Proceedings of ERES 2000 (Encuentros Relativistas Espanoles)
Eds. J.-F. Pascual-Sánchez, L. Floría, A. San Miguel, and F. Vicente

GRAVITOELECTROMAGNETISM: APPLICATIONS TO BLACK HOLE CIRCULAR ORBITS

D. BINI
Istituto per Applicazioni della Matematica, CNR, I-80131, Napoli
and International Center for Relativistic Astrophysics - I.C.R.A.
University of Rome "La Sapienza", I-00185, Roma
E-mail: binid@icra.it

R.T. JANTZEN
Department of Mathematical Sciences
Villanova University, Villanova, PA 19085, USA
and International Center for Relativistic Astrophysics - I.C.R.A.
University of Rome "La Sapienza",I-00185, Roma
E-mail: robert.jantzen@villanova.edu

The definition of inertial forces in general relativity is discussed and then illustrated for circular orbits in black hole spacetimes.

1 Introduction

Gravitoelectromagnetism has been used to analyse test particle motion (circular orbits) in black hole spacetimes, where it has been possible to study inertial forces according to the most appropriate general relativistic definition and compare with the corresponding Newtonian behaviour. Relative Frenet-Serret frames were introduced which generalize the spatial Frenet-Serret machinery needed for the analysis of test particle motion in classical mechanics. This approach and its application to general circular (nonequatorial orbits) in a Kerr spacetime are reviewed here.

2 Inertial Forces in Classical Mechanics

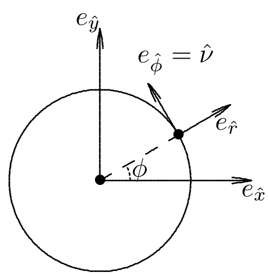
Consider the classical example of a (unit mass) particle in uniform circular motion (counter-clockwise motion for example, as indicated in Figure 1).

The orbit of the particle can be parametrized by the spatial arclength ℓ, $\ell = R\phi = R\Omega t$ so that its cartesian representation is

$$(x, y) = (R\cos\ell/R, R\sin\ell/R) \ . \tag{1}$$

The intrinsic (Frenet-Serret) frame is

$$\mathbf{t} = e_{\hat\phi} = \hat\nu, \qquad \mathbf{n} = e_{\hat r}, \qquad \mathbf{b} = e_{\hat\phi} \times e_{\hat r} = -e_{\hat z} \tag{2}$$



$$
\begin{aligned}
e_{\hat r} &= \cos\phi e_{\hat x} + \sin\phi e_{\hat y}\\
e_{\hat\phi} &= -\sin\phi e_{\hat x} + \cos\phi e_{\hat y}\\
&= de_{\hat r}/d\phi
\end{aligned}
$$

$$\dot\phi = \Omega = \text{const.}, \qquad \nu = \Omega R$$

Figure 1: Circular motion in classical mechanics.

and the associated properties can easily be obatined (curvature $\kappa = -1/R$, torsion $\tau = 0$, Frenet-Serret angular velocity $\omega_{FS} = \kappa e_{\hat z}$); the following relations are also useful:

$$\frac{de_{\hat\phi}}{d\ell} = \frac{1}{R}\frac{de_{\hat\phi}}{d\phi} = -\frac{1}{R}e_{\hat r} = \kappa\mathbf{n}, \qquad \frac{de_{\hat r}}{d\ell} = \frac{1}{R}\frac{de_{\hat r}}{d\phi} = -\frac{1}{R}e_{\hat\phi} = -\kappa\mathbf{t}\ , \tag{3}$$

and the classical notion of centripetal force (= − centrifugal force)

$$F^{(C)} = -\frac{\nu^2}{R}e_{\hat r} = \kappa\nu^2 e_{\hat r} = \nu^2\frac{de_{\hat\phi}}{d\ell} = \nu^2\frac{d\hat\nu}{d\ell}, \tag{4}$$

so that attraction from the origin of the coordinates by a constant force radial force results in a circular motion for the particle from its own point of view only if this last force is exactly balanced by the centrifugal force.

3 Inertial Forces in General Relativity

Centripetal and centrifugal forces are equivalent notions in Newtonian physics, because of its universal time shared by the reference frame and the rotating particle. In general relativity, the local rest space of the particle is distinct from that of the observer, each with its own local direction of time, so centripetal and centrifugal forces must differ. The most natural analysis on how to import these Newtonian notions into general relativity has been recently accomplished by introducing "relative Frenet-Serret" frames.

Consider a particle (a single timelike world line, U) and a family of test observers (a congruence of timelike world lines, u) using the notation introduced in the companion article in these proceedings. There arise two relative velocity unit vector fields $\hat\nu(U,u) \in LRS_u$ and $\hat\nu(u,U) \in LRS_U$, related by a boost, and defined **only** along the particle world line U.

Studying the evolution (along U) of $\hat\nu(U,u)$ as measured by its covariant derivative and projecting it onto the local rest space of the observer u using the spatial projection operator $P(u)$, one has the Fermi-Walker temporal derivative (spatial with respect to u)

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_U}\hat\nu(U,u) = P(u)\frac{D}{d\tau_U}\hat\nu(U,u) = \gamma\nu k_{(\mathrm{fw},U,u)}\hat\eta_{(\mathrm{fw},U,u)} \tag{5}$$

which naturally defines the "relative centripetal force." Studying instead the evolution (along U) of $-\hat\nu(u,U)$ (the relative velocity of U with respect to u as seen by U) and projecting its covariant derivative onto the local rest space of the particle itself U using the projection operator $P(U)$, one has the Fermi-Walker temporal derivative (spatial with respect to U)

$$\frac{D_{(\mathrm{fw},U)}}{d\tau_U}[-\hat\nu(u,U)] = P(U)\frac{D}{d\tau_U}[-\hat\nu(U,u)] = \gamma\nu\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} \tag{6}$$

which naturally defines the "relative centrifugal force".

Both the relative velocity unit vector fields, $\hat\nu(U,u)$ and $-\hat\nu(u,U)$, can be used in turn to introduce a "relative Frenet-Serret" procedure to define spatial frames $(-\hat\nu(u,U), \hat{\mathcal{N}}_{(\mathrm{fw},u,U)}, \hat{\mathcal{B}}_{(\mathrm{fw},u,U)})$ in LRS_U and $(\hat\nu(U,u), \hat\eta_{(\mathrm{fw},U,u)}, \hat\beta_{(\mathrm{fw},U,u)})$ in LRS_u satisfying the following relations

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U)}}{d\tau_U}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} &= \gamma\nu[\mathcal{K}_{(\mathrm{fw},u,U)}\hat\nu(u,U) + \mathcal{T}_{(\mathrm{fw},u,U)}\hat{\mathcal{B}}_{(\mathrm{fw},u,U)}] \ , \\
\frac{D_{(\mathrm{fw},U)}}{d\tau_U}\hat{\mathcal{B}}_{(\mathrm{fw},u,U)} &= -\gamma\nu\mathcal{T}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} \ ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U,u)}}{d\tau_U}\hat\eta_{(\mathrm{fw},U,u)} &= \gamma\nu[k_{(\mathrm{fw},U,u)}\hat\nu(U,u) + \tau_{(\mathrm{fw},U,u)}\hat\beta_{(\mathrm{fw},U,u)}] \ , \\
\frac{D_{(\mathrm{fw},U,u)}}{d\tau_U}\hat\beta_{(\mathrm{fw},U,u)} &= -\gamma\nu\tau_{(\mathrm{fw},U,u)}\hat\eta_{(\mathrm{fw},U,u)} \ .
\end{aligned}$$

Using these frames to decompose the force equation $a(U) = f(U)$ for the test particle U along the transverse directions $\hat{\mathcal{N}}_{(\mathrm{fw},u,U)}$ and $\hat\eta_{(\mathrm{fw},U,u)}$ in the local rest spaces of the observer and the particle leads to the most natural definition of relative centripetal/centrifugal forces

$$\begin{aligned}
\mathcal{F}^{(C)}_{(\mathrm{fw},U,u)} &= \gamma\nu^2 k_{(\mathrm{fw},U,u)} = [F_{(U,u)} + F^{(G)}_{(\mathrm{fw},U,u)}]\cdot\hat\eta_{(\mathrm{fw},U,u)} \ , \\
\mathcal{F}^{(C)}_{(\mathrm{fw},u,U)} &= \gamma\nu^2\mathcal{K}_{(\mathrm{fw},u,U)} = [f(U) + \mathcal{F}^{(G)}_{(\mathrm{fw},u,U)}]\cdot\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} \ ,
\end{aligned} \tag{7}$$

where the spatial gravitational forces in each local rest space are

$$
\begin{aligned}
F^{(G)}_{(\mathrm{fw},U,u)} &= -\frac{D_{(\mathrm{fw},U,u)}}{d\tau_U} u \ ,\\
\mathcal{F}^{(G)}_{(\mathrm{fw},u,U)} &= \gamma^{-1} P(U,u) F^{(G)}_{(\mathrm{fw},U,u)} = -\gamma^{-1}\frac{D_{(\mathrm{fw},U)}}{d\tau_U} u \ .
\end{aligned} \tag{8}
$$

4 Spacetime Frenet-Serret frames

The spacetime Frenet-Serret frame along a single test particle world line with 4-velocity $U = e_0$ is described by the equations

$$
\begin{aligned}
\frac{D}{d\tau} e_0 &= \kappa\, e_1 \ , & \frac{D}{d\tau} e_2 &= -\tau_1 e_1 + \tau_2 e_3 \ ,\\
\frac{D}{d\tau} e_1 &= \kappa\, e_0 + \tau_1\, e_2 \ , & \frac{D}{d\tau} e_3 &= -\tau_2 e_2 \ ,
\end{aligned} \tag{9}
$$

where $\kappa = ||a_{(U)}||$ is the curvature of the curve (magnitude of the acceleration) and τ_1 and τ_2 are the first and second torsions respectively, which determine the Fermi-Walker angular velocity

$$
\omega_{(\mathrm{FS})} = \tau_1 e_3 + \tau_2 e_1 \ , \ ||\omega_{(\mathrm{FS})}|| = (\tau_1^2 + \tau_2^2)^{1/2} \tag{10}
$$

of the Frenet-Serret spatial frame with respect to one Fermi-Walker transported along the world line, i.e., with respect to local gyro fixed axes. The angular velocity of a gyro with respect to the spatial frame then has the opposite sign.

For general test particle world lines and observers in a general spacetime, the relative Frenet-Serret frames are not closely related to the spacetime Frenet-Serret frame, but in the case of circular orbits and circularly orbiting observers in stationary axisymmetric spacetimes, they are closely related.

5 Circular orbits in stationary axisymmetric spacetimes: Kerr spacetime, nonequatorial orbits

In order to compare with the classical picture of centripetal/centrifugal forces one can study circular orbits in a familiar spacetime. Early work on circular orbits in the Schwarzschild spacetime and on the equatorial plane of the Kerr spacetime can be framed in the more general context of nonequatorial (accelerated) circular orbits in Kerr, important since a number of families of such orbits characterize various geometrical properties of the spacetime. Our intuition is not very good in this last case since rotating particles are viewed by rotating observers rather than nonrotating observers.

5.1 Spacetime description of circular orbits

Consider Boyer-Lindquist-like coordinates (t, r, θ, ϕ) for a orthogonally transitive stationary axisymmetric spacetime. The lapse/shift form of the line element is

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi + N^\phi dt)^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \ , \tag{11}$$

and an orthonormal frame $(n, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}})$ can be adapted to the zero angular momentum observers (ZAMOs) whose 4-velocity $n = N^{-1}(\partial_t - N^\phi \partial_\phi)$ is the unit normal to the time coordinate hypersurfaces. Uniformly rotating orbits have 4-velocity

$$U = \gamma(n + \nu e_{\hat{\phi}}) = \cosh\alpha\, n + \sinh\alpha\, e_{\hat{\phi}} \qquad e_{\hat{\phi}} = (\mathrm{sgn}\,\nu)\hat{\nu}_{(U,n)} \tag{12}$$

and form a 1-parameter family of orbits which can be parametrized by the magnitude of the relative velocity $\nu = \tanh\alpha$ (or equivalently by the rapidity α, both of them constant along U, $d\nu/d\tau_U = 0 = d\alpha/d\tau_U$).

By varying $\nu \in (-1, 1)$, U describes a branch of a hyperbola in the relative observer plane of n and $e_{\hat{\phi}}$ (along the t-ϕ directions). The corresponding curve of acceleration vectors is instead a branch of a hyperbola in the orthogonal r-θ tangent plane (the acceleration plane)

$$a_{(U)} = \kappa(\cos\chi\, e_{\hat{r}} + \sin\chi\, e_{\hat{\theta}}) \ , \tag{13}$$

where (κ, χ) are polar coordinates in the acceleration plane and are both constant for a given orbit. Furthermore, the Frenet-Serret angular velocity traces out a complementary hyperbola in this plane centered at the origin. The acceleration $\vec{a}$ can be expressed in terms of the ZAMO relative observer decomposition in the following form[1,2]

$$\begin{aligned}\vec{a} &= \vec{k}\sinh^2\alpha + 2\vec{\theta}_{\hat{\phi}}\sinh\alpha\cosh\alpha + \vec{A}\cosh^2\alpha \\ &= \frac{1}{4}(\vec{a}_+ + \vec{a}_-)\cosh 2\alpha + \frac{1}{4}(\vec{a}_+ - \vec{a}_-)\sinh 2\alpha + \vec{a}_0 \ ,\end{aligned} \tag{14}$$

where $\vec{\theta}_{\hat{\phi}}$ and $\vec{A} = a_{(n)}$ are components of the observer expansion tensor (contracted once with $e_{\hat{\phi}}$) and acceleration vector while $\vec{k} = k_{(\mathrm{lie},U,n)}$ is the Lie relative curvature vector, and $\vec{a}_\pm = \vec{k} + \vec{A} \pm 2\theta_{\hat{\phi}} = ||\vec{a}_\pm||\vec{n}_\pm$ are the corotating/counter-rotating photon circular orbit accelerations (with a unit energy affine parameter) which are aligned with the axes of the acceleration hyperbola and $\vec{a}_0 = \frac{1}{2}(\vec{A} - \vec{k})$ is the center of the acceleration hyperbola. The Frenet-Serret angular velocity is given instead by the formula

$$\vec{\omega}_{(\mathrm{FS})} = \frac{1}{2} e_{\hat{\phi}} \times_n \frac{d\vec{a}}{d\alpha} \ . \tag{15}$$

Observers at the vertex of this hyperbola are called MIROs because they also correspond to a minimal intrinsic rotation $\omega_{(\mathrm{FS})}$. Their defining condition is $e^{2\alpha_{(\mathrm{vert})}} = \sqrt{||\vec{a}_-||/||\vec{a}_+||}$, and introducing the quantity $\mathcal{A} = \sqrt{||\vec{a}_-||||\vec{a}_+||}/2$, these definitions lead to

$$\begin{aligned}(\vec{a} - \vec{a}_0) = \frac{1}{2}\mathcal{A}[&\cosh 2(\alpha - \alpha_{(\mathrm{vert})})(\vec{n}_+ + \vec{n}_-) + \\ &\sinh 2(\alpha - \alpha_{(\mathrm{vert})})(\vec{n}_+ - \vec{n}_-)] \ ,\end{aligned} \tag{16}$$

The photon acceleration direction vectors, which determine the common asymptotes of the acceleration and Frenet-Serret angular velocity hyperbolas, are independent of the reference observer family used to decompose the acceleration. It is natural to introduce the unit direction vectors of their sum and difference which determine the common orthogonal axes of the acceleration hyperbola.

The opening angle $\lambda \in [0, \pi]$ of the acceleration hyperbola (making $\pi - \lambda$ the opening angle of the Frenet-Serret angular velocity hyperbola) satisfies $\vec{n}_+ \cdot \vec{n}_- = \cos\lambda$, in terms of which one finds

$$||\vec{n}_+ + \vec{n}_-||/2 = \cos\frac{\lambda}{2} \ , \quad ||\vec{n}_+ - \vec{n}_-||/2 = \sin\frac{\lambda}{2} \ , \tag{17}$$

leading to the unit vectors

$$\vec{\imath} = \frac{\vec{n}_+ + \vec{n}_-}{2\cos\frac{\lambda}{2}} \ , \qquad \vec{\jmath} = \frac{\vec{n}_+ - \vec{n}_-}{2\sin\frac{\lambda}{2}} \ . \tag{18}$$

The quantities $\mathcal{A}\cos\frac{\lambda}{2}$ and $\mathcal{A}\sin\frac{\lambda}{2}$ are the semi-axes of the acceleration and the Frenet-Serret angular velocity hyperbolas, which take the form

$$\begin{aligned}\vec{a} - \vec{a}_0 &= \mathcal{A}[\cosh 2(\alpha - \alpha_{(\mathrm{vert})})\cos\frac{\lambda}{2}\,\vec{\imath} + \sinh 2(\alpha - \alpha_{(\mathrm{vert})})\sin\frac{\lambda}{2}\,\vec{\jmath}] \ , \\ \epsilon\,\vec{\omega}_{(\mathrm{FS})} &= \mathcal{A}[\cosh 2(\alpha - \alpha_{(\mathrm{vert})})\sin\frac{\lambda}{2}\,\vec{\imath} - \sinh 2(\alpha - \alpha_{(\mathrm{vert})})\cos\frac{\lambda}{2}\,\vec{\jmath}] \ .\end{aligned} \tag{19}$$

where the sign ϵ, which has the value $\mathrm{sgn}(\pi/2 - \theta)$ in the Kerr case, makes the following cross-product relation valid

$$\epsilon\,\vec{\imath} \times_n \vec{\jmath} = e_{\hat{r}} \times_n e_{\hat{\theta}} = e_{\hat{\phi}} \ . \tag{20}$$

To write the spacetime Frenet-Serret frame we choose the convention $\kappa \geq 0$; one has

$$
\begin{aligned}
e_1 &= \cos\chi\, e_{\hat r} + \sin\chi\, e_{\hat\theta}\ , \\
e_2 &= dU/d\alpha = \sinh\alpha\, n + \cosh\alpha\, e_{\hat\phi} = (\mathrm{sgn}\,\nu)\hat{\mathcal{V}}_{(n,U)}\ , \\
e_3 &= -de_1/d\chi = \sin\chi\, e_{\hat r} - \cos\chi\, e_{\hat\theta}\ .
\end{aligned} \tag{21}
$$

The spacetime torsions are given by half the sign-reversed arclength derivatives along the polar coordinate directions in the acceleration plane, leading to an alternative expression for the Frenet-Serret angular velocity

$$
\tau_1 = -\frac{1}{2} d\kappa/d\alpha\ , \quad \tau_2 = -\frac{1}{2}\kappa\, d\chi/d\alpha\ ,\ \omega_{(\mathrm{FS})} = \frac{1}{2} e_2 \times_U \frac{da_{(U)}}{d\alpha}\ , \tag{22}
$$

showing that the latter is orthogonal to the tangent to the α-parametrized acceleration hyperbola.

5.2 Relative Frenet-Serret description of circular orbits

The relative Frenet-Serret curvatures and torsions reduce to

$$
\begin{aligned}
\mathcal{K}_{(\mathrm{fw},n,U)} &= \frac{||\omega_{FS}||}{\sinh\alpha}\ , \qquad \mathcal{T}_{(\mathrm{fw},n,U)} = 0\ , \\
k_{(\mathrm{fw},U,n)} &= \frac{1}{2}\frac{\cosh^3\alpha}{\sinh\alpha}||\frac{d}{d\alpha}(\frac{a(U)}{\cosh^2\alpha})||\ , \qquad \tau_{(\mathrm{fw},U,n)} = 0\ .
\end{aligned} \tag{23}
$$

The magnitude of the relative generalized centrifugal force is then

$$
\mathcal{F}^{(C)}_{(\mathrm{fw},n,U)} = |\tanh\alpha|\, ||\omega_{FS}||\ . \tag{24}
$$

Apart from a reordering of its elements, the comoving relative Frenet-Serret frame $\{[-\hat\nu(n,U)], \hat{\mathcal{N}}_{(\mathrm{fw},n,U)}, \hat{\mathcal{B}}_{(\mathrm{fw},n,U)}\}$ at a given point on a circular orbit consists of the same vectors which belong to the Frenet-Serret frame defined along the acceleration hyperbola $a(U) = a_{(U)}(\alpha)$, provided one identifies tangent vectors to the tangent space itself with tangent vectors by translation to the origin. Analogously the relative Frenet-Serret frame $\{\hat\nu(U,n), \hat\eta_{(\mathrm{fw},U,n)}, \hat\beta_{(\mathrm{fw},U,n)}\}$ consists of the same vectors which belong to the Frenet-Serret frame defined along the rescaled acceleration curve $a(U)/\cosh^2(\alpha)$, corresponding to the second derivative proper time gamma factor rescaling.

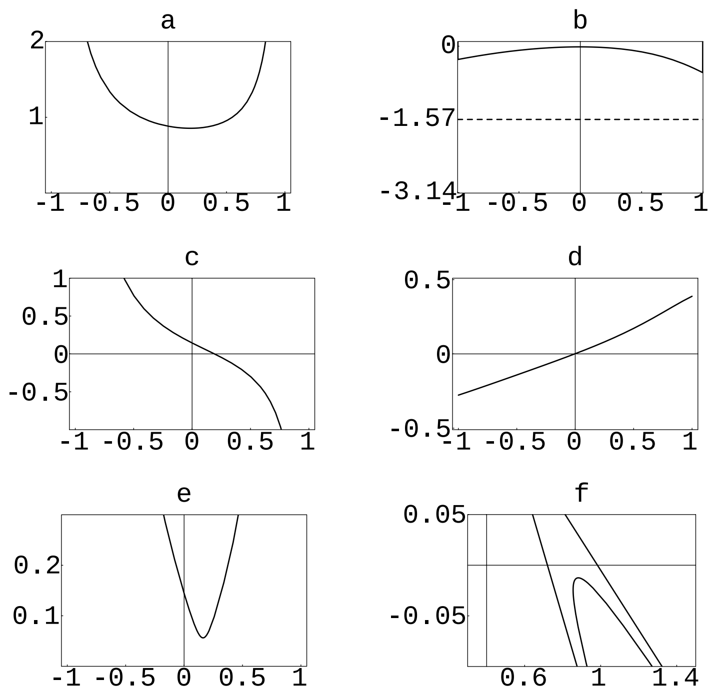


Figure 2: The following quantities are plotted versus the velocity ν for the family of all circular orbits at the Boyer-Lindquist radial coordinate values $r = 4$ and $\theta = \pi/3$ for a Kerr black hole with $a/\mathcal{M} = 0.5$: the polar coordinates in the acceleration plane: κ (a) and χ (b), the two torsions: τ_1 (c) and τ_2 (d), the magnitude of the Frenet-Serret angular velocity $||\omega_{(\mathrm{FS})}||$ (e) and the acceleration curve (f).

6 Circular orbits in stationary axisymmetric spacetimes: Kerr spacetime, equatorial orbits

Now specialize the results for general circular orbits to the equatorial plane of the Kerr spacetime where $\theta = \pi/2$[3]. One can introduce a unit vector $e_{\hat z} = -e_{\hat\theta}$ mapping the Boyer-Lindquist coordinates directions (t, r, θ, ϕ) to those of the associated flat spacetime cylindrical coordinates (t, r, ϕ, z).

The acceleration hyperbola degenerates to a half-line along the radial direction, making it convenient to allow the curvature κ to change sign while fixing the second Frenet-Serret

vector e_1 to be along the outward radial direction $e_{\hat r}$. Fix also $e_3 = -e_{\hat\theta}$

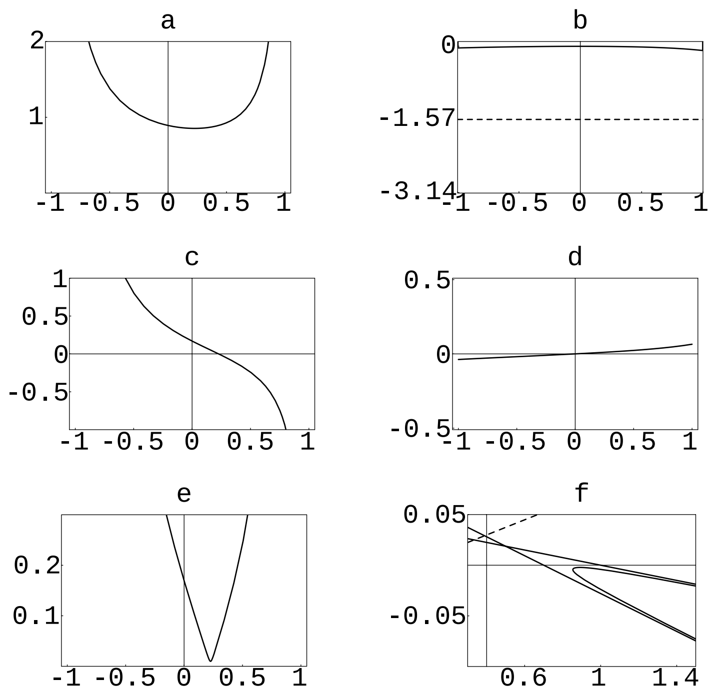


Figure 3: Figure 2 repeated for $r = 4$ and $\theta = \pi/2.1$ for the same quantities: the polar coordinates in the acceleration plane: κ (a) and χ (b), the two torsions: τ_1 (c) and τ_2 (d), the magnitude of the Frenet-Serret angular velocity $||\omega_{(\rm FS)}||$ (e) and the acceleration curve (f).

and $e_2 = (\mathrm{sgn}\,\nu)\hat{\mathcal{V}}$. Then from the Frenet-Serret relation for e_2, choosing $\hat{\mathcal{N}}_{(\rm fw,\mathit{n},\mathit{U})} = e_{\hat r} = \hat\eta_{(\rm fw,\mathit{U},\mathit{n})}$, one finds

$$-\tau_1 = \sinh\alpha\,\mathcal{K}_{(\rm fw,\mathit{n},\mathit{U})} = \frac{1}{2}\frac{d\kappa}{d\alpha}\ . \tag{25}$$

The acceleration then has the following representation in terms of ZAMOs

$$\begin{aligned}
a_{(U)} &= \kappa e_{\hat r} = \gamma^2[k_{(\mathrm{lie},U,n)}^{\hat r}\nu^2 + 2\theta_{(n)}{}^{\hat r}{}_{\hat\phi}\nu + a_{(n)}{}^{\hat r}]e_{\hat r}\\
&= \gamma^2 k_{(\mathrm{lie},U,n)}^{\hat r}(\nu-\nu_+)(\nu-\nu_-)\,e_{\hat r}\\
&= k_{(\mathrm{lie})}\frac{\sinh(\alpha-\alpha_+)\sinh(\alpha-\alpha_-)}{\cosh\alpha_+\cosh\alpha_-}e_{\hat r}\ ,
\end{aligned}$$

where the corevolving (+) and counterrevolving (−) geodesic velocities $\nu_\pm = \tanh\alpha_\pm$ have been introduced, related to the observer expansion tensor and acceleration vector. The spacetime torsions for these orbits are

$$\tau_1 = -\sinh\alpha\,\mathcal{K}_{(\mathrm{fw},n,U)} = -\frac{1}{2}\frac{d\kappa}{d\alpha} = -\frac{1}{2}k_{(\mathrm{lie})}\frac{\sinh 2(\alpha-\alpha_{(\mathrm{ext})})}{\cosh\alpha_+\cosh\alpha_-}\ ,$$
$$\tau_2 = 0\ .$$

and the resulting relative generalized centrifugal force is

$$\mathcal{F}^{(C)}_{(\mathrm{fw},n,U)} = -\gamma\mathcal{K}_{(\mathrm{fw},n,U)}\nu^2 = -\frac{1}{2}k_{(\mathrm{lie})}\frac{\sinh\alpha\sinh 2(\alpha-\alpha_{(\mathrm{ext})})}{\cosh\alpha\cosh\alpha_+\cosh\alpha_-}\ , \tag{26}$$

where $\alpha_{(\mathrm{ext})} = (\alpha_+ + \alpha_-)/2$ is the rapidity of the extremely accelerated observers (defined by the condition $d\kappa/d\alpha = 0$). For these observers

$$\mathcal{K}_{(\mathrm{fw},n,U_{(\mathrm{ext})})} = 0\ , \qquad \Longrightarrow \qquad \mathcal{F}^{(C)}_{(\mathrm{fw},n,U_{(\mathrm{ext})})} = 0\ ,$$
$$\tau_1(U_{(\mathrm{ext})}) = 0\ , \qquad \Longrightarrow \qquad \omega_{FS}(U_{(\mathrm{ext})}) = 0\ .$$

so that their orbits are comoving relatively straight and their comoving relative generalized centrifugal force vanishes, consistent with the nonrelativistic notion that a platform which is nonrotating with respect to gyroscopes should not experience a centrifugal force. The spin of a test gyro is locked to their spacetime Frenet-Serret frame and hence to the Boyer-Lindquist spatial orthonormal frame, a situation called "phase-locking" by de Felice.

The timelike geodesics instead have nonzero torsions

$$\tau_{1\,(U_\pm)} = \pm\sqrt{\mathcal{M}/r^3}, \tag{27}$$

so that their comoving relative generalized centrifugal force is nonzero.

This is also true in the Schwarzschild limit, where the extremely accelerated observers coincide with the ZAMOs ($\alpha_{(\mathrm{ext})} = 0$). Here for a general orbit one has

$$\mathcal{K}_{(\mathrm{fw},n,U)} = -\gamma\frac{1}{r}(1-3\mathcal{M}/r)(1-2\mathcal{M}/r)^{-1/2}\ , \tag{28}$$

and for the geodesics

$$\mathcal{K}_{(\mathrm{fw},n,U_\pm)} = -\frac{1}{r}(1-3\mathcal{M}/r)^{1/2}\ , \tag{29}$$

so that the generalized centrifugal force for the geodesics is then

$$-\gamma_\pm\mathcal{K}_{(\mathrm{fw},n,U_\pm)}\nu_\pm^2 = \frac{\mathcal{M}}{r^2}(1-2\mathcal{M}/r)^{-1/2}\ . \tag{30}$$

The negative sign in the comoving relative curvature formula for the Schwarzschild case indicates that for $r > 3\mathcal{M}$, the comoving relative curvature vector to the circular orbit is inward along the radial direction, while the generalized centrifugal force is radially outward. Far from the hole $|\mathcal{K}_{(\mathrm{fw},n,U_\pm)}|$ for the circular geodesics has the familiar flat space circular curvature limit $1/r$ and the generalized centrifugal force also has the familiar limiting form $-\gamma_\pm \mathcal{K}_{(\mathrm{fw},n,U_\pm)} \nu_\pm^2 \, e_{\hat r} \to \nu_\pm^2 / r \, e_{\hat r}$. Near the hole at $r = 3\mathcal{M}$, the circular geodesics become null and their orbits are comoving relatively straight, as are any accelerated orbits at this radius. Such orbits are also optically relatively straight in the optical geometry promoted by Abramowicz et al. For $r < 3\mathcal{M}$ the comoving relative curvature vector turns outward, while the generalized centrifugal force turns inward. This is a "centrifugal force reversal" of the actual generalized centrifugal force, exactly like the reversal of the optical relative curvature vector and optical relative centripetal acceleration introduced by Abramowicz et al. This reversal just reflects the fact that here the orbital angular velocity of the spherical axes relative to infinity cannot keep up with the angular velocity of the gyroscope spin as the latter rotates around the black hole at an even faster angular velocity. The extremely accelerated observers are the only ones for which the comoving relative curvature and centrifugal force have the right overall properties of the original nonrelativistic counterparts.

From equations (26) one finds that unlike the case for all other observers, the comoving curvature is exactly the relative Lie curvature modified by various gamma factors (symmetric in the relative velocity with respect to those observers) and does not vanish at any velocity unless the relative Lie curvature vanishes

$$\mathcal{K}_{(\mathrm{fw},U_{(\mathrm{ext})},U)} = k_{(\mathrm{lie})} \frac{\cosh \alpha_{(U,U_{(\mathrm{ext})})}}{\cosh \alpha_+ \cosh \alpha_-} = k_{(\mathrm{lie})} \frac{\gamma_{(U,U_{(\mathrm{ext})})}}{\gamma_+ \gamma_-} \ . \tag{31}$$

At a general radius where the Lie curvature is nonzero, the corresponding comoving relative centrifugal force $-\mathcal{K}_{(\mathrm{fw},U_{(\mathrm{ext})},U)} \nu^2_{(U,U_{(\mathrm{ext})})}$ is positive-definite (negative-definite) when the curvature is negative (positive) and is symmetric in the relative velocity, properties which hold only with respect to the extremely accelerated observers.

The symmetry means that the geodesic relative velocities for which this balances the spatial gravitational force term have the same absolute value for these observers $\nu_{(U_-,U_{(\mathrm{ext})})} = -\nu_{(U_+,U_{(\mathrm{ext})})}$. For all other observers, the generalized centrifugal force has two zeros (occurring at zero relative velocity and at the relative velocity of the extremely accelerated observer) and changes sign across each one, provided that an extremely accelerated observer exists at the given radius. This identifies the extremely accelerated observers as the only appropriate reference observers for which the generalized centrifugal force behaves as one would like it to behave.

In fact, one may express the acceleration in terms of the extremal observers with the result

$$\kappa = \gamma_{(U_{(\rm ext)},U_+)}\mathcal{K}_{({\rm fw},U_{(\rm ext)},U_+)}\frac{\nu^2_{(U,U_{(\rm ext)})} - \nu^2_{(U_+,U_{(\rm ext)})}}{1-\nu^2_{(U,U_{(\rm ext)})}}\ . \qquad (32)$$

In the nonrotating case one has $U_{(\rm ext)} = n$, $\nu_- = -\nu_+$ and this reduces to

$$\kappa = \gamma_+\mathcal{K}_{({\rm fw},n,U_+)}\frac{\nu^2-\nu_+^2}{1-\nu^2}\ , \qquad (33)$$

where $\gamma_+\mathcal{K}_{({\rm fw},n,U_+)} = k_{(\rm lie)}$. Thus the extremely accelerated observers in the equatorial plane lead to the same symmetric expression for the total acceleration as in the nonrotating case, and the corresponding curvature coefficient factor in this expression is exactly the comoving relative curvature relative to these observers.

7 Discussion

Off the equatorial plane of the Kerr spacetime, the MIROs seem to best generalize the centrifugal force properties associated with the extremely accelerated observers in the equatorial plane. Nonrotating axes are no longer possible and minimizing the intrinsic rotation is the best one can do. The centrifugal force function is symmetric and has only one zero (when the test particle is at rest with respect to the MIROs), but having one zero is now true of the centrifugal force for any other observer as well.

References

1. D. Bini, R.T. Jantzen and A. Merloni, *Class. Quantum Grav.* **16**, 1333 (1999).
2. D. Bini, F. de Felice and R.T. Jantzen, *Class. Quantum Grav.* **16**, 2105 (1999).
3. D. Bini, P. Carini and R.T. Jantzen, *Int. Jou. Mod. Phys.* **D6**, 143 (1997).