A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area \(A\) of the window as a function of the width \(x\) of the window.

This is a nice problem because it shows how domain and range are very important in a simple geometry problem. All physical dimensions must be nonnegative and this leads to a finite interval for the domain and range here.

Let \(y\) be the height of the rectangular part of the window and let \(r = x/2\) be the radius of the semicircle.

\[
\begin{align*}
C_n &= \frac{1}{2}(2\pi r) = \pi r = \frac{\pi x}{2} \\
C_u &= x + 2y \\
\text{Circumference formulas}
\end{align*}
\]

\[
P = C_n + C_u = \frac{x + 2y + \pi x}{2} = 30
\]

Constraint on \(x\) & \(y\), only one is independent; solve for one of two and eliminate:

\[
2y = 30 - x - \frac{\pi x}{2}
\]

\[
y = 15 - \frac{1}{2}(1 + \frac{\pi}{4})x \geq 0 \quad x \geq 0
\]

Constraints on \(x\)

\[
A_\triangle = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi \left(\frac{x}{2}\right)^2 = \frac{\pi x^2}{8}
\]

\[
A = xy = x \left(15 - \frac{1}{2}(1 + \frac{\pi}{4})x\right) = 15x - \frac{1}{2}(1 + \frac{\pi}{4})x^2
\]

\[
A = A_\triangle + A = 15x - \frac{1}{2}(1 + \frac{\pi}{4})x^2 + \frac{\pi x^2}{8} = 15x - \frac{1}{2}(1 + \frac{\pi}{4})x^2
\]

\[
\left(-\frac{1}{2} - \frac{\pi}{4}\right)x^2 = \left(-\frac{1}{2} - \frac{\pi}{4}\right)x^2 = -\frac{1}{2}(1 + \frac{\pi}{4})x^2
\]

So \(A(x) = 15x - \frac{1}{2}(1 + \frac{\pi}{4})x^2 \leftarrow \text{just a quadratic function, graph is U-shaped}
\]

\(\text{parabola crosses x-axis at } x = 0, \quad \frac{30}{1 + \frac{\pi}{4}} \approx 16.80
\]

\(\text{peak of parabola at halfway point: } \frac{15}{1 + \frac{\pi}{4}} \approx 8.40
\]

\(\text{maximum area: } A = \frac{225}{2(1 + \frac{\pi}{4})} \approx 63.01
\]

\(\text{endpoint configurations}
\]

\[\begin{align*}
A &= 0 \\
y &= 15 \\
x &= 0
\end{align*}\]

\[\begin{align*}
x &= 11.7 \\
y &= 0 \\
A &\approx 53.57
\end{align*}\]