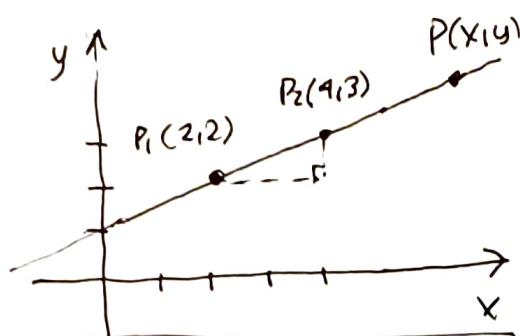


12.5a

Lines and planes in space

①

Before calc what did we know about lines in the plane?



2 points determine a line

$$\text{slope } m = \frac{\Delta y}{\Delta x} = \frac{3-2}{4-2} = \frac{1}{2}$$

$$= \frac{y-2}{x-2}$$

$$\text{so } y-2 = \frac{1}{2}(x-2) \quad (\text{pt-slope eqn})$$

solve:

$$y = 2 + \frac{1}{2}(x-2)$$

$$= \frac{1}{2}x + 2 - 1 = \frac{1}{2}x + 1 \quad (\text{slope-intercept eqn})$$

$$\left. \begin{array}{l} \downarrow \text{or} \\ 2y = x+2 \\ x = 2y-2 \quad \text{or} \quad x - 2y = -2 \end{array} \right\}$$

parametrize this line choosing x or y as the parameter t :

$$1) \ x = t \rightarrow y = \frac{1}{2}t + 1 \rightarrow \langle x, y \rangle = \langle t, \frac{1}{2}t + 1 \rangle = \vec{r}$$

$$2) \ y = s \rightarrow x = 2s - 2 \rightarrow \langle x, y \rangle = \langle 2s - 2, s \rangle = \vec{r}'$$

Two different parametrizations of same line.

How do we know they agree?

set equal: $x = 2s - 2 = t \rightarrow \text{solve } s = \frac{1}{2}(t+2) = \frac{1}{2}t + 1 \checkmark$ consistent!

$y = s = \frac{1}{2}t + 1 \leftarrow$ (otherwise not same line) ↓

either eqn gives linear relationship between two parametrizations.
(and its inverse)

Aside

Finally: $x - 2y + 2 = 0 \leftarrow$ single linear condition of $\langle x, y \rangle$
is a line in \mathbb{R}^2 but a
plane in \mathbb{R}^3 or hyperplane in \mathbb{R}^n
($n > 3$).

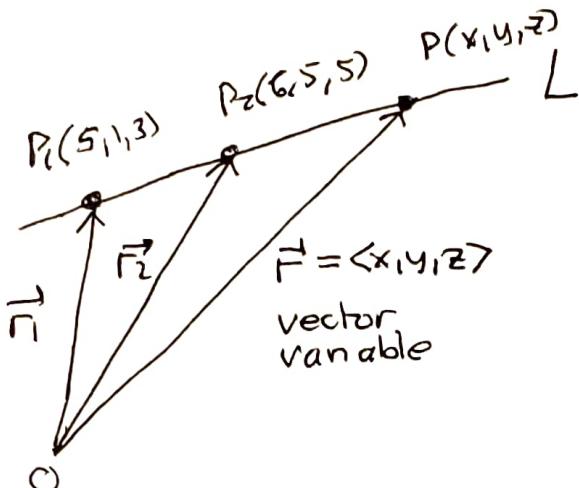
This is a key difference between 2 and 3 dimensions.

Graphs of functions are curves in \mathbb{R}^2 but surfaces
(hypersurfaces) in \mathbb{R}^n , $n > 2$.Multivariable calc in space gives us intuition for $n > 3$!

12.5a lines and planes in space

(2)

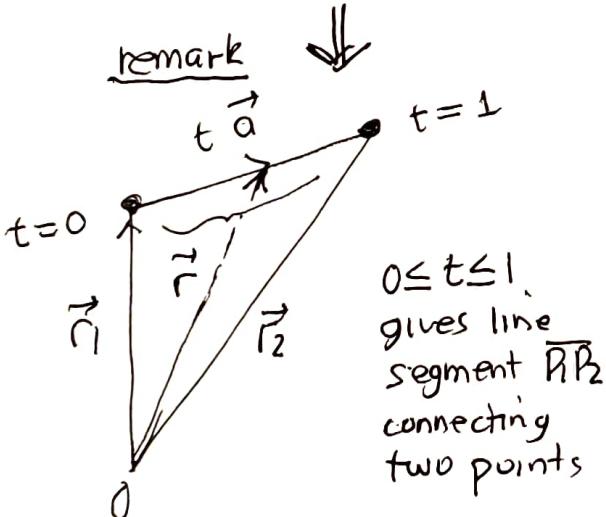
straight line in space



2 points determine a line

"parametrized
eqn(s) for the
line!"

(vector form most useful)



one point and any nonzero vector determine a line

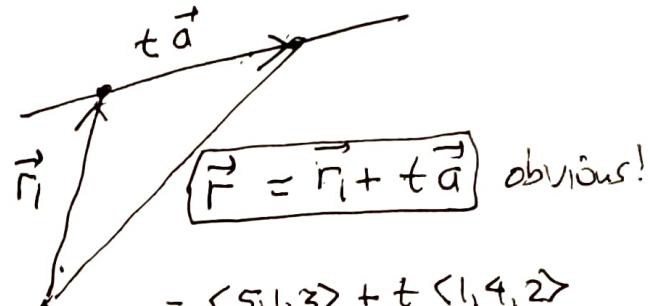
Line parallel to previous line but thru pt (1,2,3) ?

$$\langle x_1, y_1, z_1 \rangle = \vec{r} = \vec{r}_1 + t \vec{a} = \langle 1, 2, 3 \rangle + t \langle 1, 4, 2 \rangle = \boxed{\langle 1+t, 2+4t, 3+2t \rangle} \\ = \langle x_1, y_1, z_1 \rangle$$

instead of slope → "direction"

$$\vec{P_1 P_2} = \vec{r}_2 - \vec{r}_1 = \langle 6, 5, 5 \rangle - \langle 5, 1, 3 \rangle \\ = \langle 6-5, 5-1, 5-3 \rangle \\ = \langle 1, 4, 2 \rangle \equiv \vec{a}$$

pick either point, say P_1 :



$$= \langle 5, 1, 3 \rangle + t \langle 1, 4, 2 \rangle \\ \text{vector eqn: } \boxed{= \langle 5+t, 1+4t, 3+2t \rangle = \langle x_1, y_1, z_1 \rangle}$$

or scalar eqns:

$$\boxed{x = 5+t, y = 1+4t, z = 3+2t}$$

choosing P_2 would give a different parametrization

$$\vec{r} = \vec{r}_2 + s \vec{a} \quad \text{etc.}$$

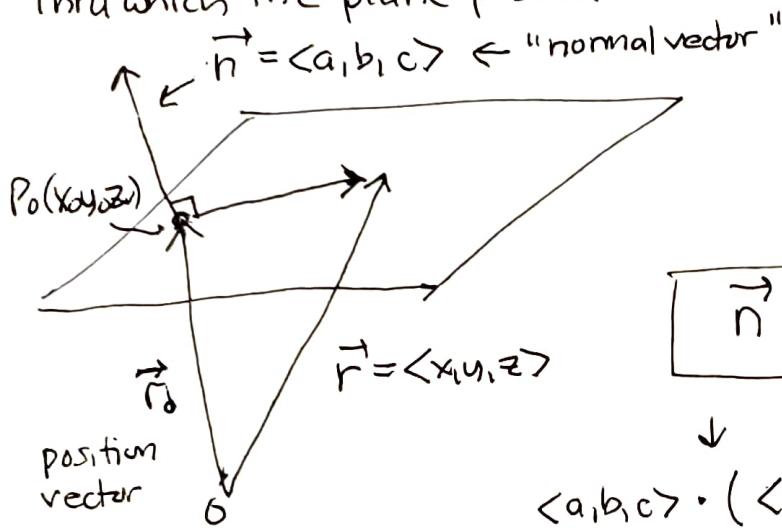
(set different params equal to see if they determine the same line)
(also length of \vec{a} doesn't matter, change length just changes the parametrization)

15.5a lines and planes in space

(3)

planes in space

are specified by a single linear condition $ax+by+cz=d$ on the points (x,y,z) . The "direction" or better "orientation" of a plane is most efficiently described by any "normal vector" which is perpendicular to the plane, while the location of the plane requires specifying one point thru which the plane passes.



orthogonality of difference vector and normal vector:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{vector form}$$

$$\langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

$\equiv d$

$$ax + by + cz = d \quad \text{simplified scalar form}$$

$\vec{n} = \langle a, b, c \rangle$ read off normal from coefficients
but position vector \vec{r}_0 lost.

any nonzero scalar multiple of \vec{n} works but simplest integer one best for "toy problems".

two unique unit vector normals:

$$\pm \hat{\vec{n}} = \pm \frac{\vec{n}}{|\vec{n}|}$$

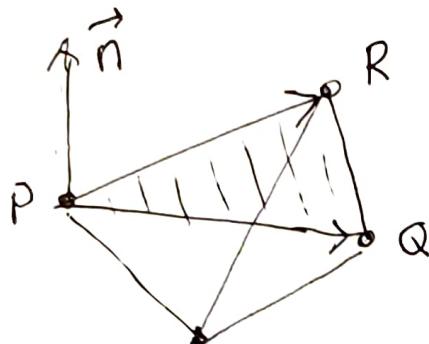
An "oriented plane" is a plane plus a choice of one of these two directions, just like an oriented line has an assigned direction.

12.5a

lines and planes in space.

(4)

BUT we are getting ahead of ourselves. Any 3 points (not collinear, distinct) determine a plane, in analogy to 2 pts determining a line.



$$P(1,3,2)$$

$$Q(3,-1,6)$$

$$R(5,2,0)$$

$$\overrightarrow{PQ} = \langle 3-1, -1-3, 6-2 \rangle = \langle 2, -4, 4 \rangle$$

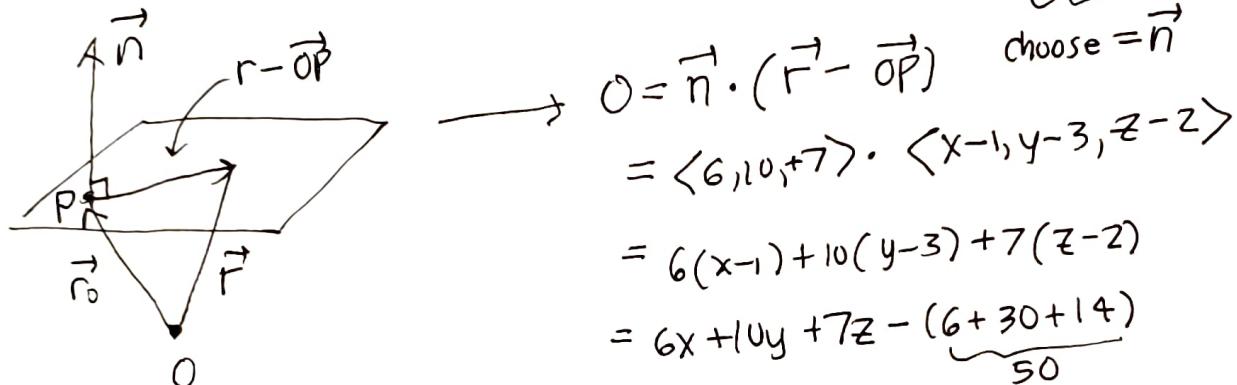
$$\overrightarrow{PR} = \langle 5-1, 2-3, 0-2 \rangle = \langle 4, -1, -2 \rangle$$

pick any vertex to
compute adjacent difference vectors

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 12, 20, 14 \rangle$$

$$\text{maple} = 2 \langle 6, 10, 7 \rangle$$

choose \vec{n}



$$0 = \vec{n} \cdot (\vec{r} - \vec{r}_0)$$

$$= \langle 6, 10, 7 \rangle \cdot \langle x-1, y-3, z-2 \rangle$$

$$= 6(x-1) + 10(y-3) + 7(z-2)$$

$$= 6x + 10y + 7z - \underbrace{(6+30+14)}_{50}$$

$$\therefore \boxed{6x + 10y + 7z = 50}$$

variation: plane thru P orthogonal to the line parallel to $\langle 1, 2, 3 \rangle$

$$\vec{r}_0 = \langle 1, 3, 2 \rangle, \quad \vec{n} = \langle 1, 2, 3 \rangle$$

$$0 = \vec{n} \cdot (\vec{r} - \vec{r}_0) = \langle 1, 2, 3 \rangle \cdot \langle x-1, y-3, z-2 \rangle$$

$$= (x-1) + 2(y-3) + 3(z-2) = x + 2y + 3z - \underbrace{1-6-6}_{-13}$$

$$\boxed{x + 2y + 3z = 13}$$

Key vectors are very different

$$0 = \vec{n} \cdot (\vec{r} - \vec{r}_0)$$

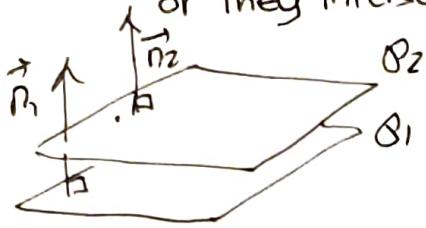
↑ orientation vector ↑ position vector

12.5a)

lines and planes in space

(5)

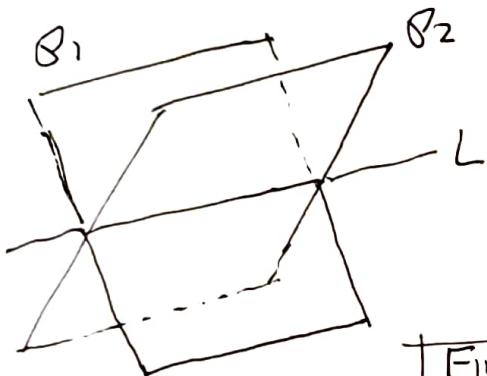
Two planes are either parallel (and possibly coincide)
or they intersect in a line



parallel planes have equations with proportional normal vectors $\vec{n}_1 \propto \vec{n}_2$ since the normals must be aligned along a common direction.

$$ax + by + cz = d$$

when scaled to be the same coefficients, the RHS constants must differ to represent distinct planes

nonparallel planes

example:

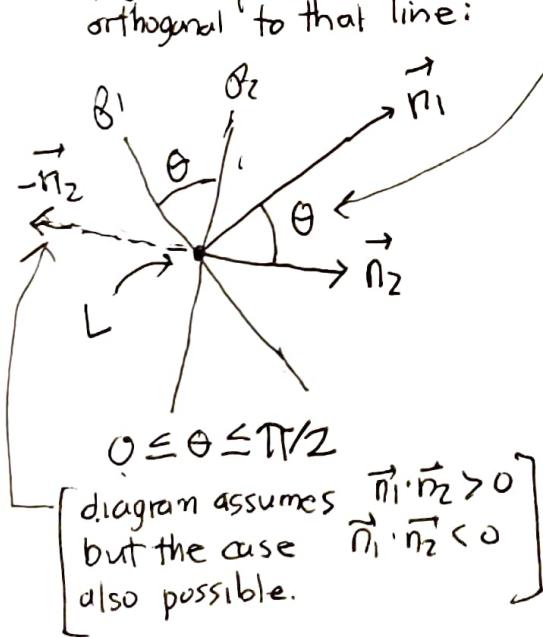
$$P_1: x + y + z = 1 \rightarrow \vec{n}_1 = \langle 1, 1, 1 \rangle$$

$$P_2: x - 2y + 3z = 4 \rightarrow \vec{n}_2 = \langle 1, -2, 3 \rangle$$

clearly not proportional
so planes must intersect.

Find the line of intersection and the (smaller) angle between the planes

For the angle look down the line of intersection to see a plane cross-section orthogonal to that line:



By HS geometry the angle between the planes agrees with the angle between the normals (modulo a reflection!)

$$\hat{\vec{n}}_1 = \frac{\langle 1, 1, 1 \rangle}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

$$\hat{\vec{n}}_2 = \frac{\langle 1, -2, 3 \rangle}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}} \langle 1, -2, 3 \rangle$$

$$\cos \theta = |\hat{\vec{n}}_1 \cdot \hat{\vec{n}}_2| \text{ to guarantee an acute angle}$$

$$\hat{\vec{n}}_1 \cdot \hat{\vec{n}}_2 = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{14}} \underbrace{\langle 1, 1, 1 \rangle \cdot \langle 1, -2, 3 \rangle}_{(1)(1) + 1(-2) + 1(3) = 2} = \frac{2}{\sqrt{3 \cdot 14}} > 0$$

$$\theta = \arccos\left(\frac{2}{\sqrt{3 \cdot 14}}\right) \approx 72^\circ$$

[diagram assumes $\vec{n}_1 \cdot \vec{n}_2 > 0$
but the case $\vec{n}_1 \cdot \vec{n}_2 < 0$ also possible.]

15.5a lines and planes in space

(6)

line of intersection : we need a point on the line & a vector parallel to the line

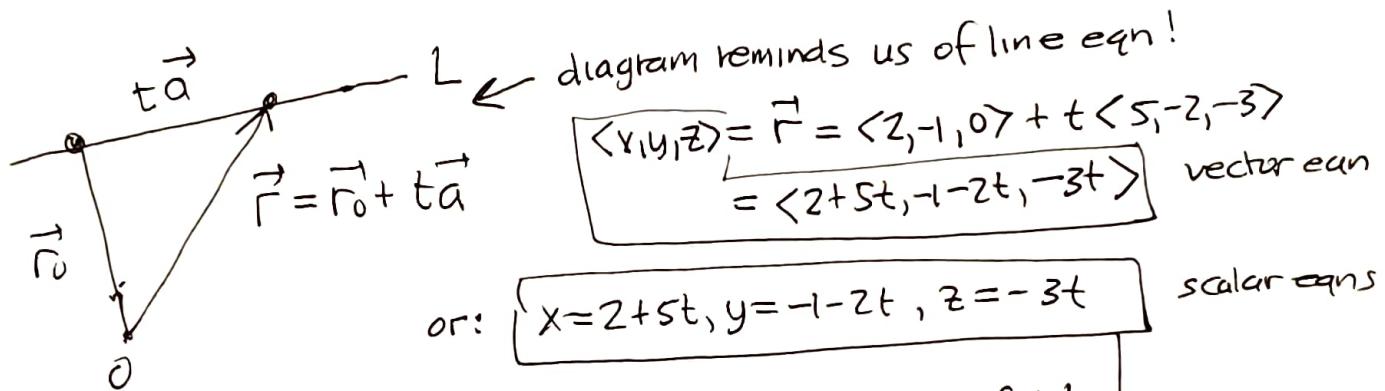
point: $x+y+z=1$ } we need a triplet (x_1, y_1, z) which solves
 $x-2y+3z=4$ } this pair of eqns.

↓
set $z=0$ to see where line intersects x-y plane

$$\begin{array}{l} x+y=1 \\ x-2y=4 \\ \hline \text{subtract } -: \quad 3y = -3 \\ \quad \quad \quad y = -1 \end{array} \rightarrow x = 1 - (-1) = 2 \quad \text{so} \quad \vec{r}_0 = \langle 2, -1, 0 \rangle \quad \begin{array}{l} \text{(locates} \\ \text{a pt on the line of intersection} \end{array}$$

direction: The line of intersection is perpendicular to both normals, so it must be along their cross-product.

$$\vec{n}_1 \times \vec{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, -2, 3 \rangle = \begin{array}{l} \text{maple} \\ \langle 5, -2, -3 \rangle \end{array} \equiv \vec{a} \quad \begin{array}{l} \text{(remove common)} \\ \text{factors to have} \\ \text{simpler vector!} \end{array}$$



symmetric equations of a line. No! not useful!

solve each equation for the parameter

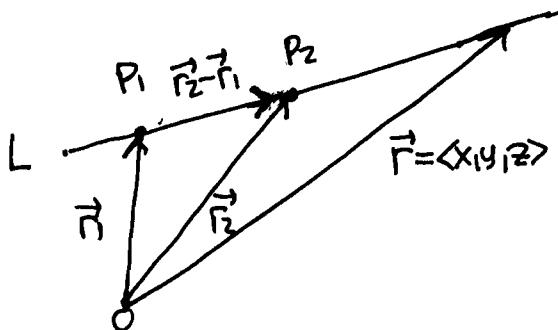
$$\frac{x-2}{5} = t, \quad \frac{y+1}{-2} = t, \quad \frac{z}{-3} = t, \quad \text{now equate:}$$

$$\frac{x-2}{5} = \frac{y+1}{-2} = \frac{z}{-3} \quad (\text{only 2 independent eqns!})$$

always write parametrized eqn(s) for a line

Describing lines and planes

2 distinct points determine a line



$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \vec{r}_2 - \vec{r}_1 \quad (\text{or } \vec{n} - \vec{r}_2)$$

$$\vec{r}_0 = \vec{r}_1 \quad (\text{or } \vec{r}_2)$$

$$\vec{r} = \vec{r}_0 + t\vec{a}$$

$$x = x_0 + t a_1$$

$$y = y_0 + t a_2$$

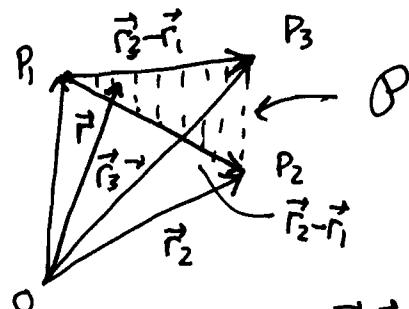
$$z = z_0 + t a_3$$

parametrized
equations of

← line

plane →

3 distinct points not on a line determine a plane



$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \vec{r}_2 - \vec{r}_1 \quad (\text{or } \dots)$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle = \vec{r}_3 - \vec{r}_1 \quad (\text{or } \dots)$$

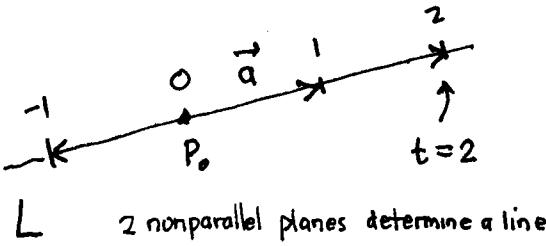
$$\vec{r}_0 = \vec{r}_1 \quad (\text{or } \vec{r}_2 \text{ or } \vec{r}_3)$$

$$\vec{r} = \vec{r}_0 + t_1 \vec{a} + t_2 \vec{b}$$

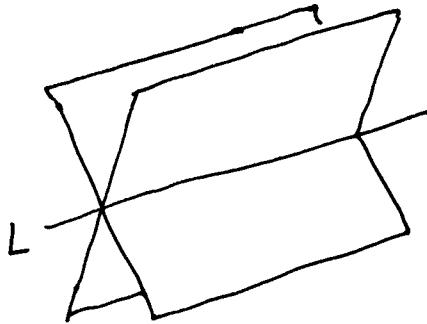
$$x = x_0 + t_1 a_1 + t_2 b_1$$

$$y = y_0 + t_1 a_2 + t_2 b_2$$

$$z = z_0 + t_1 a_3 + t_2 b_3$$



L 2 nonparallel planes determine a line



$$b_1 x + b_2 y + b_3 z = b_4$$

$$c_1 x + c_2 y + c_3 z = c_4$$

← line

unparametrized
equations of

plane →

$$\vec{n} = \vec{a} \times \vec{b}$$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{or:}$$

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

The intersection of 2 nonparallel planes is a line. The solution of these 2 linear equations for x, y, z determines the coordinates of points on the line.

The above parametrized equations of the line are a way of representing the solution of these linear equations.

The equation of a plane is a linear equation to be solved for x, y, z to yield the coordinates of a point on the plane.

The above parametrized equations of the plane are a way of representing the solution of this linear equation.

aside:

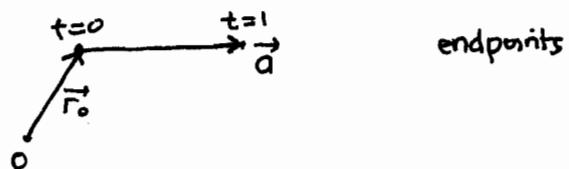
parametrized line segments, parallelograms, parallelopipeds

Using parameter values between 0 and 1 we can easily describe uniquely all the points which belong to a line segment, a parallelogram, or a parallelopiped (interior and edges).

Let $\vec{a}, \vec{b}, \vec{c}$ be any 3 nonzero vectors which are not collinear or coplanar.

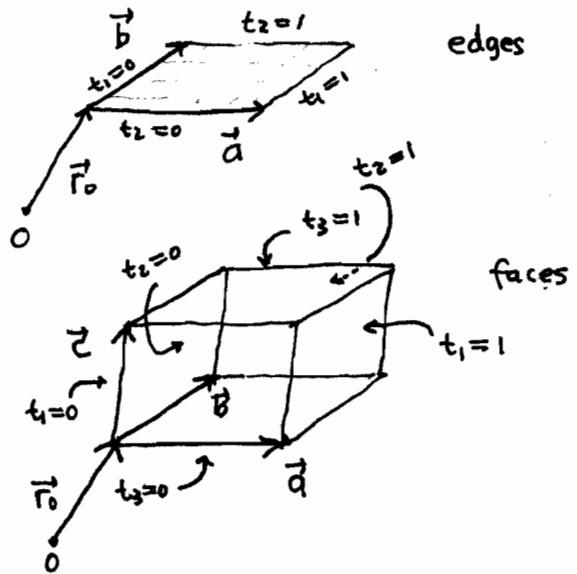
1-d: parametrized line segment

$$\vec{r} = \vec{r}_0 + t\vec{a}, \quad 0 \leq t \leq 1$$



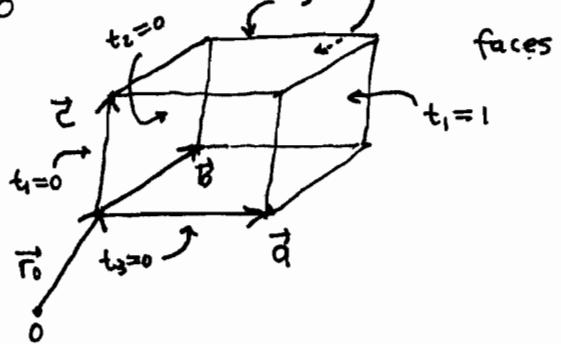
2-d: parametrized parallelogram

$$\vec{r} = \vec{r}_0 + t_1 \vec{a} + t_2 \vec{b}, \quad 0 \leq t_1 \leq 1, \quad 0 \leq t_2 \leq 1$$



3-d: parametrized parallelopiped

$$\vec{r} = \vec{r}_0 + t_1 \vec{a} + t_2 \vec{b} + t_3 \vec{c}, \quad 0 \leq t_1 \leq 1, \quad 0 \leq t_2 \leq 1, \quad 0 \leq t_3 \leq 1$$



Thus a single idea works in all 3 dimensions within space.

This is the unity of mathematics. If this does not matter to you, remember at least the first example.

→ We need this later: store in memory.