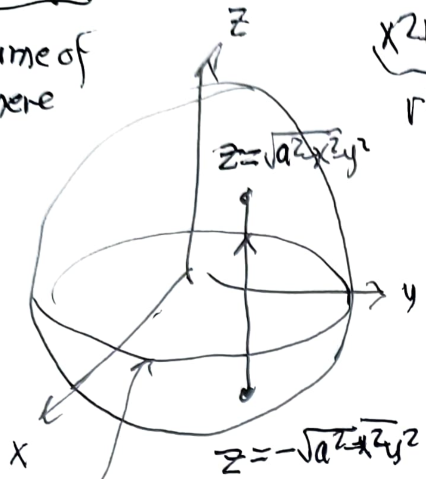
$\boxed{15.7}$ cylindrical coordinates

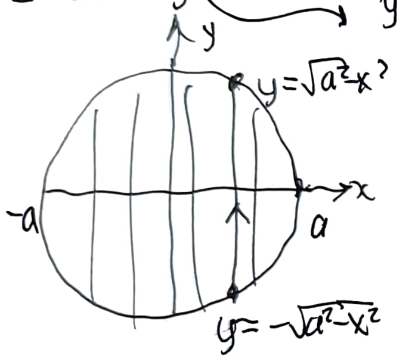
example

volume of sphere

$$\underbrace{x^2+y^2}_{r^2}+z^2=a^2 \;\longrightarrow\; z=\pm\sqrt{a^2-x^2-y^2}$$

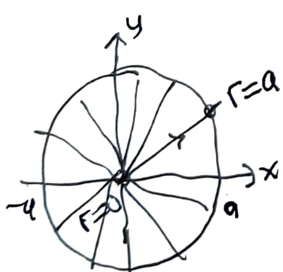


$z=0$: $x^2+y^2=a^2 \longrightarrow y=\pm\sqrt{a^2-x^2}$



$$V=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1\,dz\,\underbrace{dy\,dx}_{dA}$$

$dA \longrightarrow r\,dr\,d\theta$



$x^2+y^2=a^2 \rightarrow r^2=a^2 \rightarrow r=a$

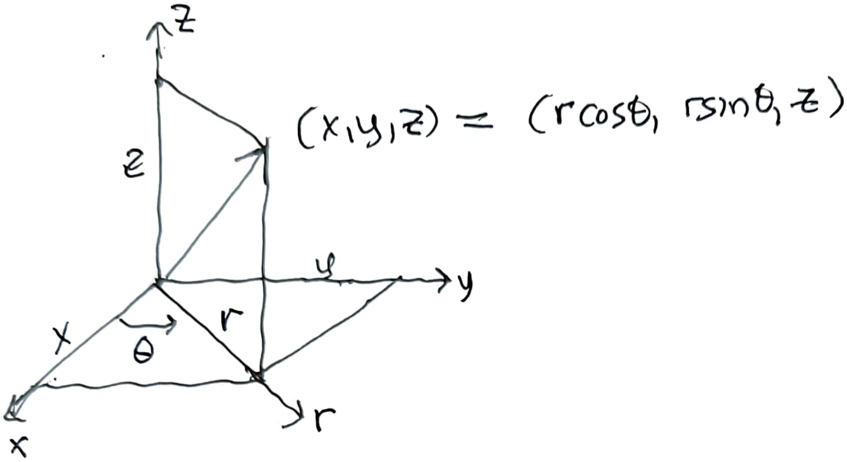
$r=0..a$ while $\theta=0..2\pi$

$$V=\int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} 1\,\underbrace{r\,dz\,dr\,d\theta}_{dV=(dz)(dr)(r\,d\theta)}$$

$\boxed{15.7}$ cylindrical coordinates

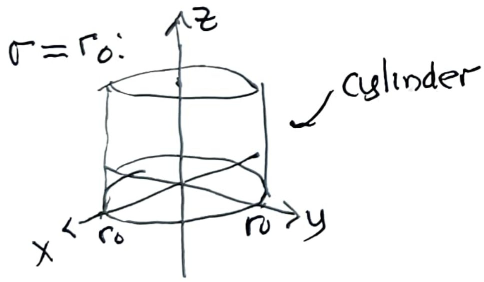
For any point $\underbrace{(x, y, z)}_{\text{"Cartesian coords"}}$ we can assign new coords by re-expressing x & y in terms of polar coords; but keep z:

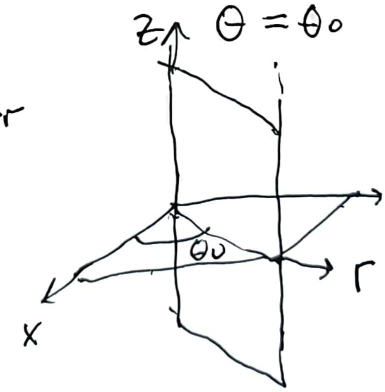
$$(x, y, z) = (r\cos\theta, r\sin\theta, z)$$



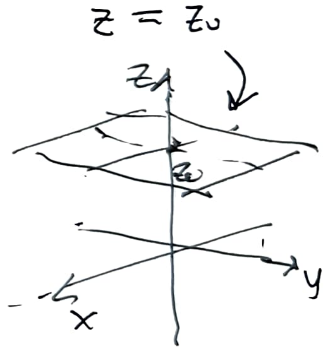
r-axis measures distance from z-axis
$r = \sqrt{x^2+y^2} \geq 0$ (not negative!)

coord surfaces (constant values):

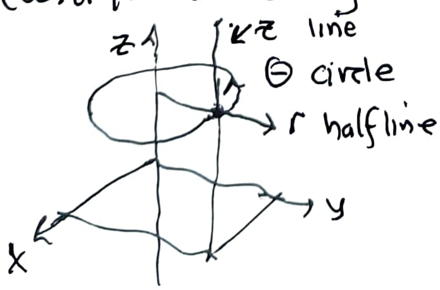




"r-z half-plane"
in every direction $\theta = \theta_0$



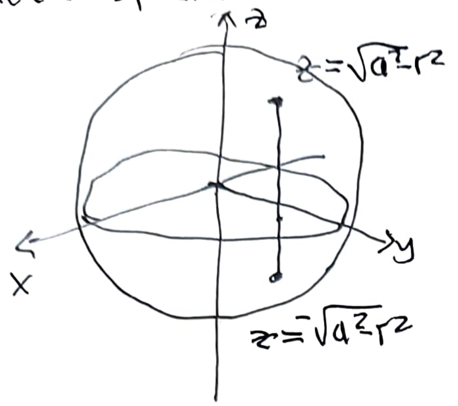
coord "lines" (only one coord varies, other two constant):



all meet at right angles
"orthogonal coords"

15.7 cylindrical coordinates

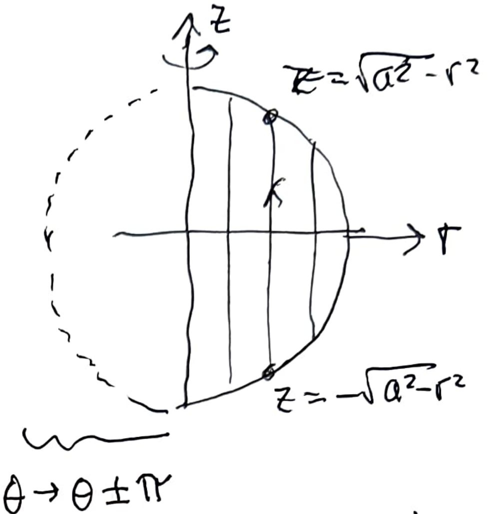
back to sphere:



$$\underbrace{x^2+y^2}_{r^2}+z^2=a^2 \rightarrow r^2+z^2=a^2$$

$$z=\pm\sqrt{a^2-r^2}$$

circle but $r\geq 0$ in r-z halfplane:



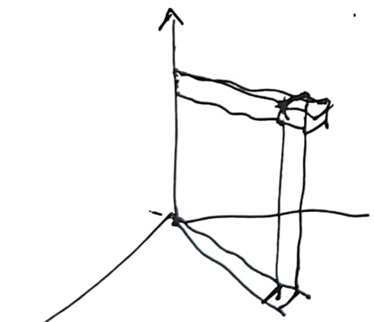
semi-circle rotated around z-axis to form sphere

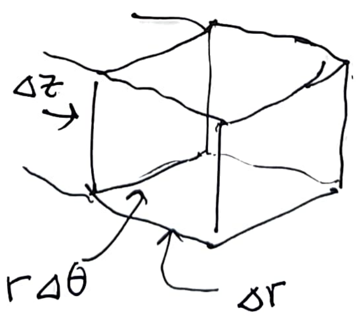
$\theta=0..2\pi$

To integrate any function over sphere:

$$\iiint_S f(x,y,z)\,dV=\int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} f(r\cos\theta, r\sin\theta, z)\,\underbrace{r\,dz\,dr\,d\theta}_{dV}$$

correction factor for differential of volume





coordinate lines orthogonal

almost rectangular box:

$$\Delta V\approx(\Delta z)(\Delta r)(r\Delta\theta)$$

$\downarrow$ limit

$$dV=dz\,dr\,r\,d\theta$$
$$=r\,dz\,dr\,d\theta$$

$\boxed{15.7}$ cylindrical coordinates

$$V = \int_0^{2\pi}\int_0^a \underbrace{\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} 1\, r\,dz}\,dr\,d\theta$$

$$rz\Big|_{z=-\sqrt{a^2-r^2}}^{z=\sqrt{a^2-r^2}} = r\left(\sqrt{a^2-r^2}-\left(-\sqrt{a^2-r^2}\right)\right)$$

$$= \sqrt{a^2-r^2}\,(2r)$$

$$= \int_0^{2\pi}\int_0^a \underbrace{(a^2-r^2)^{1/2}}_{u = a^2-r^2}\,\underbrace{2r\,dr}_{=-du}\,d\theta$$

$u = a^2-r^2$

$du = -2r\,dr$

$$\int_{r=0}^{r=a} -u^{1/2}\,du = -\frac{u^{3/2}}{3/2}\Big|_{r=0}^{r=a} = -\frac{2}{3}(a^2-r^2)^{3/2}\Big|_{r=0}^{r=a}$$

$$= -\frac{2}{3}(0) + \frac{2}{3}(a^2)^{3/2} = \frac{2}{3}a^3$$

$$= \int_0^{2\pi} \frac{2}{3}a^3\,d\theta = \frac{2}{3}a^3 \underbrace{\int_0^{2\pi} d\theta}_{\theta\big|_0^{2\pi}} = \frac{2a^3}{3}(2\pi) = \frac{4\pi a^3}{3} \checkmark$$

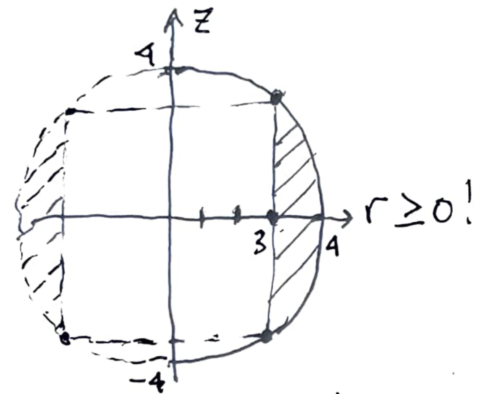
Wedding ring setup (1)

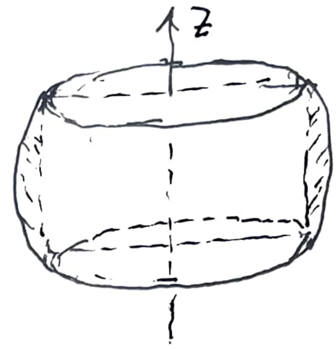
Drill a 3 unit radius hole through the center of a 4 unit radius sphere.
set up with center at origin.

sphere: $x^2+y^2+z^2=4^2 \longrightarrow \rho^2=4^2 \to \rho=4$

cylinder (hole): $x^2+y^2=3^2 \longrightarrow r^2=3^2 \to r=3$

$\to r^2+z^2=4^2$





solid of revolution

$0 \le \theta \le 2\pi$

integrate last in triple integral

revolved around to rest of plane (mirror reflection)

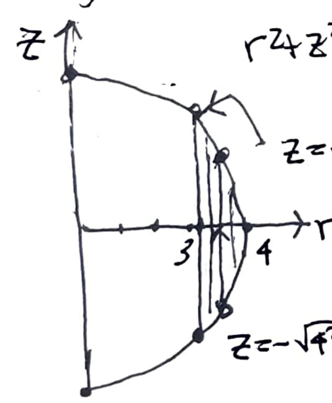
r-z halfplane (at fixed angle θ)

intersection points:

$r^2+z^2=4^2 \to z^2=4^2-3^2=7 \to z=\pm\sqrt{7}$

$r=3$

Integration order in r-z half plane: z-first (vertical) or r-first (horizontal)

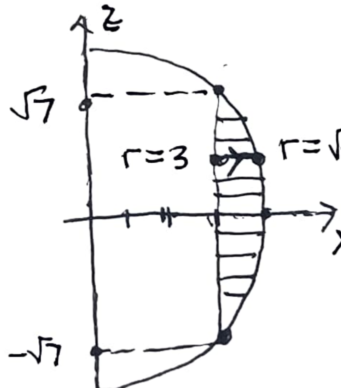


$r^2+z^2=4^2$

$z=\pm\sqrt{4^2-r^2}$

$z=\sqrt{4^2-r^2}$

$z=-\sqrt{4^2-r^2}$



$r^2+z^2=4^2$

$r=\sqrt{4^2-z^2} \ge 0$

$r=3$

$r=\sqrt{4^2-z^2}$

$z=-\sqrt{4^2-r^2} \,..\, \sqrt{4^2-r^2}$

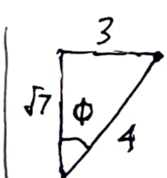
while $r=3 \,..\, 4$

vertical linear cross-section ⇒ cylinder

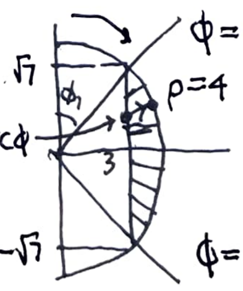
$r=3 \,..\, \sqrt{4^2-z^2}$

while $z=-\sqrt{7} \,..\, \sqrt{7}$

horizontal linear cross-section ⇒ annulus



OR POLAR COORDS in r-z halfplane (spherical coords)



$\rho=3\csc\phi$

$\phi=\phi_1$ (acute)

$\phi=\phi_2=\pi-\phi_1$ (supplementary angles)

$\rho=3\csc\phi \,..\, 4$

while $\phi=\arcsin\frac{3}{4} \,..\, \pi-\arcsin\frac{3}{4}$

while $\theta=0 \,..\, 2\pi$

$z=\rho\cos\phi$

$r=\rho\sin\phi=3$

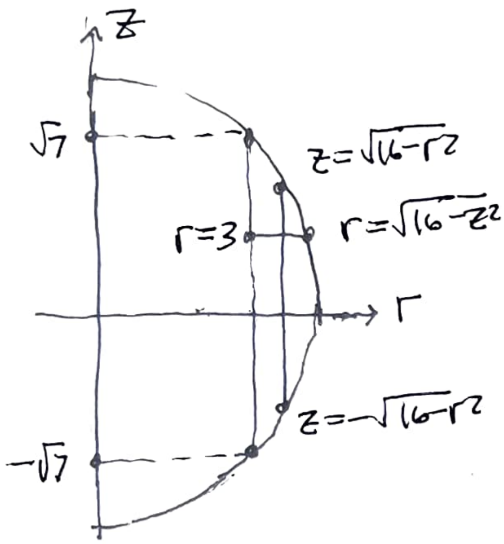
$\frac{r}{z}=\tan\phi$

$\rho=3\csc\phi$

$\phi=\arctan\left(\frac{3}{\sqrt{7}}\right)=\arccos\left(\frac{\sqrt{7}}{4}\right)$

$=\arcsin(3/4)$

Wedding Ring setup (2): integration



inner double integral diagram

z-first (vertical)

always do θ integral last

$$\int_0^{2\pi}\int_3^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} f\ \underbrace{r\,dz\,dr\,d\theta}$$

$$dV = dz\,dr\,(r\,d\theta)$$

nothing depends on θ, factors out. do at any time $\left(\int_0^{2\pi} 1\,d\theta = 2\pi\right)$ ← (if revolved 360°, otherwise limit range of θ)

r-first (horizontal)

$$\int_0^{2\pi}\int_{-\sqrt{7}}^{\sqrt{7}}\int_3^{\sqrt{16-z^2}} f\ \underbrace{r\,dr\,dz\,d\theta}$$

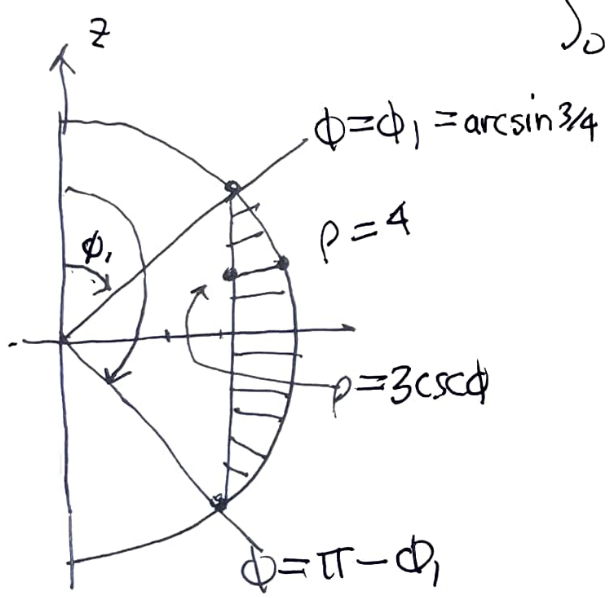
$$dV = \underbrace{(dr)(dz)}\,\underbrace{(r\,d\theta)}$$

$$\underbrace{d\rho(\rho\,d\phi)\,(\rho\sin\phi)\,d\theta}$$

$$\rho^2\sin\phi\ d\rho\,d\phi\,d\theta$$

geometric correction factor = $\rho\cdot r$ product of two arc radii converting $d\phi$ and dr to arclength differentials.

do ρ first (like do r first)



ρ-first (radial)

$$\int_0^{2\pi}\int_{\arcsin 3/4}^{\pi-\arcsin 3/4}\int_{3\csc\phi}^{4} f\ \rho^2\sin\phi\ d\rho\,d\phi\,d\theta$$

radial integral always done first

since $\rho=\rho(\phi)$ always defines curves of variable ϕ.

Cartesian to cylindrical / spherical integration

convert $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} x\,y \, dz\,dy\,dx \equiv Q$

$$\underbrace{\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{2-x^2-y^2}} x\,y\,dz\,dy\,dx$$

$z^2 = 2-x^2-y^2$

$\underbrace{x^2+y^2}_{r^2} + z^2 = 2 = \rho^2$

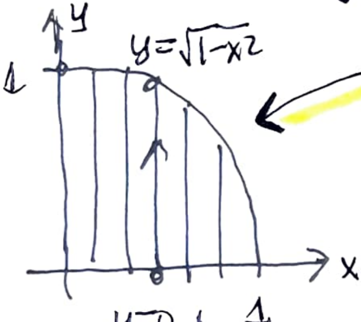
$r^2+z^2 = 2 \rightarrow z = \pm\sqrt{2-r^2}$

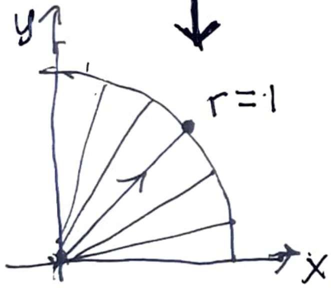
$y^2 = 1-x^2$

$x^2+y^2 = 1$

$z^2 = x^2+y^2$ cone

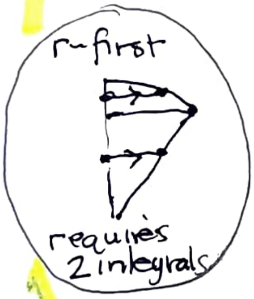
$z = r$

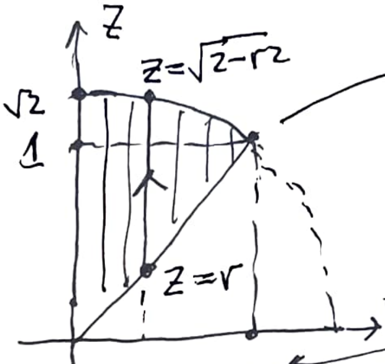




$r = 0..1$ while $\theta = 0..\frac{\pi}{2}$

outer integral





intersect:

$r = \sqrt{2-r^2}$

$r^2 = 2-r^2 \rightarrow r^2 = 1$

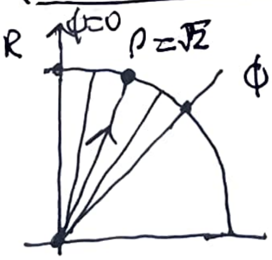
$r = 1$

$z = r = 1$

inner double integral

$z = r..\sqrt{2-r^2}$ while $r = 0..1$

OR



$\phi = \pi/4 \leftarrow (\tan\phi = 1)$

$\rho = 0..\sqrt{2}$ while $\phi = 0..\pi/4$.

3d!

1/4 of a snow cone!

$dA = r\,dr\,d\theta$

$$Q = \int_0^{\frac{\pi}{2}} \int_0^1 \int_r^{\sqrt{2-r^2}} \underbrace{\overbrace{r(\cos\theta)}^{x}\overbrace{(r\sin\theta)}^{y}\, r}_{3 \text{ factors of } r}\; \underbrace{dz\,dr}_{d\rho\,(\rho\,d\phi)}\,d\theta$$

$dV = \underbrace{r}_{\rho\sin\phi}\rho\,d\rho\,d\phi\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \underbrace{(\rho\sin\phi)^3 \quad \theta\sin\theta\,\rho}_{\underbrace{(\rho\sin\phi\cos\theta)}_{x}\underbrace{(\rho\sin\phi\sin\theta)}_{y}\,(\rho^2\sin\phi)}\; d\rho\,d\phi\,d\theta$$