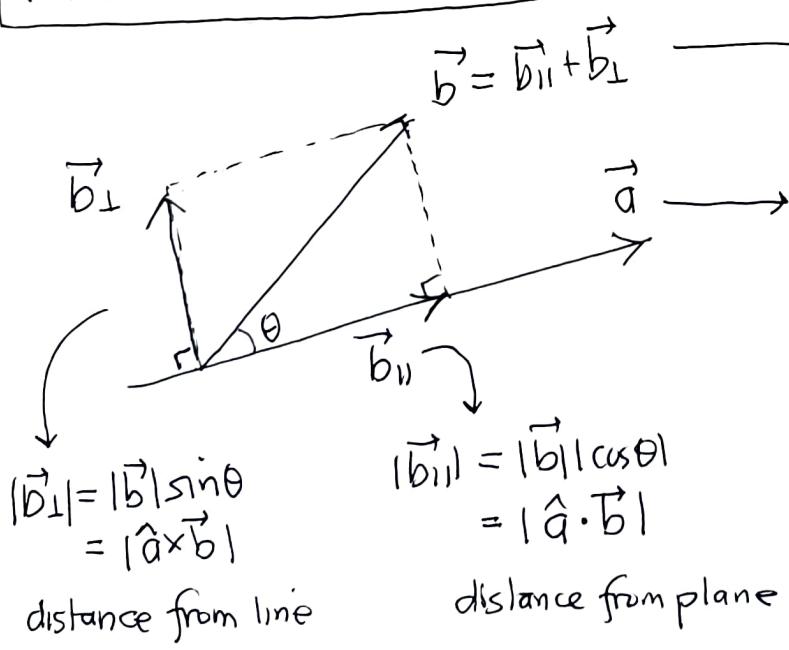


$$\gamma_1 - \gamma_2$$

$$L_1 - L_2$$

In each case we introduce [any] separation vector between points in the two objects, and project it perpendicularly to the objects (perp to line direction, parallel to plane normal).

The orthogonal projection process to resolve a vector



$$|\vec{b}_{\perp}| = |\vec{b}| \sin \theta$$

$$= |\hat{\alpha} \times \vec{b}|$$

distance from line

$$|\vec{b}_{\parallel}| = |\vec{b}| \cos \theta$$

$$= |\hat{\alpha} \cdot \vec{b}|$$

distance from plane

$$\left\{ \begin{array}{l} \vec{P}_1 \vec{P}_2 \text{ separation vector:} \\ d = |\vec{P}_1 \vec{P}_2| \text{ or } |\vec{P}_1 \vec{P}_2| \sin \theta \end{array} \right.$$

$$\left\{ \begin{array}{l} \vec{a} \text{ for } L \\ \vec{n} \text{ for } \gamma \end{array} \right.$$

To get a pt on a parametrized line, pick the parameter = 0.

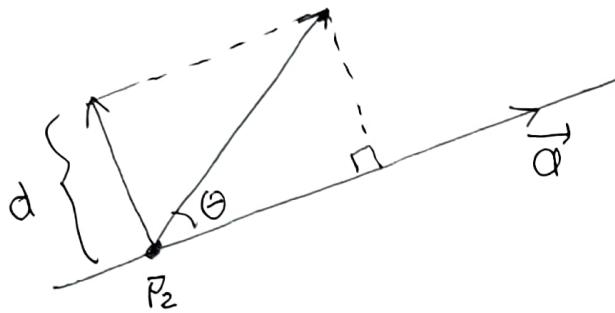
To get a point on a plane, choose the z-intercept
(set x=y=0)

[unless parallel to z-axis]
choose another axis.

12.5b) distances between points, lines, planes

(3)

example: point & line



$$d = |\hat{a}| |\vec{P_2P_1}| \sin \theta$$

$$= |\hat{a} \times \vec{P_2P_1}|$$

$$\vec{r}_1 = <4, -1, 2>$$

$$\vec{a} = <1, -2, -3>$$

$$L: \vec{r} = <1+t, 3-2t, 4+3t>$$

$$\hookrightarrow t=0: \vec{r}_2 = <1, 3, 4>$$

$$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$$

$$\hat{a} = \frac{1}{\sqrt{14}} <1, -2, -3>$$

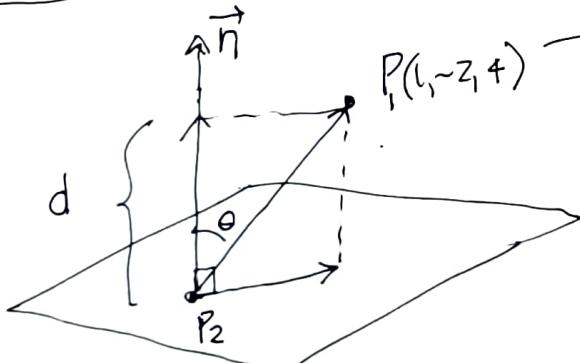
$$\begin{aligned} \vec{r}_2 - \vec{r}_1 &= <1, 3, 4> - <4, -1, 2> \\ &= <-3, 4, 2> \end{aligned}$$

$$\hat{a} \times (\vec{r}_2 - \vec{r}_1) = \frac{1}{\sqrt{14}} <1, -2, -3> \times <-3, 4, 2>$$

$$= \frac{1}{\sqrt{14}} <6, -3, 4>$$

$$d = |\hat{a} \times (\vec{r}_2 - \vec{r}_1)| = \frac{1}{\sqrt{14}} \sqrt{36+9+16} = \sqrt{\frac{61}{14}}$$

example: point & plane



$$d = |\hat{n}| |\vec{P_2P_1}| \cos \theta$$

$$= |\hat{n} \cdot \vec{P_2P_1}|$$

$$\vec{r}_1 = <1, -2, 4>$$

$$\vec{n} = <3, 2, 6>$$

$$\text{Plane: } 3x + 2y + 6z = 5$$

$$x=y=0 \rightarrow z=\frac{5}{6}: \vec{r}_2 = <0, 0, \frac{5}{6}>$$

$$\begin{aligned} \vec{r}_2 - \vec{r}_1 &= <0, 0, \frac{5}{6}> - <1, -2, 4> \\ &= <-1, 2, -\frac{19}{6}> \end{aligned}$$

$$|\vec{n}| = \sqrt{9+4+36} = \sqrt{49} = 7$$

$$\hat{n} = \frac{1}{7} <3, 2, 6>$$

$$\begin{aligned} \hat{n} \cdot (\vec{r}_2 - \vec{r}_1) &= \frac{1}{7} <3, 2, 6> \cdot \frac{1}{6} <-1, 2, -\frac{19}{6}> \\ &= \frac{1}{42} (-18 + 24 - 114) = -\frac{108}{42} = -\frac{18}{7} \end{aligned}$$

$$d = |\hat{n} \cdot (\vec{r}_2 - \vec{r}_1)| = \frac{18}{7}$$

12.5b

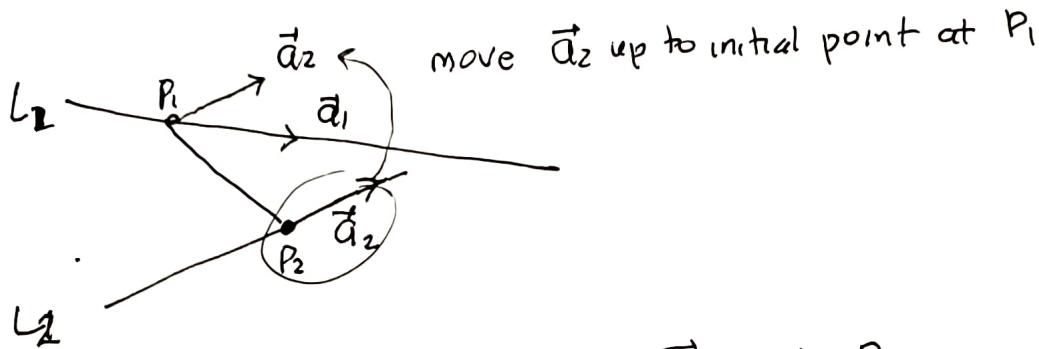
distances between points, lines, planes

4a

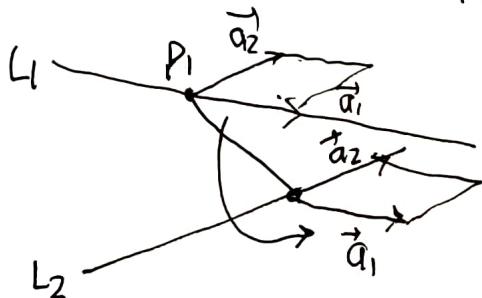
a bit more on "skew" lines.
 [skew referring to lines or directions means "not parallel"]

Given 2 such skew lines

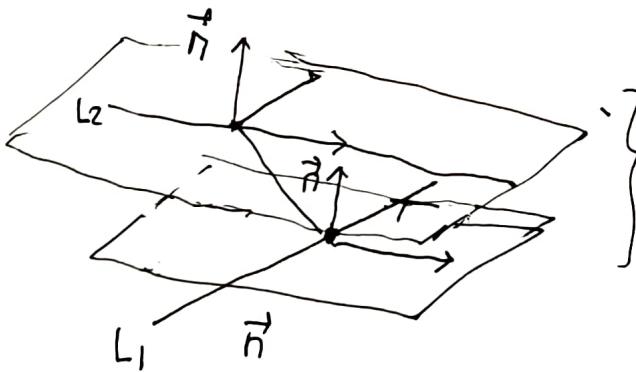
$$\left. \begin{array}{l} L_1: \vec{r} = \vec{r}_1 + t \vec{a}_1 \\ L_2: \vec{r} = \vec{r}_2 + s \vec{a}_2 \end{array} \right\} \text{ where } \vec{a}_1 \times \vec{a}_2 \neq \vec{0} \text{ (not parallel)}$$



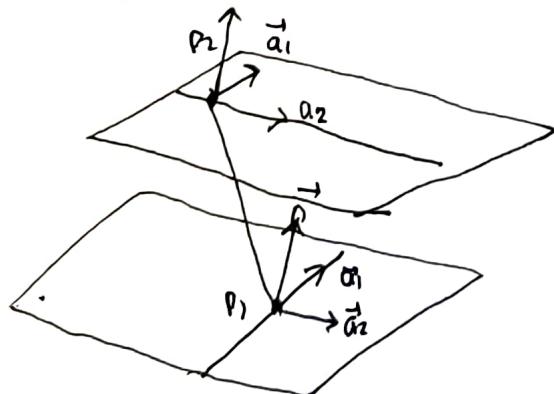
then bring \vec{a}_1 down to P_2



now we have $\vec{a}_1 \times \vec{a}_2 = \vec{n}$
 for the orientation of the pair of parallel planes that contain the pair of lines.



bad view since upper plane obscures lower plane.
 but \vec{n} at P_1 and P_2 is clear.



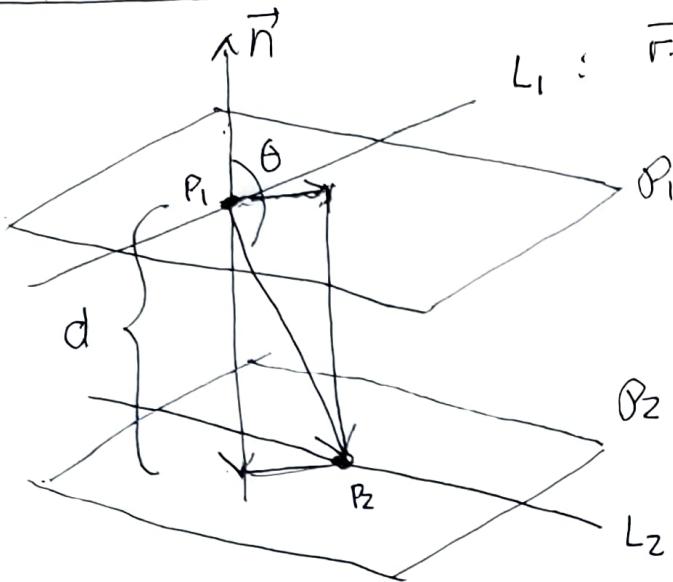
so we try again, separating the two planes.

All plotting with technology requires thinking about the view and often some trial and error.

12.5b) distances between points, lines, planes

(4)

Example: skew lines



$$L_1: \vec{r} = \langle 1+t, 1+6t, 2t \rangle$$

$$\downarrow \vec{r}_1 = \langle 1, 1, 0 \rangle$$

$$\downarrow \vec{q}_1 = \langle 1, 6, 2 \rangle$$

$$L_2: \vec{r} = \langle 1+2s, 5+15s, -2+6s \rangle$$

$$\downarrow \vec{r}_2 = \langle 1, 5, -2 \rangle$$

$$\downarrow \vec{q}_2 = \langle 2, 15, 6 \rangle$$

$$d = |\hat{n}| \|(\vec{P}_1\vec{P}_2)|(\cos\theta)$$

$$= |\hat{n} \cdot \vec{P}_1\vec{P}_2|$$

The two planes are:

$$Q_1: \vec{n} \cdot (\vec{r} - \vec{r}_1) = 0$$

Maple:

$$Q_1: 6x - 2y + 3z = 4$$

$$Q_2: 6x - 2y + 3z = -10$$

Example: parallel planes

Find z-intercepts

$$\vec{r}_3 = \langle 0, 0, 4/3 \rangle$$

$$\vec{r}_4 = \langle 0, 0, -10/3 \rangle$$

$$\vec{r}_3 - \vec{r}_4 = \langle 0, 0, 14/3 \rangle$$

$$\vec{n} = \langle 6, -2, 3 \rangle \quad |\vec{n}| = \dots = 7 \quad \hat{n} = \frac{1}{7} \langle 6, -2, 3 \rangle$$

$$\hat{n} \cdot (\vec{r}_3 - \vec{r}_4) = \frac{1}{7} \langle 6, -2, 3 \rangle \cdot \langle 0, 0, 14/3 \rangle = \frac{14}{7} = 2 \checkmark$$

$$\vec{a}_1 \times \vec{a}_2 = \langle 1, 6, 2 \rangle \times \langle 2, 15, 6 \rangle$$

$$= \langle 6, -2, 3 \rangle = \vec{n}$$

Maple

$$|\vec{n}| = \sqrt{36+4+9} = \sqrt{49} = 7$$

$$\hat{n} = \frac{1}{7} \langle 6, -2, 3 \rangle$$

$$\vec{r}_1 - \vec{r}_2 = \langle 1, 1, 0 \rangle = \langle 0, -4, 2 \rangle$$

$$- \langle 1, 5, -2 \rangle$$

$$\hat{n} \cdot (\vec{r}_1 - \vec{r}_2) = \frac{1}{7} \langle 6, -2, 3 \rangle \cdot \langle 0, -4, 2 \rangle$$

$$= \frac{1}{7} (0 + 8 + 6) = \frac{14}{7} = 2$$

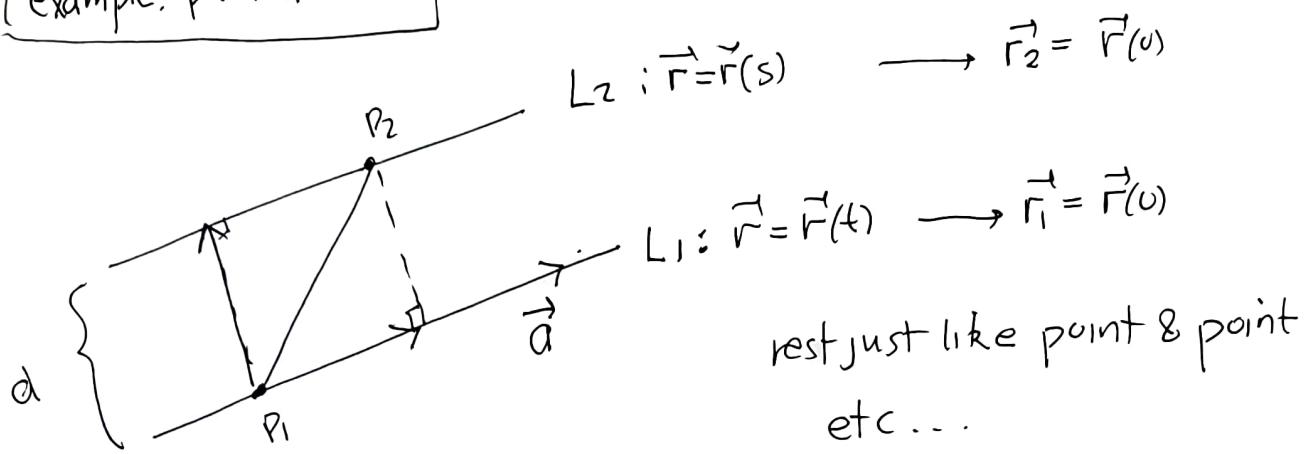
$$= d$$

12.5b

distances between points, lines, planes

(5)

example: parallel lines



Recipe

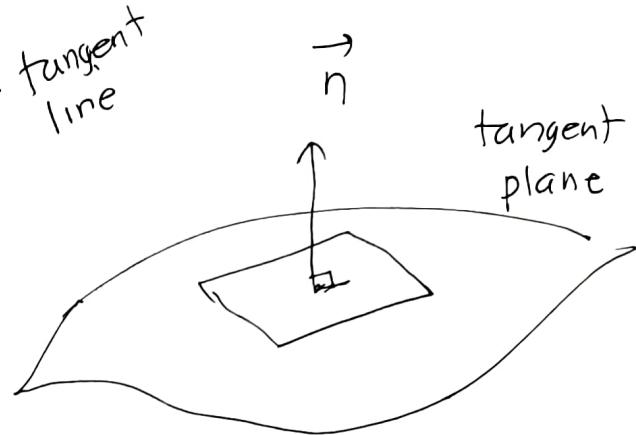
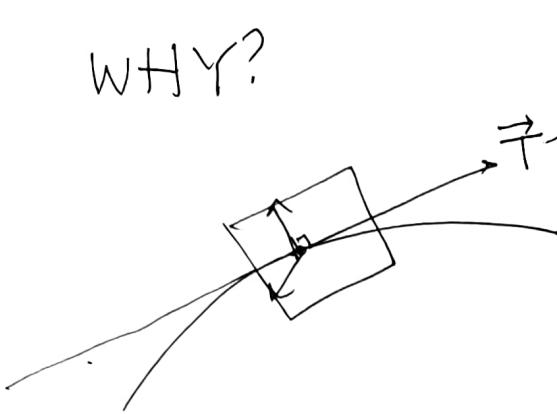
- Draw a generic diagram.
- Locate a point in each object.
- Draw in separation vector.
- Complete to a projection rectangle.
- Evaluate appropriate scalar projection.

12.5b

distances between pts, lines, planes in space

(6)

WHY?



CHAPTER 13

geometry of curves

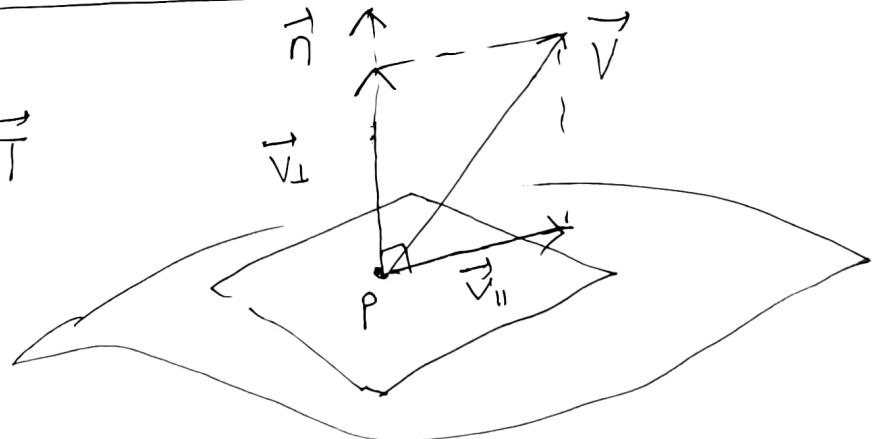
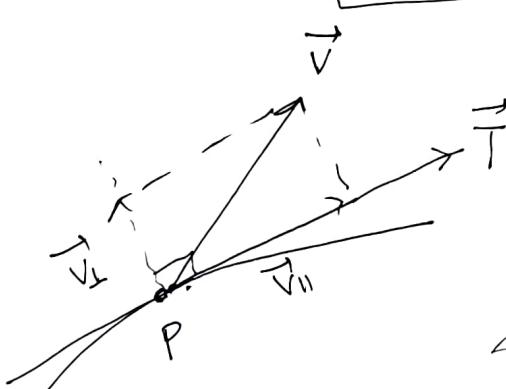
how does a curve
"curve" and "twist"
as one moves along it?

CHAPTER 14

calculus of surfaces & surface graphs

CHAPTER 16

vector fields and curves and surfaces



vector integration
& vector differentiation

CHAPTER 15
is multivariable scalar integration
so no vectors needed

12.5b

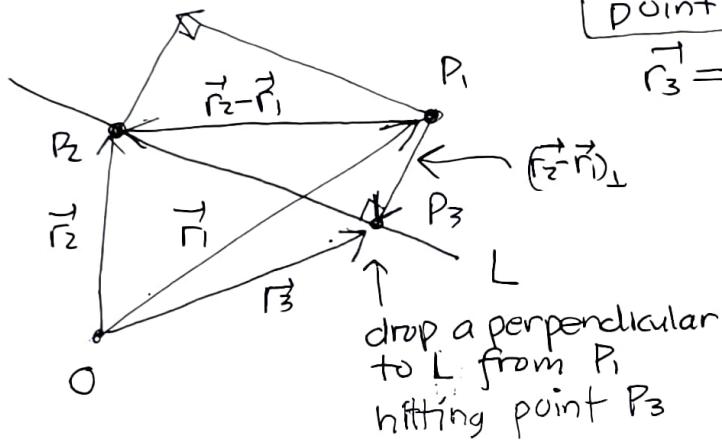
distances between pts, lines, planes

you can **SAFELY IGNORE** this page.

Most of us go into STEM fields (not all of us) because we like math and math can easily feed our curiosity. Those of us who are curious wonder how we can push on a problem harder to understand more aspects of it.

For example "Which points are closest?"

If one of the two endpoints of a shortest connecting line segment is fixed, the vector components of the nonorthogonal connecting separation vector help us move from the second endpoint to the new endpoint that does the trick.



Point P_1 , line L :

$$\vec{r}_3 = \vec{r}_1 + (\vec{r}_2 - \vec{r}_1)_\perp$$

This also works for cases of parallel objects if you fix the original point, since there are an infinite number of connecting shortest line segments.

For skew lines, neither point we start with will be on the unique shortest separation vector (in general).

Instead we must actually minimize the distance between 2 arbitrary points on the two lines (distance squared is easier):

? $\vec{r} = \vec{r}_1(t)$ $D(t,s) = (\vec{r}_1(t) - \vec{r}_2(s)) \cdot (\vec{r}_1(t) - \vec{r}_2(s))$

? $\vec{r} = \vec{r}_2(s)$

clearly it should be a minimum in both variables simultaneously, so we can find the unique soln by setting the two partial derivatives to zero.

This is chapter 14 material.