Solving a functional relationship between 2 variables for the input variable $m$ (1)

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$  (equation)

$f(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ is the definition of a function which produces a number (with units) from the input; $f$ is the name of this function.

To find a formula for the inverse function, we must solve the equation in which $v$ and $m$ appear together for $v$. The expression we find for it will then define the formula for the inverse function.

$f$ is the following sequence of operations:

- **input:** $v \geq 0$
- **square:** $v^2$
- **divide by $c^2$:** $\frac{v^2}{c^2}$
- **change sign:** $-\frac{v^2}{c^2}$
- **add 1:** $1 - \frac{v^2}{c^2}$
- **take square root:** $\sqrt{1 - \frac{v^2}{c^2}}$
- **take reciprocal:** $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$
- **multiply by $m_0$:** $\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$
- **multiply by $m$:** $\frac{m_0}{m} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

**simplification steps:**

- take square root: $\sqrt{c^2(1 - \frac{m_0^2}{m^2})} = \sqrt{v^2} = v$ (since $v \geq 0$)
- multiply by $c^2$: $c^2(1 - \frac{m_0^2}{m^2}) = v^2$
- change sign: $1 - \frac{(m_0)^2}{m^2} = -\frac{v^2}{c^2}$
- subtract 1: $\frac{(m_0)^2}{m^2} - 1 = -\frac{v^2}{c^2}$
- **square:** $\left(\frac{m_0}{m}\right)^2 = 1 - \frac{v^2}{c^2}$
- **take reciprocal:** $\frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$
- **divide by $m_0$:** $\frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

**reverse process: to go from output back to input:** one-by-one apply inverse operations in reverse order

**final result:**

$$v = c\sqrt{1 - \frac{m_0^2}{m^2}}$$  (function of variable $m$: $f^{-1}(m) = c\sqrt{1 - \frac{m_0^2}{m^2}}$)

**domain of $f^{-1}$**

what are the allowed input values $m$?

$$1 - \frac{m_0^2}{m^2} \geq 0 \quad \text{solve inequality} \quad 1 \geq \frac{m_0^2}{m^2} \quad m^2 \geq m_0^2 \quad m \geq m_0$$

Same for domain of $f$:

$$1 - \frac{v^2}{c^2} > 0 \rightarrow 1 > \frac{v^2}{c^2} \rightarrow c^2 > v^2 \rightarrow c > v \quad \text{or} \quad v < c$$

since by assumption $v \geq 0$:

$$0 \leq v < c$$
Graphs

These are the same curves and same relationship between the two variables, but with the axes interchanged.

While the axes are still labeled \( v \) and \( m \), it makes no sense to graph them together with these orientations since one then cannot use either variable name to label the axes.

The two constants \( c \) (with the same velocity units as the variable \( v \), so that their ratio \( v/c \) is dimensionless) and \( m_0 \) (with the same mass units as the variable \( m \), so that their quotient is dimensionless) set the scale for the two axes.

We could re-express these two equations in dimensionless form:

\[
\frac{m}{m_0} = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \rightarrow \quad M = \frac{1}{\sqrt{1-v^2}} = F(v)
\]

\[
\frac{v}{c} = \sqrt{1-(\frac{m_0}{m})^2} \rightarrow \quad \frac{v}{c} = \sqrt{1-\frac{1}{M^2}} = F^{-1}(M)
\]

This corresponds to measuring the variables by multiples of these 2 constants:

"Half the speed of light" : \( v = \frac{1}{2}c \rightarrow \frac{v}{c} = \frac{1}{2} \)

"twice the rest mass" : \( m = 2m_0 \rightarrow M = 2 \).

We can study the dimensionless mathematical functions with the default names \( x \) and \( y \) for inputs and outputs and graph them on the same axes:

\[
y = F(x) = \frac{1}{\sqrt{1-x^2}}
\]

\[
y = F^{-1}(x) = \sqrt{1-x^2}
\]

Now we don't have to worry about conflicting interpretations of variable names. \( x \) is just the input, \( y \) the output.

[By introducing different variable names \( x \) and \( y \) we could even make such a graph for the original functions.]

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