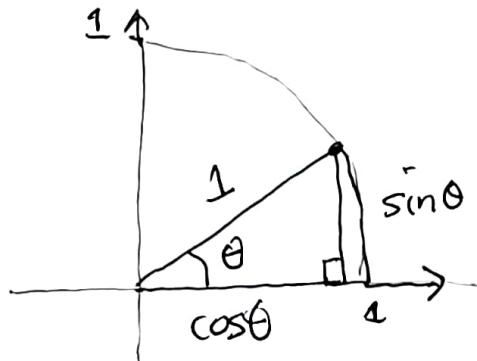


[12.3a] dot product

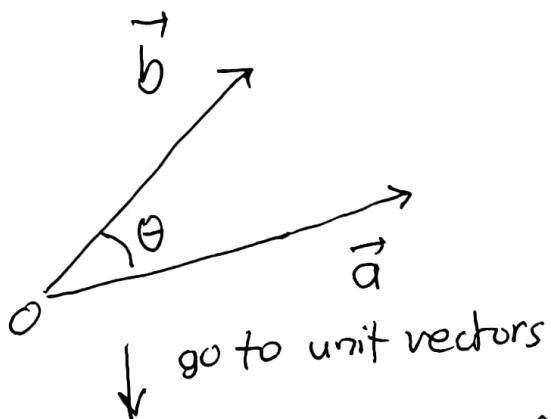
(1)

Where are we going?

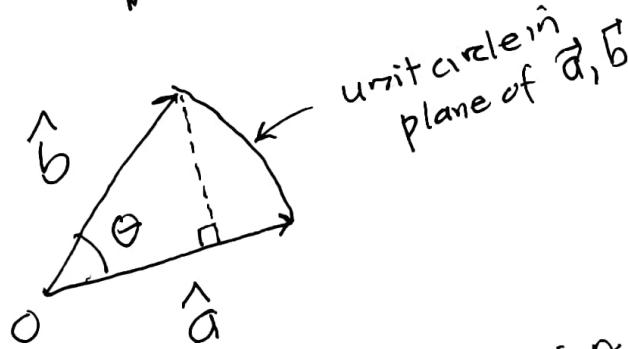


right triangle geometry in  
unit circle = trigonometry  
unit hypotenuse!

↓ generalize to pair of nonzero vectors in any dimension



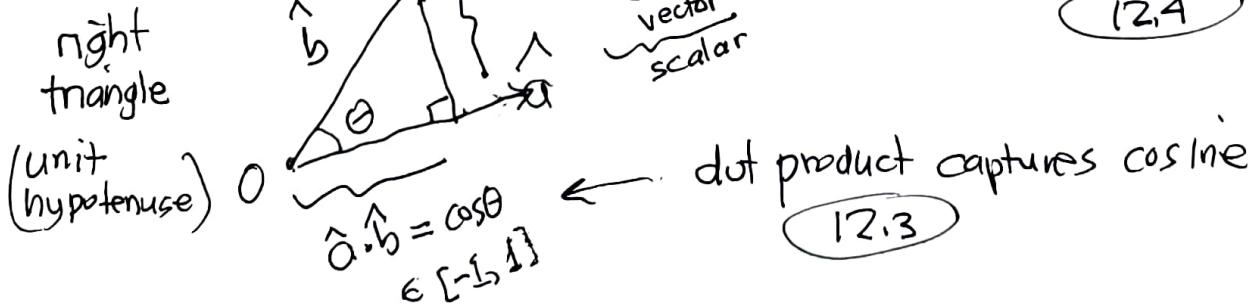
↓ go to unit vectors



unit circle in  
plane of  $\vec{a}, \vec{b}$

$|\hat{a} \times \hat{b}| = \sin \theta \geq 0$  ← later:  
cross product  
captures sine

(12.4)



dot product captures cosine

(12.3)

## 12.3a] dot product

(3)

dot product of two vectors in  $\mathbb{R}^3$  (same in  $\mathbb{R}^n$ ,  $n \geq 1$ )

### component definition

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

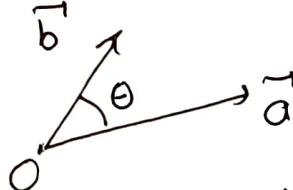
$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \cdot \vec{b} = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\text{sum of products of corresponding components}} \text{ (scalar)}$$

$$\begin{aligned} \mathbb{R}^n & \\ \vec{a} &= \langle a_1, \dots, a_n \rangle \\ \vec{b} &= \langle b_1, \dots, b_n \rangle \\ \vec{a} \cdot \vec{b} &= a_1 b_1 + \dots + a_n b_n \end{aligned}$$

### geometric definition (lengths and angles)

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$



examples

$$\langle 1, 2, -1 \rangle \cdot \langle 0, 3, 5 \rangle = 1(0) + 2(3) - 1(5) = 0 + 6 - 5 = 1$$

$$\begin{array}{c} 2 \rightarrow \vec{b} \\ \swarrow 45^\circ \\ 1 \quad \vec{a} \end{array} \quad \left. \begin{array}{l} |\vec{a}| = 1 \\ |\vec{b}| = 2 \\ \theta = \frac{\pi}{4} = 45^\circ \end{array} \right\} \quad \vec{a} \cdot \vec{b} = 1(2) \cos 45^\circ = \frac{2}{\sqrt{2}} = \sqrt{2}$$

remember

normally vectors are specified by their components and the geometric definition helps us interpret what the components tell us.

### obvious properties

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3 = \vec{b} \cdot \vec{a} \quad \left. \begin{array}{l} (\text{commutative}) \\ (\text{order does not matter}) \end{array} \right\}$$

$$\text{or } \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta = |\vec{b}| |\vec{a}| \cos \theta = \vec{b} \cdot \vec{a}$$

$$\text{self-dot product is length squared!} \quad \left. \begin{array}{l} |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} \end{array} \right\}$$

$$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$$

$$\text{and } c \vec{a} \cdot \vec{b} = c(a_1 b_1 + a_2 b_2 + a_3 b_3) = \underbrace{c a_1 b_1}_c + \underbrace{c a_2 b_2}_c + \underbrace{c a_3 b_3}_c \quad \text{associative!}$$

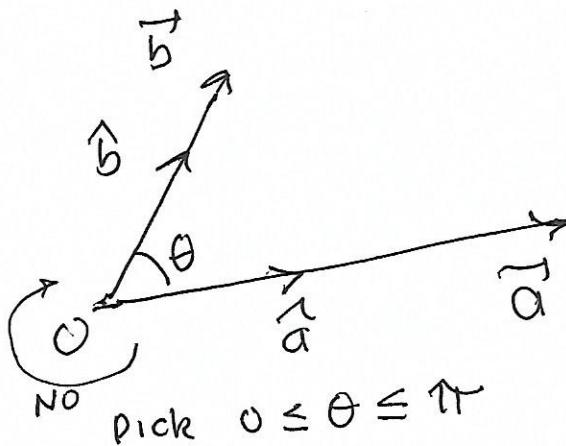
$$= (c \vec{a}) \cdot \vec{b} = \vec{a} \cdot (c \vec{b}) \quad \text{scalar factors can go anywhere}$$

### 12.3a) dot product

(5)

How to find angle between 2 vectors?

Lengths clearly don't matter so the answer can only involve their directions identified with unit vectors.

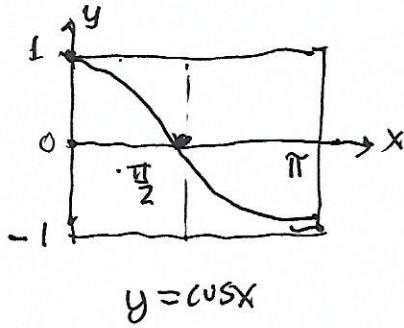


$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

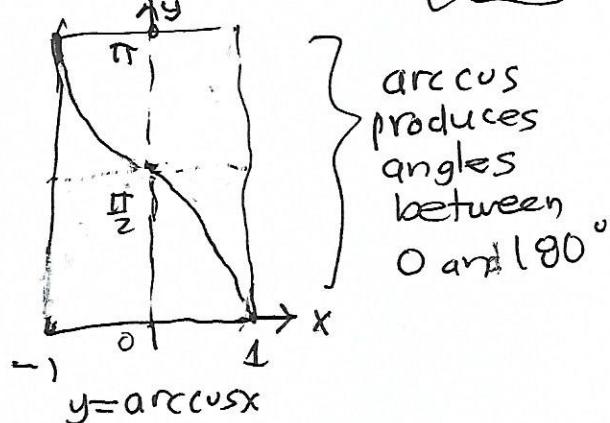
$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \left( \frac{\vec{a}}{|\vec{a}|} \right) \cdot \left( \frac{\vec{b}}{|\vec{b}|} \right) = \hat{a} \cdot \hat{b}$$

The dot product between two unit vectors is the cosine of the included angle.

$$\theta = \arccos(\hat{a} \cdot \hat{b}) \in [0, \pi]$$



invert by exchanging x & y  
reflect across  $y=x$



example

$$\vec{a} = \langle 1, 2, -1 \rangle \quad |\vec{a}| = \sqrt{1+4+1} = \sqrt{6} \quad \hat{a} = \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle$$

$$\vec{b} = \langle 0, 3, 5 \rangle \quad |\vec{b}| = \sqrt{0+9+25} = \sqrt{34} \quad \hat{b} = \frac{1}{\sqrt{34}} \langle 0, 3, 5 \rangle$$

$$\cos \theta = \hat{a} \cdot \hat{b} = \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle \cdot \frac{1}{\sqrt{34}} \langle 0, 3, 5 \rangle = \frac{1}{\sqrt{6}} \frac{1}{\sqrt{34}} \frac{(1(0) + 2(3) - 1(5))}{\sqrt{6}\sqrt{34}} = \frac{1}{\sqrt{6}\sqrt{34}}$$

$$\theta = \arccos(\hat{a} \cdot \hat{b}) = \arccos\left(\frac{1}{\sqrt{6}\sqrt{34}}\right)$$

$$\approx 1.500725 \text{ radians}$$

math angles always in radians!

$$\approx 85.985^\circ \approx 86.0^\circ \leftarrow \text{never need more than tenth of degree}$$

degrees are only for interpretation

mentally comparing to  $0, 30^\circ, 45^\circ, 60^\circ, 90^\circ$  etc

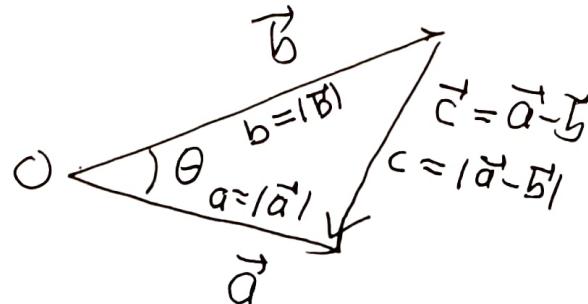
NOT for math calculations!

# 12.3a dot product

4

Why do the two definitions agree?

LAW OF COSINES!



$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

if  $\theta = \frac{\pi}{2}$ ,  $\rightarrow 0$  giving Pythagorean Thm for right triangle

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2ab \cos \theta$$

$$\underbrace{(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})}_{\text{multiply out}} = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2 |\vec{a}| |\vec{b}| \cos \theta \quad \text{self dotprod!}$$

[ ASIDE: distributive law  
 $\vec{a} \cdot (\vec{b} + \vec{c}) = a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$   
 then iterate:  $(a+b) \cdot (c+d)$  etc (minus signs) ]

$$\underbrace{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a}}_{-2 \vec{a} \cdot \vec{b}} = \underbrace{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b}}_{= -2 |\vec{a}| |\vec{b}| \cos \theta} - 2 |\vec{a}| |\vec{b}| \cos \theta$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \quad \checkmark$$

(derived only using properties of component definition)

Conclusion: the "dot" product vectorizes the law of cosines thus incorporating the cosine of the included angle into its information content

Self dot product produces lengths, what about angles?

12.3a) dot product

(6)

~~$$\theta = \arccos \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$~~

do not remember or plug  
into a formula like this

just remember the cosine of the angle is the  
dot product of the unit vectors.

understand, don't memorize.

use hat notation to signal unit vectors!

$\hat{a}$   $\leftarrow$  tells you immediately it is a unit vector.

sign of dot product alone is useful:  $\hat{a} \cdot \hat{b}$  &  $\vec{a} \cdot \vec{b}$  have  
same sign =  
 $\text{sign}(\cos \theta)$

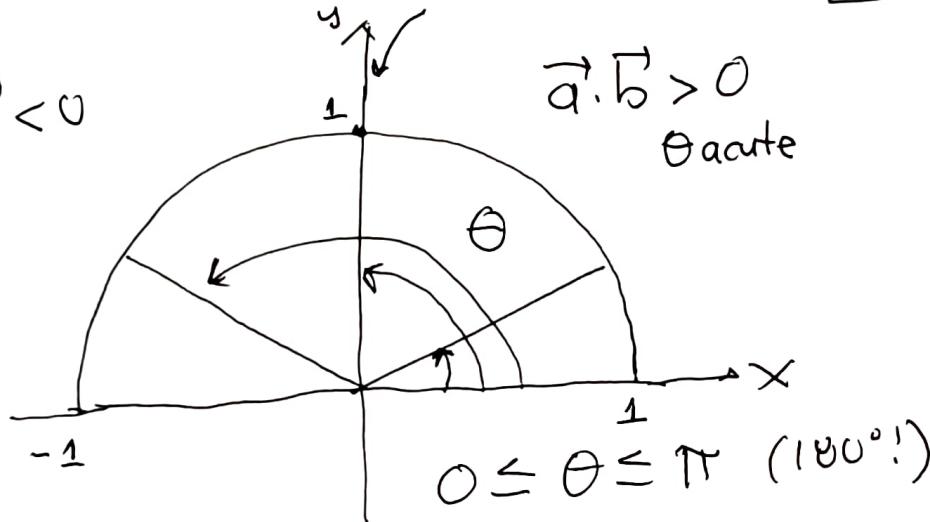
$$\vec{a} \cdot \vec{b} < 0$$

$\theta$  obtuse

$$\vec{a} \cdot \vec{b} = 0 \text{ right angle}$$

$$\vec{a} \cdot \vec{b} > 0$$

$\theta$  acute



Two vectors satisfying  $\vec{a} \cdot \vec{b} = 0$  are called "orthogonal".

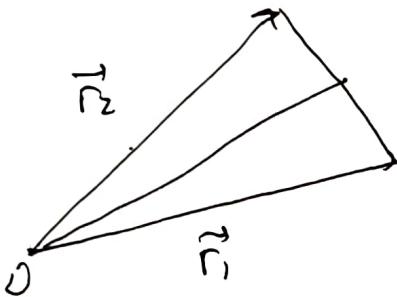
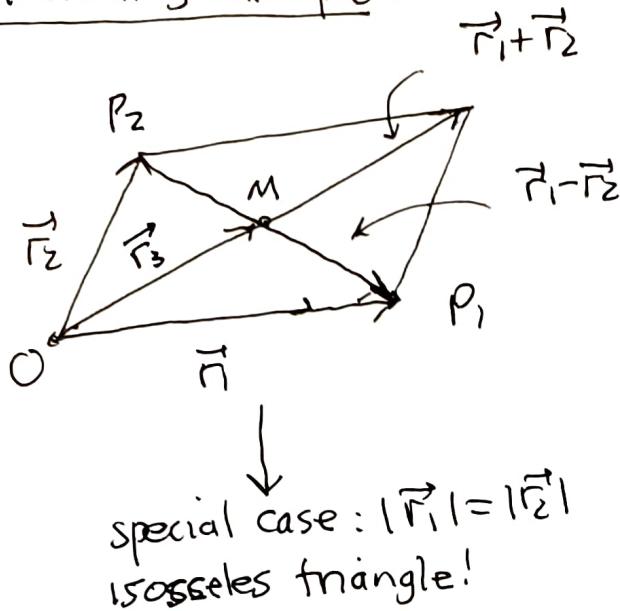
If  $\vec{a} \neq \vec{0}$  and  $\vec{b} \neq \vec{0}$ , then they are perpendicular.

"orthogonality" allows one vector factor to vanish

vanishing dot product is a test for orthogonality

12.3a) dot product

Motivating example:



high  
school  
geometry

midpoint:  $\vec{r}_3 = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$

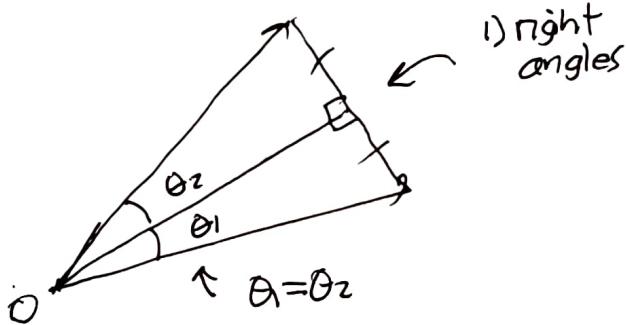
remember:

$$\vec{r}_3 = \vec{r}_2 + \frac{1}{2}(\vec{r}_1 - \vec{r}_2)$$

$$= \frac{1}{2}\vec{r}_1 + \left(-\frac{1}{2}\right)\vec{r}_2$$

$$= \frac{1}{2}\vec{r}_1 + \frac{1}{2}\vec{r}_2 = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

]



Facts. Opposite side bisector is perpendicular to the opp side.  
Opposite side bisector bisects the angle.

How to derive these results? (without going back to H.S. !)

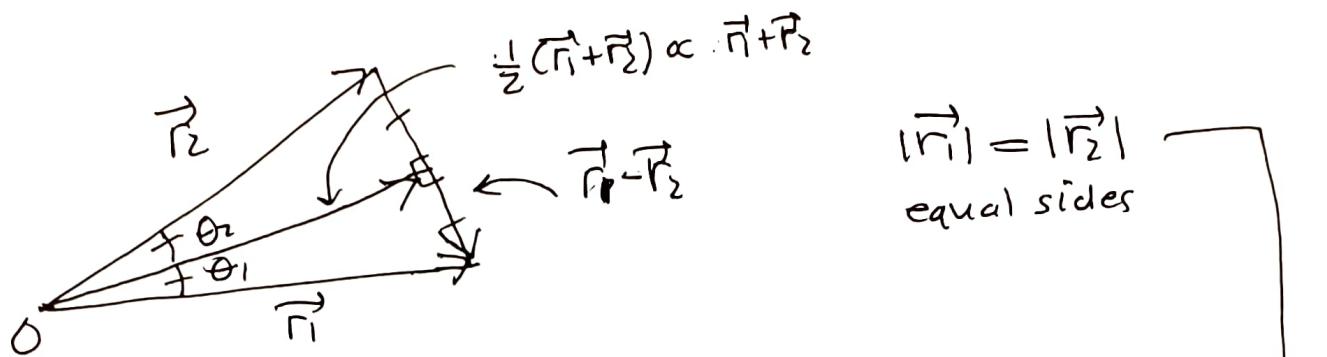
We can use the dot product to evaluate these angles.

With vectors, this works in any dimension ( $\mathbb{R}^n$ )!

12.3a) dot product

(7)

Back to our isosceles triangle.



$$1) \quad \vec{r}_1 - \vec{r}_2 \perp \vec{r}_1 + \vec{r}_2 \therefore$$

$$\begin{aligned} (\underbrace{\vec{r}_1 - \vec{r}_2}_{\perp}) \cdot (\overbrace{\vec{r}_1 + \vec{r}_2}^{\text{base}}) &= \vec{r}_1 \cdot \vec{r}_1 - \vec{r}_2 \cdot \vec{r}_2 + \cancel{\vec{r}_1 \cdot \vec{r}_2} - \cancel{\vec{r}_2 \cdot \vec{r}_1} \\ &= |\vec{r}_1|^2 - |\vec{r}_2|^2 \\ &= 0 \quad \checkmark \end{aligned}$$

$$2) \quad \theta_1 = \theta_2 \leftrightarrow \cos \theta_1 = \cos \theta_2 \therefore$$

$$\begin{aligned} \cos \theta_1 &= \hat{r}_1 \cdot \widehat{(\vec{r}_1 + \vec{r}_2)} = \frac{\vec{r}_1}{|\vec{r}_1|} \cdot \frac{(\vec{r}_1 + \vec{r}_2)}{|\vec{r}_1 + \vec{r}_2|} = \frac{\vec{r}_1 \cdot \vec{r}_1 + \vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1| |\vec{r}_1 + \vec{r}_2|} \\ &= \frac{|\vec{r}_1|^2 + \vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1| |\vec{r}_1 + \vec{r}_2|} \end{aligned}$$

$$\begin{aligned} \cos \theta_2 &= \hat{r}_2 \cdot \widehat{(\vec{r}_1 + \vec{r}_2)} = \frac{\vec{r}_2}{|\vec{r}_2|} \cdot \frac{(\vec{r}_1 + \vec{r}_2)}{|\vec{r}_1 + \vec{r}_2|} = \frac{\vec{r}_2 \cdot \vec{r}_1 + \vec{r}_2 \cdot \vec{r}_2}{|\vec{r}_2| |\vec{r}_1 + \vec{r}_2|} \\ &= \frac{|\vec{r}_2|^2 + \vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_2| |\vec{r}_1 + \vec{r}_2|} = \frac{|\vec{r}_1|^2 + \vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1| |\vec{r}_1 + \vec{r}_2|} \end{aligned}$$

Easy dot product calculation.