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SPACETIME SPLITTING TECHNIQUES AND GRAVITOELECTROMAGNETISM IN GENERAL RELATIVITY

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A review of spacetime splitting techniques and gravitoelectromagnetism is presented.

1 Spacetime splitting techniques: an introduction

While it is difficult to associate mental images with concepts on a general spacetime manifold, by making a local identification with Minkowski spacetime, we can transfer our space-plus-time experience in this simpler context, already upgraded to special relativity from Newtonian physics, to a general curved spacetime through a spacetime splitting scheme. This locally decomposes spacetime into "space plus time" and permits the splitting of spacetime tensors and equations into the more familiar counterparts grounded in our Newtonian intuition.

However, there exist different approaches to this problem whose common features have only been considered recently in a unified framework often associated with the term gravitoelectromagnetism. Each splitting point of view or approach is based on two fundamental concepts: **measurement** and **evolution**, the realization of which differ for each of the possible choices.

One can associate a "measurement process" with each of the two complementary ways in which the notion of time can be introduced: 1) time as "time elapsing at a point fixed in space," represented by a "time line;" 2) time as "space at a moment of time," associated with some kind of synchronization of times at different points of space and represented by a "time hypersurface."

In the first case ("time splitting") a local time direction is fundamental, while in the second ("space splitting") a nonlocal correlation of local times, i.e., space is fundamental. More precisely, a "time splitting" is based on the existence on the spacetime of a "time congruence" of curves, while a "space splitting" on the existence on the spacetime of a foliation by "space hypersurfaces."

In addition to these two types of measurement, one may consider a partial splitting or a full splitting of spacetime[1,2].

1.1 Partial Splitting

Assume only one structure is defined on the spacetime: either a timelike congruence or a spacelike foliation. Using only this structure splits off either the time or the space alone, leading to a partial splitting of spacetime.

In the "time splitting" approach, here called the "**congruence point of view**," one has only a timelike congruence at one's disposal, with a (future-pointing) unit timelike tangent vector field u. It may then be interpreted as the 4-velocity of a family of test observers whose worldlines are the curves of the congruence, and it determines the *local time direction* at each point of spacetime. The orthogonal complement of this local time direction in the tangent space is the *local rest space* LRS_u of the test observer at that event. It is exactly this structure that one needs for the *measurement process*. The orthogonal decomposition of the tensor algebra induced by this decomposition of the tangent space at each event defines the measurement process, yielding a collection of "*spatial tensor fields*" of different rank for each spacetime tensor field that is split.

The "space splitting" approach, that of a spacelike slicing of spacetime with no additional structure, is essentially equivalent to the special case of a nonrotating congruence since every spacelike slicing admits a family of timelike orthogonal trajectories. These are the integral curves of the (rotation-free) unit normal vector field n to the slicing, which may be assumed to be future-pointing. The accompanying point of view, for the sake of completeness, might be called the "**hypersurface point of view**". Its measurement process is associated with the normal congruence, taking $u = n$ as the 4-velocity of the family of test observers who do the measuring. The local rest spaces of this family are integrable and coincide with the subspaces of the tangent space which are tangent to the slicing.

1.2 Full Splitting

A full splitting of spacetime requires both a *slicing* of the spacetime and a congruence, to be referred to as a "*threading*" of the spacetime, together with a compatibility condition that the two families be everywhere transversal. Such a structure will be called a "*nonlinear reference frame*" in order to distinguish it from the various distinct usages of the term "reference frame" that one finds in the literature.

A "*parametrized nonlinear reference frame*" consists of a nonlinear reference frame together with a choice of parametrization of the family of slices; in such a way one defines a specific "time function" t on the spacetime which in turn provides an obvious parametrization for each curve in the threading congruence.

Full splittings differ according to the causality condition imposed on the nonlinear reference frame. In the "**slicing point of view**" the slicing is assumed to be spacelike, but no assumption is made about the causality properties of the threading, which serves only as a way of identifying the points on different slices. In the "**threading point of view**," the threading is assumed to be timelike, but no assumption is made about the causality properties of the slicing, which serves only to synchronize in some arbitrary fashion points on different curves in the congruence. If both causality conditions hold, then both points of view hold and one can transform between them.

In the special case of an *orthogonal nonlinear reference frame* (one for which both the causality conditions hold and the slicing and threading are everywhere orthogonal), then the two points of view coincide.

1.3 Evolution

Evolution is defined first by a choice of a 1-parameter group of diffeomorphisms of the spacetime into itself which in some sense advances into the future (either its orbits are timelike or it pushes certain spacelike hypersurfaces into their future), and second by a choice of transport along its orbits for the spatial fields of the given point of view.

For a partial splitting only one congruence is available and it is timelike. In the absense of additional structure one can take u or n respectively in the congruence or hypersurface points of view as the generator of such a group, and choose either spatially-projected Lie transport or spatially-projected parallel transport along this congruence. The latter transport of spatial fields coincides with Fermi-Walker transport which defines locally nonrotating axes along a worldline. Each of these choices may be extended to the corresponding full splitting but it is the spatial Lie transport along the threading congruence which defines the evolution relative to the nonlinear reference frame, since fields which are "rigidly" attached to this frame do not evolve with this choice. However, unlike Fermi-Walker transport, spatial Lie transport is in general incompatible with orthonormal frames. A compromise between the two kinds of transport leads to *co-rotating Fermi-Walker transport*, which is the closest one can get to attaching an orthonormal frame to the nonlinear reference frame.

For a full splitting of spacetime, usually accomplished via adapted coordinates so the time coordinate lines are the threading congruence and the time coordinate hypersurfaces are the slicing foliation, the situation is slightly more complicated. For threading point of view, one simply uses the timelike congruence itself to evolve fields, leading to a representation of the congruence point of view in this setting. For the slicing point of view, one can describe evolution either with respect to the timelike normal congruence or the more general threading congruence of the time coordinate lines. The following introduction will limit itself to the partial splitting case and the measurement process for it.

2 Historical Background

The slicing and threading points of view today have been popularized through two leading textbooks, respectively *Gravitation* by Misner, Thorne, and Wheeler [1973] and *The Classical Theory of Fields* by Landau and Lifshitz [1975], each of which carefully avoids mention of the "competing" point of view. Both points of view can be traced back to the early forties, the history of which can be summarized as follows.

- Landau and Lifshitz [1941] first introduced the threading point of view splitting of the spacetime metric and later in the stationary case, of the spacetime connection to define spatial gravitational forces.

- Lichnerowicz [1944] introduced the slicing point of view with an article discussing the initial value problem in an orthogonal nonlinear reference frame.

- Møller [1952] discussed a parametrization-dependent definition of spatial gravitational forces for a general spacetime.

- Zel'manov [1956] refined the approach by Møller to a parametrization-independent splitting but unfortunately most of his work was originally published only in Russian.

- Cattaneo[1958] developed most of the foundations of the threading point of view splitting of the spacetime metric and connection, presumably unaware of Zel'manov's work.

- Dirac [1959] recognized the significance of the slicing point of view metric decomposition for the Hamiltonian dynamics of general relativity and its relation to his theory of constrained Hamiltonian systems.

- Arnowit, Deser and Misner [1962] used the Hamiltonian formulation permitted by the slicing point of view to study the true degrees of freedom of the gravitational field, culminating in an often cited review article [1962]. The approach of Arnowit, Deser and Misner together with Wheeler's lapse and shift terminology soon found widespread acceptance.

- Ferrarese [1963] continued the work of Cattaneo to the curvature level and studied the Cauchy problem for the Einstein equations in the threading point of view, which still remains an open problem (in contrast with the slicing case which is rather well understood).

- Hawking and Ellis at the beginning of the seventies introduced the congruence point of view, following a previous work in German by Ehlers [1961] which had been unavailable in English until 1994.

- Massa [1974] reformulated the approach of Cattaneo and Ferrarese to the threading point of view in a more modern language and used it to discuss gyroscope precession.

- Braginsky, Caves and Thorne [1977] introduced an "electric-type" gravitational field and a "magnetic-type" gravitational field, the latter of which became the "gravitomagnetic" field of the eighties. The introduction of the terminology "gravitoelectric" field and the first discussion of slicing spatial gravitational forces finally appeared in the text *Black Holes: The Membrane Paradigm* by Thorne et al [1986], but only in the case of a stationary gravitational field in their treatment of black hole spacetimes.

This brief history must be complemented with the remark that an enormous language barrier existed over many years between the congruence, slicing and threading points of view, preventing those versed in the formalism and notation of one from easily penetrating the others or understanding how they are related to each other. In the early nineties Jantzen, Carini and Bini [1992] made an effort to find a common precise language to discuss all of the spacetime splitting approaches and use it to study properties of test particle motion. The present article is a brief introduction to this line of work. The measurement process is described here for a single test observer congruence as well as for relating measurements made by two or more such observers, one of which may be associated with a single test particle world line as seen by one or more observers.

3 The measurement process for a single observer family

3.1 Notation and conventions

Let ${}^{(4)}\mathrm{g}$ (signature `-+++` and components ${}^{(4)}g_{\alpha\beta}$, $\alpha, \beta, \ldots = 0,1,2,3$) be the spacetime metric, ${}^{(4)}\nabla$ its associated covariant derivative operator, and ${}^{(4)}\eta$ the unit volume 4-form which orients spacetime (${}^{(4)}\eta_{0123} = {}^{(4)}g^{1/2}$ in an oriented frame, where ${}^{(4)}g \equiv |\det({}^{(4)}g_{\alpha\beta})|$). Assume the spacetime is also time oriented and let u be a future-pointing unit timelike vector field ($u^\alpha u_\alpha = -1$) representing the 4-velocity field of a family of test observers filling the spacetime (or some open submanifold of it).

Denote by $S^\flat$ ($S^\sharp$) the totally covariant (contravariant) form with respect to the metric index-shifting operations of an arbitrary tensor field S and introduce the right contraction notation $[S \llcorner X]^\alpha = S^\alpha{}_\beta X^\beta$ for the contraction of S with a vector field X (left contraction notation being analogous).

3.2 Observer-orthogonal splitting

The observer-orthogonal decomposition of the tangent space, and in turn of the algebra of spacetime tensor fields, is accomplished by the temporal projection operator $T(u)$ along u and the spatial projection operator $P(u)$ onto LRS_u, which may be identified with mixed second rank tensors acting by contraction

$$\begin{aligned} \delta^\alpha{}_\beta &= T(u)^\alpha{}_\beta + P(u)^\alpha{}_\beta \ , \\ T(u)^\alpha{}_\beta &= -u^\alpha u_\beta \ , \qquad P(u)^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta \ . \end{aligned} \tag{1}$$

These satisfy the usual orthogonal projection relations $P(u)^2 = P(u)$, $T(u)^2 = T(u)$, and $T(u) \llcorner P(u) = P(u) \llcorner T(u) = 0$. Let

$$[P(u)S]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_\gamma \cdots P(u)^\delta{}_\beta \cdots S^{\gamma\ldots}{}_{\delta\ldots} \tag{2}$$

denote the spatial projection of a tensor S on all indices.

The *measurement of* S by the observer congruence is the family of spatial tensor fields which result from the spatial projection of all possible contractions of S by any number of factors of u. For example, if S is a $\binom{1}{1}$-tensor, then its measurement

$$S^\alpha{}_\beta \leftrightarrow (u^\delta u_\gamma S^\gamma{}_\delta, P(u)^\alpha{}_\gamma u^\delta S^\gamma{}_\delta, P(u)^\delta{}_\alpha u_\gamma S^\gamma{}_\delta, P(u)^\alpha{}_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta) \tag{3}$$

results in a scalar field, a spatial vector field, a spatial 1-form and a spatial $\binom{1}{1}$-tensor field. It is exactly this family of fields which occur in the (orthogonal) "decomposition of S" with respect to the observer congruence

$$\begin{aligned} S^\alpha{}_\beta &= [T(u)^\alpha{}_\gamma + P(u)^\alpha{}_\gamma][T(u)^\delta{}_\beta + P(u)^\delta{}_\beta] S^\gamma{}_\delta \\ &= [u^\delta u_\gamma S^\gamma{}_\delta] u^\alpha u_\beta + \cdots + [P(u)S]^\alpha{}_\beta \ . \end{aligned} \tag{4}$$

The same decomposition can be applied to spacetime differential operators.

3.3 Examples

1. Measurement of the spacetime metric and volume 4-form
   - spatial metric $[P(u)^{(4)}\mathrm{g}]_{\alpha\beta} = P(u)_{\alpha\beta}$
   - spatial unit volume 3-form $\eta(u)_{\alpha\beta\gamma} = u^{\delta\,(4)}\eta_{\delta\alpha\beta\gamma} = [P(u)\, u \,\lrcorner\, {}^{(4)}\eta]_{\alpha\beta\gamma}$
2. Measurement of the Lie, exterior and covariant derivative
   - spatial Lie derivative $\pounds(u)_X = P(u)\pounds_X$
   - the spatial exterior derivative $d(u) = P(u)d$
   - the spatial covariant derivative $\nabla(u) = P(u)^{(4)}\nabla$
   - the spatial Fermi-Walker derivative (or Fermi-Walker temporal derivative) $\nabla_{(\mathrm{fw})}(u) = P(u)^{(4)}\nabla_u$
   - the Lie temporal derivative $\nabla_{(\mathrm{lie})}(u) = P(u)\pounds_u = \pounds(u)_u$

   Note that these spatial differential operators do not obey the usual product rules for nonspatial fields since undifferentiated factors of u are killed by the spatial projection.
3. 3-dimensional notation

   If X and Y are two spatial vector fields one can define their
   - spatial inner product $X \cdot_u Y = P(u)_{\alpha\beta} X^\alpha Y^\beta$
   - spatial cross product $[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma} X^\beta Y^\gamma$.

   and spatial gradient, curl and divergence operators for functions f and spatial vector fields X become

$$\begin{aligned} \mathrm{grad}_u\, f &= \nabla(u) f = [d(u)f]^\sharp \ , \\ \mathrm{curl}_u\, X &= \nabla(u) \times_u X = [{}^{*(u)}d(u)X^\flat]^\sharp \ , \\ \mathrm{div}_u\, X &= \nabla(u) \cdot_u X = {}^{*(u)}[d(u)^{*(u)}X^\flat] \ , \end{aligned} \tag{5}$$

where ${}^{*(u)}$ is the spatial duality operation for antisymmetric tensor fields associated with the spatial volume form $\eta(u)$ in the usual way. These definitions enable one to mimic all the usual formulas of 3-dimensional vector analysis.

4. Measurement of the covariant derivative of the observer 4-velocity

   Measurement of the covariant derivative $[{}^{(4)}\nabla u]^\alpha{}_\beta = u^\alpha{}_{;\beta}$ leads to two spatial fields, the acceleration vector field $a(u)$ and the kinematical mixed tensor field $k(u)$

$$\begin{aligned} &u^\alpha{}_{;\beta} = -a(u)^\alpha u_\beta - k(u)^\alpha{}_\beta \ , \\ &a(u) = \nabla_{(\mathrm{fw})}(u)u \ , \qquad k(u) = -\nabla(u)u = \omega(u) - \theta(u) \ . \end{aligned} \tag{6}$$

   The kinematical tensor field may be decomposed into its antisymmetric and symmetric parts

$$\begin{aligned} [\omega(u)^\flat]_{\alpha\beta} &= P(u)^\sigma_\alpha P(u)^\delta_\beta u_{[\delta;\sigma]} = \tfrac{1}{2}[d(u)u^\flat]_{\alpha\beta} \ , \\ [\theta(u)^\flat]_{\alpha\beta} &= P(u)^\sigma_\alpha P(u)^\delta_\beta u_{(\delta;\sigma)} = \tfrac{1}{2}[\nabla_{(\mathrm{lie})}(u)P(u)^\flat]_{\alpha\beta} = \tfrac{1}{2}\pounds(u)_u{}^{(4)}g_{\alpha\beta} \ , \end{aligned} \tag{7}$$

   defining the mixed rotation or vorticity tensor field $\omega(u)$ (whose sign depends on convention) and the mixed expansion tensor field $\theta(u)$, the latter of which may itself be decomposed into its tracefree ($\sigma(u)$, the shear tensor field) and pure trace ($\Theta(u)$, the expansion scalar) parts.

   Define also the rotation or vorticity vector field $\omega(u) = \frac{1}{2}\,\mathrm{curl}_u\, u$ as the spatial dual of the spatial rotation tensor field

$$\omega(u)^\alpha = \tfrac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} = \tfrac{1}{2}{}^{(4)}\eta^{\alpha\beta\gamma\delta}u_\beta u_{\gamma;\delta} \ . \tag{8}$$

5. Lie, Fermi-Walker and co-Fermi-Walker derivatives

   The kinematical tensor describes the difference between the Lie and Fermi-Walker temporal derivative operators when acting on spatial tensor fields. For example, for a spatial vector field X

$$\begin{aligned} \nabla_{(\mathrm{fw})}(u)X^\alpha &= \nabla_{(\mathrm{lie})}(u)X^\alpha - k(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha - \omega(u)^\alpha{}_\beta X^\beta + \theta(u)^\alpha{}_\beta X^\beta \ , \end{aligned} \tag{9}$$

   where $\omega(u)^\alpha{}_\beta X^\beta = -[\omega(u) \times_u X]^\alpha$. The kinematical quantities associated with u may be used to introduce two spacetime temporal derivatives, the Fermi-Walker derivative and the co-rotating Fermi-Walker derivative along u

$$\begin{aligned} {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha &= {}^{(4)}\nabla_u X^\alpha + [a(u) \wedge u]^{\alpha\beta} X_\beta \ , \\ {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \ . \end{aligned} \tag{10}$$

These may be extended to arbitrary tensor fields in the usual way. For an arbitrary vector field X the following relations hold

$$\begin{aligned} \pounds_u X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u) X^\alpha + [\omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta] X^\beta \\ &= {}^{(4)}\nabla_{(\mathrm{cfw})}(u) X^\alpha + [-\theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta] X^\beta \ . \end{aligned} \tag{11}$$

A spatial co-rotating Fermi-Walker derivative $\nabla_{(\mathrm{cfw})}(u)$ ("co-rotating Fermi-Walker temporal derivative") may be defined in a way analogous to the ordinary one, such that the three temporal derivatives have the following relation when acting on a spatial vector field X

$$\begin{aligned} \nabla_{(\mathrm{cfw})}(u) X^\alpha &= \nabla_{(\mathrm{fw})}(u) X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u) X^\alpha + \theta(u)^\alpha{}_\beta X^\beta \ . \end{aligned} \tag{12}$$

It is convenient to use an index notation to handle these three operators simultaneously

$$\{\nabla_{(\mathrm{tem})}(u)\}_{\mathrm{tem=fw,cfw,lie}} = \{\nabla_{(\mathrm{fw})}(u), \nabla_{(\mathrm{cfw})}(u), \nabla_{(\mathrm{lie})}(u)\} \ . \tag{13}$$

4 The measurement process for two observers in relative motion

Suppose U is another unit timelike vector field representing a different family of test observers. One can then consider relating the "observations" of each to the other. Their relative velocities are defined by

$$U = \gamma(U,u)[u + \nu(U,u)] \ , \qquad u = \gamma(u,U)[U + \nu(u,U)] \ , \tag{14}$$

where the relative velocity $\nu(U,u)$ of U with respect to u is spatial with respect to u and vice versa, both of which have the same magnitude $||\nu(U,u)|| = [\nu(U,u)_\alpha \nu(U,u)^\alpha]^{1/2}$, while the common gamma factor is related to that magnitude by $\gamma(U,u) = \gamma(u,U) = [1 - ||\nu(U,u)||^2]^{-1/2} = -U_\alpha u^\alpha$. Let $\hat\nu(U,u)$ be the unit vector giving the direction of the relative velocity $\nu(U,u)$.

In addition to the natural parametrization of the worldlines of U by the proper time τ_U, one may introduce two new parametrizations: a) by a (Cattaneo) relative standard time $\tau_{(U,u)}$

$$d\tau_{(U,u)}/d\tau_U = \gamma(U,u) \ , \tag{15}$$

which corresponds to the sequence of proper times of the family of observers from the u congruence which cross paths with a given worldline of the U congruence, and b) by a relative standard length $\ell_{(U,u)}$

$$d\ell_{(U,u)}/d\tau_U = \gamma(U,u)||\nu(U,u)|| = ||\nu(U,u)||d\tau_{(U,u)}/d\tau_U \ , \tag{16}$$

which corresponds to the spatial arclength along U as observed by u.

Eqs. (14) describe a "relative observer boost" $B(U,u)$ in the "relative observer plane" spanned by u and U such that

$$B(U,u)u = U\ , \quad B(U,u)\nu(U,u) = -\nu(u,U) \tag{17}$$

and which acts as the identity on the common subspace of the local rest spaces $LRS_u \cap LRS_U$ orthogonal to the direction of motion. The inverse boost $B(u,U)$ "brings U to rest" relative to u.

4.1 Maps between the LRSs of different observers

The projection $P(U)$ restricts to an invertible map

$$P(U,u) = P(U) \circ P(u) : LRS_u \to LRS_U \tag{18}$$

with inverse $P(U,u)^{-1} : LRS_U \to LRS_u$ and vice versa; these maps also act as the identity on the common subspace of the local rest spaces. Similarly the boost $B(U,u)$ restricts to an invertible map

$$B_{(\mathrm{lrs})}(U,u) \equiv P(U) \circ B(U,u) \circ P(u) \tag{19}$$

between the local rest spaces which also acts as the identity on their common subspace. One can explictly evaluate expressions for the tensors representings these maps

$$P(U,u)^{-1} = [P(U) + U \otimes \nu(u,U)^\flat]\ , \qquad P(U,u) = P(u) + \gamma U \otimes \nu(U,u)\ . \tag{20}$$

They appear in the transformation law for the electric and magnetic fields:

$$\begin{aligned} E(u) &= \gamma P(U,u)^{-1}[E(U) + \nu(u,U) \times_U B(U)]\ ,\\ B(u) &= \gamma P(U,u)^{-1}[B(U) - \nu(u,U) \times_U E(U)]\ . \end{aligned} \tag{21}$$

One can also consider the individual projections parallel and perpendicular to the direction of relative motion between the local rest spaces and within each local rest space which allow a further "1+2" splitting of the local rest space of the observer along the (longitudinal) direction of the relative velocity and into the (transverse) orthogonal plane

$$P^{(||)}(U,u) = -\gamma \hat\nu(u,U) \otimes \hat\nu(U,u)^\flat\ ,\ P^{(\perp)}(U,u) = P(U,u) - P^{(||)}(U,u)\ , \tag{22}$$

where $P(U,u)\hat\nu(u,U) = -\gamma\hat\nu(U,u)$ explains the γ factor in the first relation. Of course analogous considerations hold for the boost map. One can finally study composition of these projection maps; for example $P(u,U)P(U,u)$ is an isomorphism of LRS_u into itself which turns up in manipulations with these maps. It and its inverse have the following expressions

$$\begin{aligned} P(u,U,u) &= P(u,U)P(U,u) = P(u) + \gamma^2 \nu(U,u) \otimes \nu(U,u)^\flat \ , \\ P(u,U,u)^{-1} &= P(U,u)^{-1}P(u,U)^{-1} = P(u) - \nu(U,u) \otimes \nu(U,u)^\flat \ . \end{aligned} \tag{23}$$

5 The measurement process for three or more observers in relative motion

A typical situation is that of a fluid/particle which is observed by two differrent families of observers. In this case one deals with three timelike congruences: the 4-velocity of the fluid/particle U and the two observer families, u and u'. All the previous formalism can be easily generalized. One has

$$\begin{aligned} U &= \gamma(U,u)[u + \nu(U,u)] = \gamma(U,u')[u' + \nu(U,u')] \ , \\ u' &= \gamma(u',u)[u + \nu(u',u)] \ , \qquad u = \gamma(u,u')[u' + \nu(u,u')] \ , \end{aligned} \tag{24}$$

and composition of projectors involving the various 4-velocities can be introduced. For example for $P(u,U,u') = P(u,U)P(U,u')$ one has

$$\begin{aligned} P(u,U,u') &= P(u,u') + \gamma(U,u)\gamma(U,u')\nu(U,u) \otimes \nu(U,u') \\ P(u,U,u')^{-1} &= P(u',u) + \gamma(u,u')[(\nu(u,u') - \nu(U,u')) \otimes \nu(U,u) \\ &\quad + \nu(U,u') \otimes \nu(u',u)] \ . \end{aligned}$$

6 Derivatives

Consider the "total covariant derivative" along U

$$^{(4)}D(U)/d\tau_U = {}^{(4)}\nabla_U \ . \tag{25}$$

Its spatial projection with respect to u and rescaling corresponding to the reparametrization of Eq. (16) is then given by the "Fermi-Walker total spatial covariant derivative," defined by

$$\begin{aligned} D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)} &= \gamma^{-1} D_{(\mathrm{fw},U,u)}/d\tau_U = \gamma^{-1} P(u)^{(4)}D(U)/d\tau_U \\ &= \nabla_{(\mathrm{fw})}(u) + \nabla(u)_{\nu(U,u)} \ . \end{aligned} \tag{26}$$

Extend this to two other similar derivative operators (the co-rotating Fermi-Walker and the Lie total spatial covariant derivatives) by

$$D_{(\mathrm{tem},U,u)}/d\tau_{(U,u)} = \nabla_{(\mathrm{tem})}(u) + \nabla(u)_{\nu(U,u)} \ , \qquad \mathrm{tem}{=}\mathrm{fw,cfw,lie} \ , \tag{27}$$

which are then related to each other in the same way as the corresponding temporal derivative operators

$$\begin{aligned} D_{(\mathrm{cfw},U,u)}X^\alpha/d\tau_{(U,u)} &= D_{(\mathrm{fw},U,u)}X^\alpha/d\tau_{(U,u)} + \omega(u)^\alpha{}_\beta X^\beta \\ &= D_{(\mathrm{lie},U,u)}X^\alpha/d\tau_{(U,u)} + \theta(u)^\alpha{}_\beta X^\beta \end{aligned} \tag{28}$$

when acting on a spatial vector field X. All of these derivative operators reduce to the ordinary parameter derivative $D/d\tau_{(U,u)} \equiv d/d\tau_{(U,u)}$ when acting on a function and extend in an obvious way to all tensor fields.

Introduce the ordinary and co-rotating Fermi-Walker and the Lie "relative accelerations" of U with respect to u by

$$a_{(\mathrm{tem},U,u)} = D_{(\mathrm{tem},U,u)}\nu(U,u)/d\tau_{(U,u)} \ , \qquad \mathrm{tem}{=}\mathrm{fw,cfw,lie} \ . \tag{29}$$

These are related to each other in the same way as the corresponding derivative operators in Eq. (12). For a test particle world line as seen by an observer congruence, one can then study the relative kinematics, including further orthogonal decompositions in either local rest space along and orthogonal to the direction of relative motion, leading to the introduction of various centripetal and centrifugal forces. Their influence on the test particle motion and spin-evolution along it can then be investigated.

7 Conclusions

The general features of the observer-dependent measurement process have been considered in detail for spacetime splittings in general relativity. All the mathematical tools needed to consider a generic observer for whatever measurement one has in mind (particle or field) are described in a unified framework, paving the way to a better understanding of the role of special observer families in special spacetimes from a geometrical point of view.

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