changing the variable in an integral

integration techniques so far

unlike differentiation which has a well defined set of rules that enable us to differentiate any even very complicated function expressed by combining the familiar elementary functions, integration is a collection of an enormous list of special recipes that are not straightforward to apply.

Because we now have "computer algebra systems" which can integrate any expression that can be integrated in terms of familiar and even unfamiliar functions, we only need to "learn" a few basic techniques to appreciate how the game is played and use Maple when we need antiderivatives.

In addition to the basic properties of antiderivatives we already know at this point, we will only consider two special techniques: 1) "u-substitution" (change of variable)
2) integration by parts (reversing the product rule).

→ 7.1 integration by parts

7.2–7.5 waste of time
comparable to long division
(will not be covered)

The least important part of calculus is evaluating derivatives or antiderivatives. Knowing what to do with the results is what is really important.

5.5a changing the variable in an integral

What do we know so far? We have a short list of basic functions and their antiderivatives.

SHORT LIST

$\frac{d}{dx}(x^p) = px^{p-1} \longrightarrow \int x^p\,dx = \frac{x^{p+1}}{p+1} + C \quad (p \neq -1)$ (power rule)

$\frac{d}{dx} e^x = e^x \longrightarrow \int e^x\,dx = e^x + C$

$\frac{d}{dx} \ln|x| = \frac{1}{x} \longrightarrow \int \frac{1}{x}\,dx = \ln|x| + C$

trig $\frac{d}{dx}\sin x = +\cos x \longrightarrow \int \cos x\,dx = \sin x + C$

$\frac{d}{dx}\cos x = -\sin x \longrightarrow \int \sin x\,dx = -\cos x + C$

$\frac{d}{dx}\tan x = \sec^2 x \longrightarrow \int \sec^2 x\,dx = \tan x + C$

$\frac{d}{dx}\sec x = \sec x \tan x \longrightarrow \int \sec x \tan x\,dx = \sec x + C$

[etc cot, csc]

arctrig $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \longrightarrow \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

etc...

We can combine these with sums/differences/constant multipliers:

$$\frac{d}{dx}\left(c_1 F(x) + c_2 G(x)\right) = c_1 F'(x) + c_2 G'(x) = c_1 f(x) + c_2 g(x)$$

$$\int c_1 f(x) + c_2 g(x)\,dx = c_1 \underbrace{\int f(x)\,dx}_{F(x)} + c_2 \underbrace{\int g(x)\,dx}_{G(x)}$$

(integrate "term by term") ← adjust for arbitrary constants

We can also expand out or rewrite an integrand first:

Example: $\int \frac{1+x}{x^{1/2}}\,dx = \int \frac{1}{x^{1/2}} + \frac{x}{x^{1/2}}\,dx = \int x^{-1/2} + x^{1/2}\,dx$

$$= \frac{x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} + C$$

In the same way that the sum rule for differentiation extends to the sum rule for integration (ditto for the constant factor rule), the chain rule for composed functions leads to an integration method.

changing the variable in an integral

The $\boxed{\text{chain rule}}$ for differentiation can be reversed to make a method for antidifferentiation:

$$\frac{d}{dx} F(\underbrace{g(x)}_{\text{identify inner function as new variable}}) = F'(g(x))\, g'(x)$$

$$\begin{cases} u = g(x) \\ \frac{du}{dx} = g'(x) \\ du = g'(x)\,dx \end{cases}$$

Multiply by dx :

$$\frac{d}{dx} F(g(x))\, dx = F'(\underbrace{g(x)}_{u})\, \underbrace{g'(x)\,dx}_{du} = F'(u)\, du$$

Integrate:

$$\underbrace{\int \frac{d}{dx} F(g(x))\, dx}_{= F(g(x)) + C\,!} = \underbrace{\int F'(g(x))\, g'(x)\, dx}_{} = \underbrace{\int F'(u)\, du}_{= F(u) + C}$$

this is a complicated product integrand but if we can "identify" F and g in it, we can "unlock" the integral and find an antiderivative.

Consistent with LHS!

simplified integral easy to integrate if recognize F(u)

BUT not that difficult! All we need to do is identify an inner function of a composed function to change the variable $u = g(x)$ and see where it goes by re-expressing the integrand and differential in terms of the new variable.

An example shows how straightforward this can be.

5.5a Changing the variable in an integral

Example $\int 2x\sqrt{1+x^2}\,dx = \int (1+x^2)^{1/2}\, 2x\,dx$

obvious composition

$u = 1+x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$ = du!

$= \int u^{1/2}\,du$

$= \frac{u^{3/2}}{3/2} + c = \frac{2}{3}u^{3/2} + c \underset{\text{go back}}{=\!=} \frac{2}{3}(1+x^2)^{3/2} + C$

EASY! But the integrand does not have to be precisely of this chain rule form.

Example $\int x^3\sqrt{1+x^2}\,dx = \int (1+x^2)^{1/2}\, x^2\, x\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$\frac{du}{2}$

leftover factor, just re-express in terms of $u = 1+x^2$

$u-1 = x^2$

$x^2 = u-1$!

$= \int u^{1/2}(u-1)\frac{du}{2}$ expand out

$= \frac{1}{2}\int u^{1/2}(u-1)\,du$

$= \frac{1}{2}\int u^{1/2}u - u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du$

$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + c = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + c$

$= \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + c$ (simplified in new variable)

$= \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + c$ (return to original variable)

changing the variable in an integral

What about definite integrals? The only extra piece is adding the limits of integration.

$$\int x^3\sqrt{1+x^2}\,dx = \int u^{3/2}-u^{1/2}\,du =$$
$$= \frac{1}{5}u^{5/2}-\frac{1}{3}u^{3/2}+C$$
$$= \frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C$$

But $\int_0^1 x^3\sqrt{1+x^2}\,dx = \int_{x=0}^{x=1} x^3\sqrt{1+x^2}\,dx = \int_{u=?}^{u=?} u^{3/2}-u^{1/2}\,du$

identifies variable whose values are limits } more explicit

here we need u values

Calculate corresponding values:

$u=1+x^2$
- $x=0 \to u=1+0^2=1$ (lower)
- $x=1 \to u=1+1^2=2$ (upper)

$$= \boxed{\int_1^2 u^{3/2}-u^{1/2}\,du}$$

new form of definite integral

We have 2 choices.
1) go back to original variable
2) stay with new variable

$$\int_0^1 x^3\sqrt{1+x^2}\,dx = \frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}\Big|_0^1$$
$$= \frac{1}{5}(2)^{5/2}-\frac{1}{3}(2)^{3/2}-\left[\frac{1}{5}(1)^{5/2}-\frac{1}{3}(1)^{3/2}\right]$$

$$= \frac{1}{5}u^{5/2}-\frac{1}{3}u^{3/2}\Big|_1^2 = \frac{1}{5}(2)^{5/2}-\frac{1}{3}(2)^{3/2}-\left[\frac{1}{5}(1)^{5/2}-\frac{1}{3}(1)^{3/2}\right]$$

same just different order of operations

finish evaluation $= 2^{3/2}\left(\frac{2}{5}-\frac{1}{3}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)$

$$= \frac{2^{3/2}}{15}+\frac{2}{15} = \boxed{\frac{2}{15}(\sqrt{2}+1)}$$

always simplify, check with Maple!

5.5a Changing the variable in an integral

Example $\displaystyle\int \frac{dx}{(3-5x)^2} = \int \underbrace{(3-5x)}^{-2}\,dx$

$u = 3-5x$ (linear change)

$\frac{du}{dx} = -5$

$du = -5\,dx$ (differential times constant)

$dx = -\frac{1}{5}\,du$ (divide by constant)

$$= \int u^{-2}\left(-\tfrac{1}{5}du\right) = -\frac{1}{5}\int u^{-2}\,du = -\frac{1}{5}\frac{u^{-1}}{-1} + C = \frac{1}{5u} + C$$

$$= \frac{1}{5(3-5x)} + C$$

[clearly this generalizes: $\displaystyle\int \underbrace{(ax+b)}^{p}\,dx = \int u^p \frac{du}{a} = \frac{1}{a}\frac{u^{p+1}}{p+1} + C = \frac{1}{a}\frac{(ax+b)^{p+1}}{p+1} + C$]

$u = ax+b$

$du = a\,dx$

just divide by coefficient of x

Example Pick any function from our short list and multiply x by a:

$f(x) = \cos ax \rightarrow \displaystyle\int \cos \underbrace{ax}_{u}\,\underbrace{dx}_{\frac{du}{a}} = \frac{1}{a}\int \cos u\,du = \frac{1}{a}\sin u + C$

$= \frac{1}{a}\sin ax + C$

differentiation: multiply by a

integration: divide by a

Example $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_{x=1}^{x=e} \frac{\ln x}{x}\,dx$ ($\ln x \to u$, $dx \to du$)

$\rightarrow$ here composition with outer function not obvious since it is the identity:

$g(\ln x) = \ln x$

$g(x) = x$

so "inner function" is

$u = \ln x$ $\begin{cases} x=1 \rightarrow u = \ln 1 = 0 \\ x=e \rightarrow u = \ln e = 1 \end{cases}$

$\frac{du}{dx} = \frac{1}{x} \rightarrow du = \frac{dx}{x}$

$$= \int_{x=1}^{x=e} u\,du = \int_0^1 u\,du$$

new limits for u

return to x

$$= \frac{u^2}{2}\Big|_{x=1}^{x=e} = \frac{(\ln x)^2}{2}\Big|_1^e$$

$$= \frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} = \frac{1}{2}$$

$$= \frac{u^2}{2}\Big|_0^1$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$