# OPTIMIZATION

$\boxed{\text{Word Problem}} \xrightarrow{\text{translate}} \boxed{\text{Math Problem}}$

**Word Problem**

maximize / minimize some quantity

may involve more than 1 variable

but "constraints" must eliminate extras

(geometry easy to set up such problems)

↓

plain English response (no variable names typically)

**Math Problem**

find global max/min of $f(x)$

on some interval: $[a,b]$, $(a,b)$ $(a,\infty)$, etc

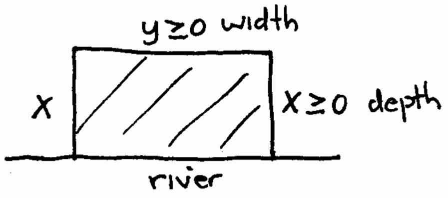
final output: $x$ or $f(x)$ or both (or extra vars)

interpret ← result

> A farmer has 2400 ft of fencing,
>
> and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river.
>
> What are the dimensions of the field that has the largest area?

↓ find $(x,y)$

diagram + variables →



"constraint": $2x+y=2400$

↓ eliminate 1 variable

$y=2400-2x\ge 0$

$1200\ge x$

so $0\le x\le 1200$ domain

$A=xy$ ← maximize

make ↓ function of one variable

$A(x)=x(2400-2x)$

$=2x(1200-x)$ factor for zeros

$=2400x-2x^2$ expand for D

maximize $A(x)$ on interval $[0,1200]$:

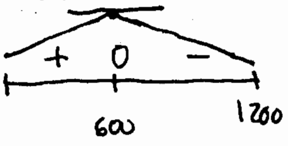
$A'(x)=2400-4x=4(600-x)=0$

$\to x=600 \to y=2400-2(600)=1200$

$A''(x)=-4<0$ $\frown$ everywhere

2nd der test: $\frown$ local max

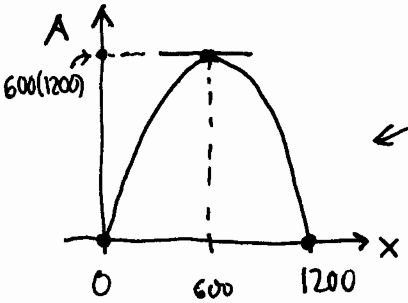
1st der test:



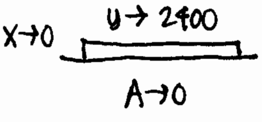
must be global max

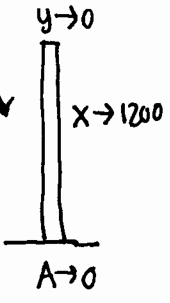
$(x,y)=(600,1200)$

sketch graph on domain:



limiting configurations



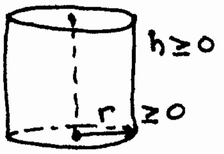


A field with depth 600 ft away from the river and width 1200 ft along the river has the largest area.

# OPTIMIZATION (2)

A cylindrical can is to be made to hold 1 L of oil ($1L = 1000 cm^3$). Find the dimensions that will minimize the cost of the metal to manufacture the can

diagram variables



$V = \pi r^2 h = 1000$ constraint of fixed volume

eliminate h to avoid square roots of r

$h = \frac{1000}{\pi r^2}$

find $(r, h)$

i.e. area

$A = 2(\pi r^2) + (2\pi r)h$

bot + top side

$2\pi r\left(\frac{1000}{\pi r^2}\right) = \frac{2000}{r}$

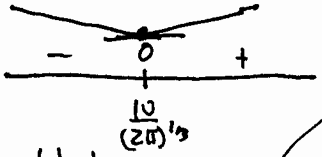
$$\text{minimize: } A(r) = 2\pi r^2 + \frac{2000}{r}, \quad r > 0$$

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = 0$$

$A''(r) = 4\pi + \frac{4000}{r^3} > 0$, ∪ everywhere.

2nd der test ∪ local min

1st der:



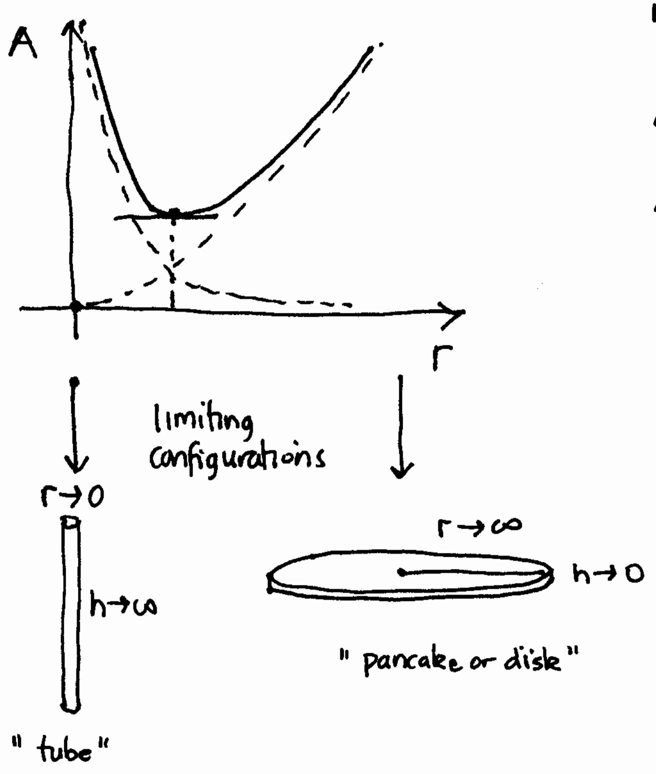
global min

$r^3 = \frac{1000}{2\pi}$

$r = \frac{10}{(2\pi)^{1/3}}$

$$h = \frac{1000}{\pi\left(\frac{10}{(2\pi)^{1/3}}\right)^2} = \frac{1000\,(2\pi)^{2/3}}{100\;\pi} = 2\cdot 10\,\frac{(2\pi)^{2/3}}{2\pi} = 2\cdot\frac{10}{(2\pi)^{1/3}} = 2r$$

= diameter



The can with the least area and hence the least cost for the metal needed to manufacture it has height and diameter both equal to

$\underbrace{\frac{20}{(2\pi)^{1/3}}}_{} \approx 10.84$ cm $\approx 4.27$ in.

exact answer nice but need decimal equivalent for interpretation with units

$\left[\frac{2\cdot 1000^{1/3}}{(2\pi)^{1/3}} = 2\left(\frac{V}{2\pi}\right)^{1/3}\right.$ in terms of an arbitrary volume $\left.\right]$