

12.2a

## Vectors

(1)

Points versus vectors

$\mathbb{R}^n$  consists of ordered  $n$ -tuples of real numbers

"the plane"     $\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$

"space"     $\mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$

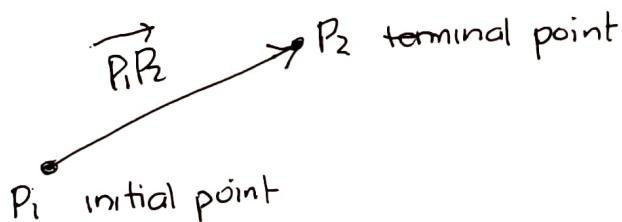
$\mathbb{R}^n = \underbrace{\{(x_1, \dots, x_n) | x_i \in \mathbb{R}, i=1, \dots, n\}}$

elements of these spaces are called "points"

"coordinates" of the point

↑  
[rectangular or Cartesian coordinate systems]

"vectors" are directed line segments in these spaces:

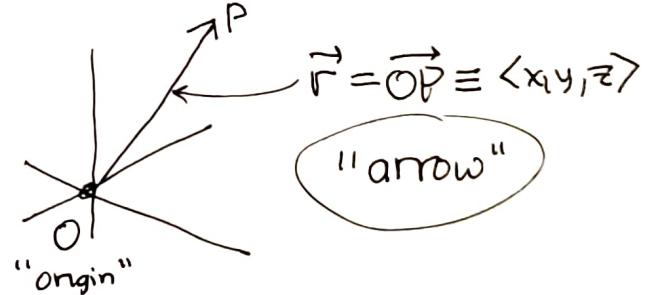
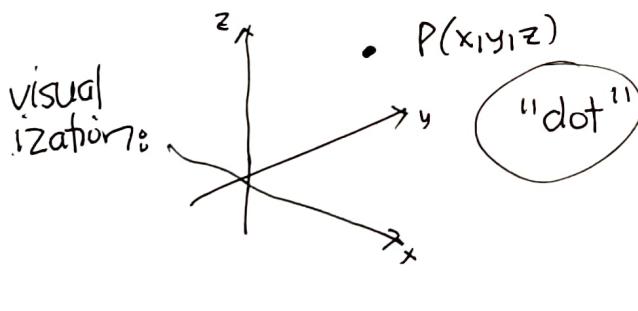


Every point in  $\mathbb{R}^n$  is associated with its position vector

3-D:

[point:]  $P(x_1, y_1, z_1)$     point delimiters  
               "coordinates"

[vector:]  $\vec{OP} = \langle x_1, y_1, z_1 \rangle \equiv \vec{r}$     vector delimiters  
               arrow over vector symbols!  
               "components" (for "radius" vector)  
               ALWAYS!!



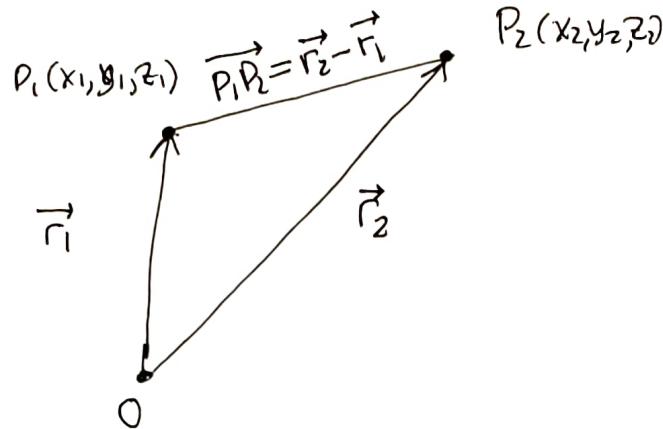
same triplet of numbers but different mathematical interpretation—  
     angle brackets  $\langle, \rangle$  remind us to interpret as an arrow.

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Vectors

(2)

Directed line segments are difference vectors



They are an instruction  
on how to move from  
one point to another point.

$$\begin{aligned} \overrightarrow{P_1P_2} &= \langle x_2, y_2, z_2 \rangle - \langle x_1, y_1, z_1 \rangle \\ &\equiv \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \end{aligned}$$

difference vector

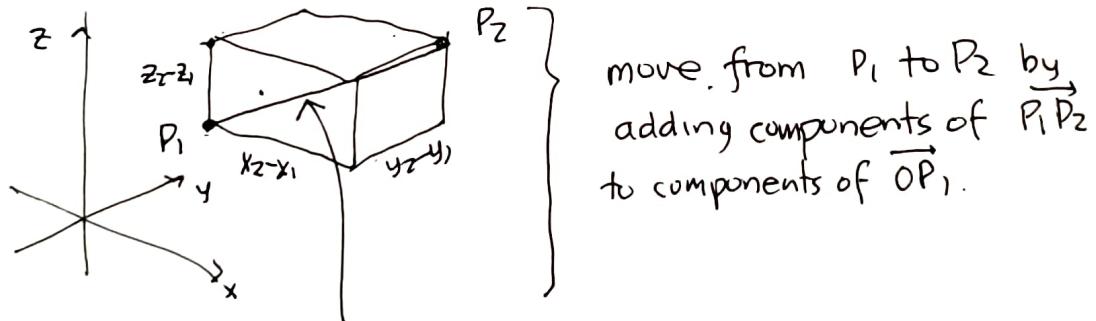
↓

↑  
definition  
(see next page)

If  $\overrightarrow{P_1P_2} = \overrightarrow{r_2} - \overrightarrow{r_1}$ , adding  
it to  $\overrightarrow{OP_1}$  takes you to  $\overrightarrow{OP_2}$ :

$$\begin{aligned} \overrightarrow{r_2} &= \overrightarrow{r_1} + \overbrace{(\overrightarrow{r_2} - \overrightarrow{r_1})} \\ &= \overrightarrow{r_1} - \overrightarrow{r_1} + \overrightarrow{r_2} = \overrightarrow{r_2} \quad \checkmark \end{aligned}$$

vector addition corresponds  
to "tip to tail" joining of  
the two arrows



move from  $P_1$  to  $P_2$  by  
adding components of  $\overrightarrow{P_1P_2}$   
to components of  $\overrightarrow{OP_1}$ .

Main diagonal of rectangular box  
oriented by coordinate "grid"

## 12.2a Vectors

(3)

An abstract vector is not located anywhere particular in space.  
It is instead an instruction on how to move from any initial point to a terminal point.

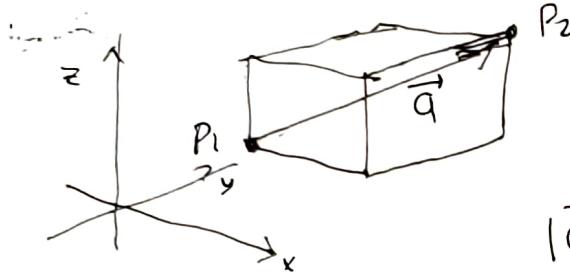
$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

↑      ↑  
numbered components

length or magnitude:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

defined by  
distance  
formula:



$$\overrightarrow{OP_1} + \overrightarrow{q} = \overrightarrow{OP_2}$$

$\overbrace{\overrightarrow{P_1P_2}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

$$|\vec{a}| = |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

magnitude of vector

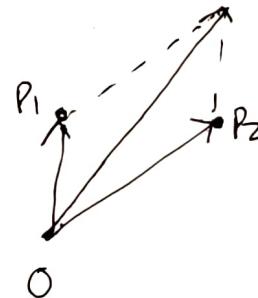
distance between points

points cannot be added or subtracted or multiplied by numbers

$$P_1 \circ P_2 \rightarrow ?$$

dots don't do it!

vectors  
can be  
added



once you have  
an origin,  
position  
vectors are  
defined and  
vector ops  
can be  
implemented

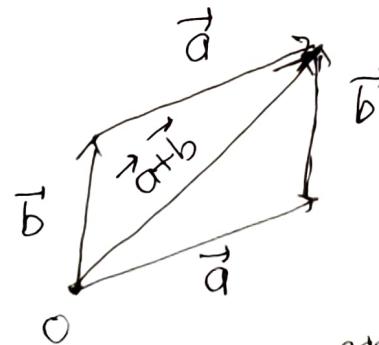
### Vector operations

#### abstractly: vector addition

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$



parallelogram  
vector addition  
= tip to tail  
addition in  
2 ways

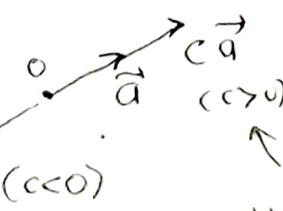
add  $\vec{a}$  to tip of  $\vec{b}$  (top)  
add  $\vec{b}$  to tip of  $\vec{a}$  (bottom)

#### scalar multiplication

$$c\vec{a} = c \langle a_1, a_2, a_3 \rangle$$

$$= \langle ca_1, ca_2, ca_3 \rangle$$

(single real #'s are "scalars")



"scales vector up or down"  
(same direction, different length)

$$\text{NOTE: } |c\vec{a}| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = \dots = \sqrt{c^2(a_1^2 + a_2^2 + a_3^2)} = |c| |\vec{a}|$$

length scales  
by  $|c|$

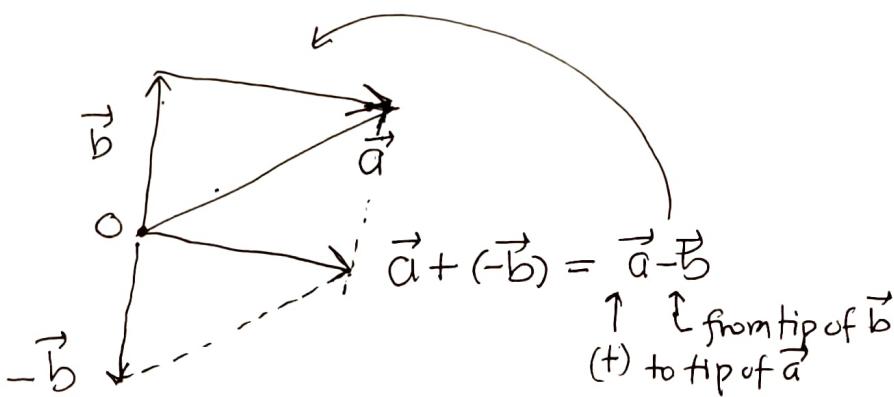
12.2a

vectors

④

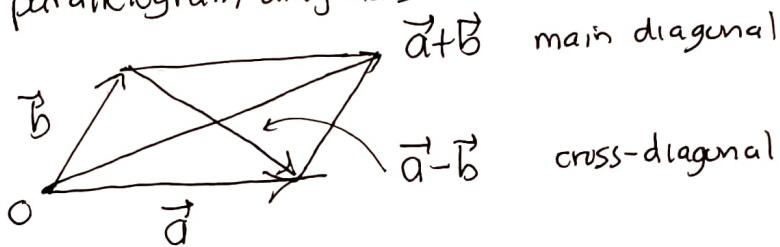
vector subtraction

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$



Thus  $\overrightarrow{P_1 P_2} = \vec{OP}_2 - \vec{OP}_1$   
is consistent  
with these  
vector operations

parallelogram diagonals:



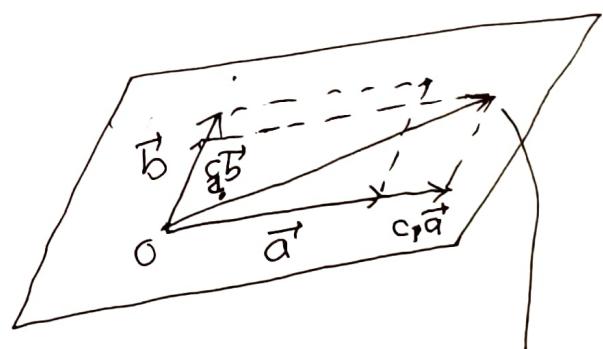
2 nonzero, noncollinear (non parallel) vectors determine a plane through arbitrary linear combination

linear combination

$$c_1 \vec{a} + c_2 \vec{b}$$

with coefficients  $(c_1, c_2)$

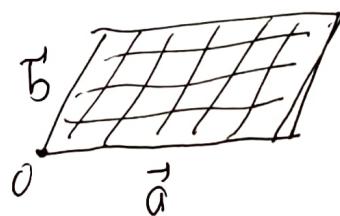
vary arbitrarily  
to sweep out a plane  
containing both vectors



tip moves  
around in  
a plane  
containing  
 $\vec{a}$  and  $\vec{b}$

$$\begin{cases} 0 \leq c_1 \leq 1 \\ 0 \leq c_2 \leq 1 \end{cases}$$

sweeps out  
unit parallelogram



nice representation of a piece of the plane they determine

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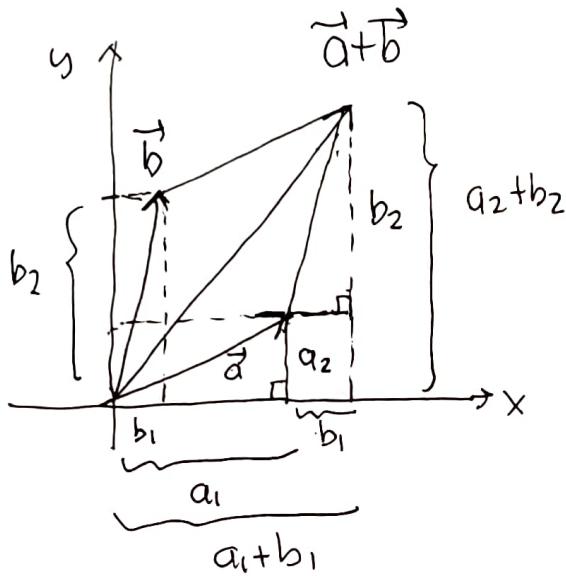
## vector addition

geometrical definition : parallelogram or  
(interpretation) tip to tail arrows (5)

component definition : add corresponding components  
(evaluation)

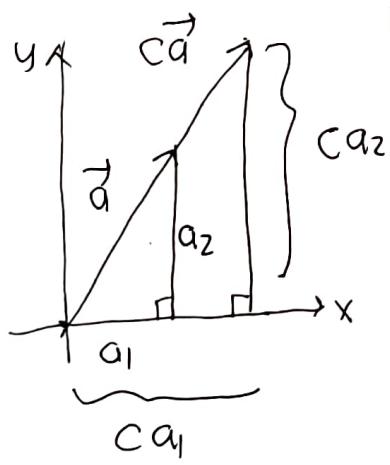
why do these agree?

Easiest to see key points in the plane.



coords of  $\vec{a} + \vec{b}$   
are clearly  
 $\langle a_1 + b_1, a_2 + b_2 \rangle$

scalar multiplication works the same way



similar triangles tells us that scaling the hypotenuse scales the sides by the same factor so

$$\vec{a} = \langle a_1, a_2 \rangle \rightarrow \\ c\vec{a} = \langle ca_1, ca_2 \rangle$$

we will introduce two vector multiplications (dot and cross products)  
similarly with

← a geometrical definition for interpretation  
a component definition for evaluation

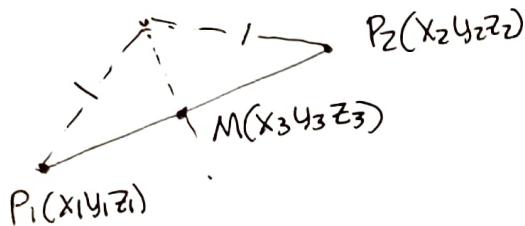
In each case we must show that the two definitions agree.

## 12.2a Vectors

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Why vectors are efficient for calculations: Example

How to find midpoint of a directed line segment?

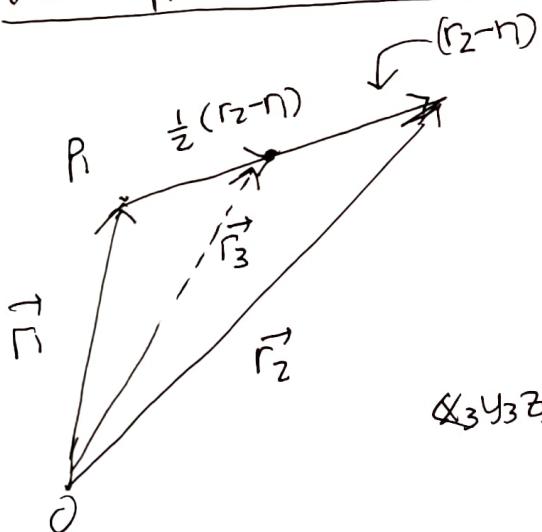


If we only can use the distance formula  
we have only 2 independent quadratic  
equations:

$$|\vec{P_1M}| = |\vec{P_2M}|, \quad |\vec{P_1M}| = \frac{1}{2} |\vec{P_1P_2}| \\ (= \frac{1}{2} \vec{P_2M}) \text{ implied by first eqn.}$$

namely:  $\begin{cases} (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = \frac{1}{4} [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \end{cases}$

Even just checking the answer derived below is tedious!

vector approach is trivialGo halfway from  $P_1$  to  $P_2$ :

$$\begin{aligned} \vec{r}_3 &= \vec{r}_1 + \frac{1}{2}(\vec{r}_2 - \vec{r}_1) \\ &= (1 - \frac{1}{2})\vec{r}_1 + \frac{1}{2}\vec{r}_2 \\ &= \frac{1}{2}\vec{r}_1 + \frac{1}{2}\vec{r}_2 = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \end{aligned}$$

average vector

$$\langle x_3 y_3 z_3 \rangle = \frac{1}{2} (\langle x_1 y_1 z_1 \rangle + \langle x_2 y_2 z_2 \rangle) \\ = \left\langle \frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2) \right\rangle$$

average components

coords of midpoint = average of endpoint coords.

textbook exercise 12.1.21a asks you to  
verify this soln by plugging into the above distance  
equations. (But even Maple cannot solve them!)

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## vectors

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 $\mathbb{R}^n$  coords and vectors

vectors

2d  $\vec{a} = \langle a_1, a_2 \rangle$

3d  $\vec{a} = \langle a_1, a_2, a_3 \rangle$

⋮  $\vec{a} = \langle a_1, \dots, a_n \rangle$

$|\vec{a}| = \sqrt{a_1^2 + \dots + a_n^2} \geq 0$

$\vec{0} = \langle \emptyset, \dots, \emptyset \rangle$  zero vector

coords

(x, y)

(x, y, z)

(x<sub>1</sub>, ..., x<sub>n</sub>)

most of our work  
is done here  
but all  
ideas apply to  
 $\mathbb{R}^n$

unit vectors have unit length

Define  $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$  for  $\vec{a} \neq \vec{0}$   $\longrightarrow$

"direction" of  $\vec{a}$   
can be identified with  $\hat{a}$  ("a hat")  
(length irrelevant)

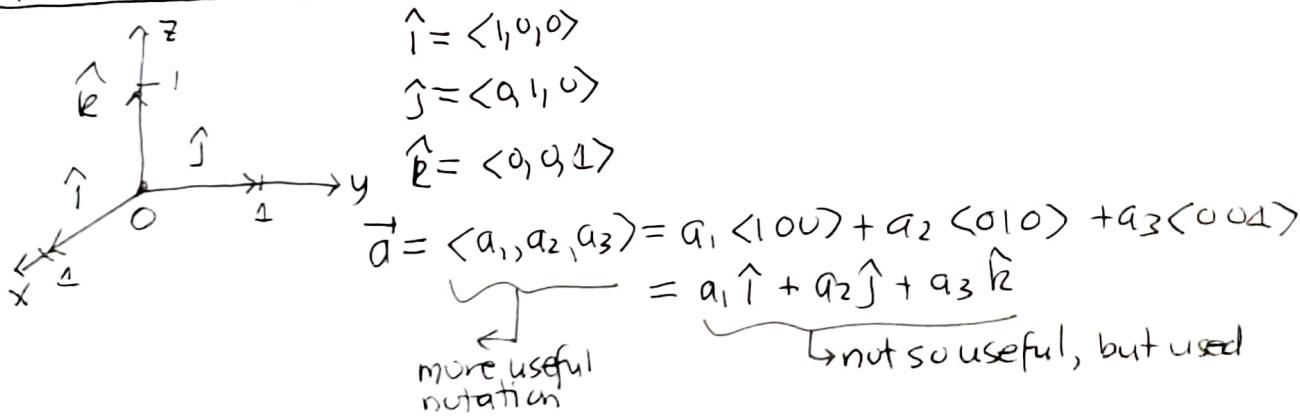
Note  $|\hat{a}| = \left| \frac{\vec{a}}{|\vec{a}|} \right| = \frac{|\vec{a}|}{|\vec{a}|} = 1$  ✓

$\vec{a} = |\vec{a}| \hat{a}$   
decomposes vector into  
length and direction  
(geometrical properties)

example  $\vec{a} = \langle 3, 4 \rangle \rightarrow |\vec{a}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\hat{a} = \frac{1}{5} \langle 3, 4 \rangle = \langle \frac{3}{5}, \frac{4}{5} \rangle$

more useful in intermediate calculations (often)

special unit vectors (3-D)

example  $|\langle 3, 4, 12 \rangle| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{52 + 12^2} = 13$

$\langle 3, 4, 12 \rangle = \frac{1}{13} \langle 3, 4, 12 \rangle = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \rangle$

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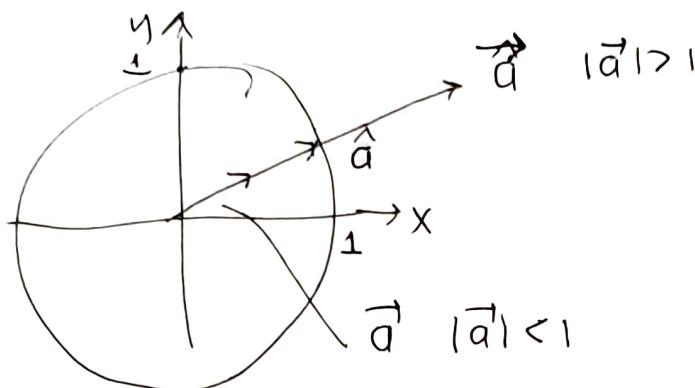
vectors

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$$\vec{a} \mapsto \hat{\vec{a}} = \frac{\vec{a}}{|\vec{a}|}$$

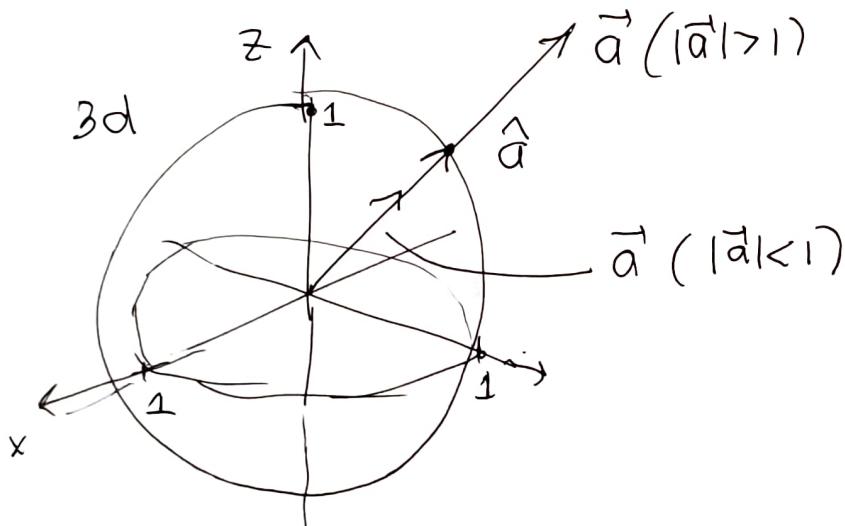
"normalize" the vector (math lingo):  
visualized by projection

2d



project along  
radial direction  
to unit circle  
 $x^2+y^2=1$

3d



project along  
radial direction  
to unit sphere  
 $x^2+y^2+z^2=1$