1-D probability distributions

Given that a dart does hit the dartboard from a random dart toss (all locations on dartboard equally likely to be hit), what is the probability that a dart hits:

- A strip between \( x = x_1 \) and \( x = x_2 \) on a square vertically striped dartboard (5 equal width strips)?
- A ring between \( r = r_1 \) and \( r = r_2 \) on a circular dartboard (5 equal radius increment rings)?

**Goal:**

Find:

\[
P(x_1 \leq x \leq x_2) = ?
\]

**Calculate:**

\[
\text{area (strip)} = \left( x_2 - x_1 \right) \frac{R}{2} = \frac{x_2 - x_1}{2} R
\]

\[
= \frac{x_2 - x_1}{2} \int_{x_1}^{x_2} \frac{x}{R} \, dx
\]

\[
\implies \mathcal{O}(x) \leftarrow \text{probability distribution}
\]

\[
\int_{0}^{R} \mathcal{O}(x) \, dx = \int_{0}^{R} \frac{1}{R} \, dx = 1
\]

**Check:**

\[
\text{area (total prob.)} \approx \frac{1}{5} R = \frac{1}{5} \times \frac{R}{2} = \frac{R}{10}
\]

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**Expected position ("average" position)?**

- **Discrete:**
  \[
  \langle x \rangle = \frac{1}{5} \left( \frac{5}{10} x_1 + \frac{5}{10} x_2 + \frac{5}{10} x_1 + \frac{5}{10} x_2 + \frac{5}{10} x_1 \right)
  \]
  \[
  = \frac{1}{5} \times (x_1 + x_2) = \frac{x_1 + x_2}{2}
  \]

- **Continuous:**
  \[
  x^* \Delta R \rightarrow \int_{x_1}^{x_2} \frac{1}{R} \, dx = \frac{x_2 - x_1}{R}
  \]

**Uniform prob. distribution:**

\[
\text{sum} = \frac{5}{10} = \frac{1}{2}
\]

\[
\text{"biased towards larger } r\text{"}
\]

**Expected Value:**

\[
\langle r \rangle = \frac{1}{5} \left( \frac{5}{10} r_1^2 + \frac{5}{10} r_2^2 + \frac{5}{10} r_1^2 + \frac{5}{10} r_2^2 + \frac{5}{10} r_1^2 \right)
\]

\[
= \frac{1}{5} \times (r_1^2 + r_2^2 + r_1^2 + r_2^2 + r_1^2) = \frac{5}{5} R^2 = \frac{R^2}{2}
\]

\[
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1-D probability distributions (2)

b) semi-infinite "Poisson" distribution

\[ \int_{0}^{\infty} A e^{-c x} dx = -\frac{A}{c} e^{-c x} \bigg|_{0}^{\infty} = \frac{A}{c} e^{0} = \frac{A}{c} = A = C \]

\[ \langle x \rangle = \int_{0}^{\infty} x C e^{-c x} dx = -\left( \frac{c x + 1}{c} \right) e^{-c x} \bigg|_{0}^{\infty} = \frac{1}{c} (1 - \frac{1}{c}) \equiv \mu \]

by limit test rule

y +
\[ y = \delta(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

spread 2.0

peak

\[ \int_{-\infty}^{\infty} \delta(x) dx = 1 \] (we can't show this but Calc 3 can)

probability of being one "standard deviation" from the "average" value \( \mu \):

\[ \int_{\mu - \sigma}^{\mu + \sigma} \delta(x) \, dx \]

variable change

\[ u = \frac{x - \mu}{\sigma} \]

\[ \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du \]

"Standard normal curve" in units of standard deviation from expected value

\[ \sigma \]

\[ \mu \]

\[ y = \delta(x) \geq 0 \]

\[ P(a \leq x \leq b) = \int_{a}^{b} \delta(x) dx = \frac{1}{b-a} \delta(x) \]

\[ \langle x \rangle = \frac{1}{b-a} \int_{a}^{b} x \delta(x) dx = \frac{x^2}{2(b-a)} \bigg|_{a}^{b} = \frac{b^2 - a^2}{2(b-a)} = \frac{ab}{2} \]

(midpoint)