Drill a 3 unit radius hole through the center of a 4 unit radius sphere. How can we describe this solid using coordinates? Put the sphere at the origin and align the hole with the z-axis. Then it is a solid of revolution so we can look at its cross-section in the r-z plane.

\[ x^2 + y^2 + z^2 = q^2 \iff \rho^2 = q^2 \Rightarrow \rho = q \]

\[ r^2 = 3^2 \Rightarrow r = 3 \]

\[ z^2 + 2^2 = 4^2 \Rightarrow z = \pm 2 \]

This is a ring shaped solid. The range \( 0 \leq \theta \leq 2\pi \) corresponds to revolving the right half of this diagram about the z-axis.

- **in terms of cylindrical coordinates** \( r, \theta, z \):
  \[ r^2 + z^2 = 4^2 \]
  \[ 3 \leq r \leq \sqrt{16-2^2} \text{ (inner)} \]
  \[ z = \pm 2 \text{ (upper)} \]
  \[ r = 3 \text{ (lower)} \]

We have two ways of describing the solid’s cross-section in the r-z half-plane.

- **in terms of spherical coordinates** \( \rho, \theta, \phi \):
  \[ \phi_1 \leq \phi \leq \phi_2 \]
  \[ 3 \leq \rho \leq 4 \text{ (outer)} \]
  \[ \rho = 3 \text{ (inner)} \]

\[ \phi_1 = \text{arcsin} \frac{2}{4} = 30^\circ \]
\[ \phi_2 = \text{arcsin} \frac{4}{4} = 90^\circ \]

\[ q_{\text{range}}: \cos \phi_1 = \frac{\sqrt{3}}{2} \implies \phi_1 = \text{arcsin} \frac{\sqrt{3}}{2} \approx 108^\circ \]
\[ q_{\text{range}}: \cos \phi_2 = 0 \implies \phi_2 = 90^\circ \]

Now we need to describe the cylinder \( r=3 \) in terms of spherical coordinates using \( r = \rho \sin \phi \), \( z = \rho \cos \phi \) (see next page).

\( \rho = 3 \sin \phi \Rightarrow \rho = 3 / \csc \phi = 3 \csc \phi \)
Solid regions in space can often be described as between two graphs of functions of two of the three coordinates, as in the previous example, i.e., between two surfaces which can be described by expressing one coordinate as a function of the other two (even if it depends only on one of them).

Here is another example of describing a surface in the three coordinate systems:

\[ x^2 + y^2 - z^2 = 16 \]

- Surface of revolution since it only depends on \( x^2 + y^2 = r^2 \)

- Hyperbola in \( r^2 \)-plane only has:
  - R-intercepts: \( z = 0 \rightarrow r = 4 \)
  - So opens up about \( z = 0 \)-axis.

When \( r, z \) are very large compared to \( 16 \):

\[ r^2 - z^2 \approx 0 \rightarrow z = \pm r \text{ (asymptotes)} \]

It is natural to express \( r \geq 0 \) in terms of \( z \):

\[ \begin{align*}
  r &= \sqrt{16 + z^2} \\
  -\infty &< z \leq \infty \\
  0 &\leq \theta \leq 2\pi
\end{align*} \]

- Cylindrical coordinate description

We can also describe this in spherical coordinates, expressing \( \rho \geq 0 \) in terms of \( \phi \):

\[ \rho^2 - \rho^2 \sin^2 \phi = 16 \]

\[ \rho^2 \sin^2 \phi = \rho^2 \cos^2 \phi = 16 \]

\[ \rho^2 (\sin^2 \phi - \cos^2 \phi) = 16 \]

\[ \rho^2 = \frac{16}{\sin^2 \phi - \cos^2 \phi} = \frac{16}{(\cos^2 \phi) - \cos^2 \phi} = \frac{16}{1 - 2 \cos^2 \phi} \]

Allowed values of \( \phi \):

\[ 1 - 2 \cos^2 \phi \leq 0 \]

\[ \frac{1}{2} \leq \cos^2 \phi \leq 1 \]

\[ 0 \leq \phi \leq \frac{\pi}{3} \]

Also clear from asymptotes: \( z = \pm r \) (slope \( \pm 1 \)).

- Spherical coordinate description

\[ \rho = 4 \]

\[ 0 \leq \phi \leq \frac{\pi}{3} \]

\[ 0 \leq \theta \leq 2\pi \]