Understanding Spacetime Splittings and Their Relationships

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Abstract

A single mathematical framework is introduced to discuss both the hypersurface and congruence approaches to splitting spacetime and to clarify their relationship to each other and to the invariant 4-geometry.

1 Introduction

From the presentation of Cattaneo's spacetime splitting formalism discussed by Professor Massa, one thing is very evident. To those of us who are better acquainted with the so-called "ADM formalism", the Cattaneo approach seems entirely foreign, while those of us who have instead developed more familiarity with the latter approach probably find the ADM one equally alien. This is exactly the problem I wish to address, namely the lack of a common mathematical framework to discuss both approaches on an equal footing. In fact both of these splitting formalisms can be developed in a completely parallel way as complementary aspects of a single geometrical structure imposed on spacetime, aspects which in a close way are related by the same duality that links contravariant and covariant fields on the spacetime manifold. The style of the Cattaneo approach as usually presented [1–3] is somewhat more cumbersome than the ADM one, so I will recast it in the ADM style, generalizing from adapted coordinate systems to adapted local frames. In the special case of a "threading" of the spacetime slicing by an orthogonal congruence of time lines, the two descriptions will then coincide.

The Cattaneo congruence approach was developed independently by Zel'manov [4,5] in the Soviet Union, expanding upon a framework apparently originally due to Landau and Lifshitz in their well known text *The Classical Theory of Fields* [6,7], and then extended by Møller in the first edition of his text *The Theory of Relativity* [8]. The "ADM hypersurface approach" was not developed by Arnowit, Deser and Misner [9] but merely used by them to discuss the true degrees of freedom of the gravitational field in a notation that has been widely used since then and which is discussed in the text *Gravitation* by Misner, Thorne and Wheeler [10]. The hypersurface approach is instead originally due to Lichnerowicz [11] and Choquet-Bruhat [12], successively refined and extended by many others. (See Thorne and MacDonald [13] for a summary.)

Unfortunately much of this history is nearly inaccessible to the generation to which I belong, due to combined difficulties of published work which is not well known or easily located and the frequent lack of an English translation. Among non-Italian relativists who are not of the generation which had direct or indirect contact with the work of Cattaneo during the renaissance of relativity which took place some decades ago, his work is virtually unknown. I became aware of it only a year ago when Remo Ruffini lent me a book of published lecture notes [3] from Cattaneo's course on relativity at the University of Rome dating back to that period. The work of Zel'manov is also somewhat inaccessible. Only during the course of rewriting this present article with Paolo did we become aware of the work of Massa [14–16] in reformulating the Cattaneo approach and analyzing the spin precession formula with Zordan [17].

Since I am in no position to reconstruct the sequence of contributions made by many individuals to the various splitting approaches described in this lecture over a period of time dating back to the forties, I will not attempt to do so. Instead I will try to break down the the formalism barriers that have developed over the years by introducing a common mathematical language, the general idea of which I hope to indicate in this lecture. The details [18] and their application to electromagnetism and to the Sagnac effect and synchronization questions can be examined later by those interested in doing so.

Of course the first question to address is "Why split spacetime at all?" Certainly the idea of a four-dimensional spacetime and its local Lorentzian geometry has been an important advance of this century. However, in many applications a splitting naturally occurs, and this has the advantage of allowing our three-dimensional intuition and experience to interface better with the four-dimensional information. Many spacetimes of interest have either a preferred slicing by a family of spacelike hypersurfaces or a preferred congruence of timelike curves or both. The two formalisms which are built around these geometrical structures correspond to the two independent notions of "time", one involving the nonlocal concept of time as a hypersurface of simultaneity – some kind of synchronization of a family of arbitrarily running clocks, and another involving the local concept of time as the proper time measured by test observers along their worldlines independently of one another. Of the two formalisms, the hypersurface approach is much more developed and well known due to the Hamiltonian formulation it gives to the Einstein equations, a formulation used as a starting point for much of the work in quantum gravity. The congruence approach on the other hand is perhaps more natural for stationary spacetimes and the exact solution industry active in that area.

The present work was initiated with the aim of clarifying the relationship between the various ways of treating Maxwell's equations in three-dimensional form [18]. The need for this clarification comes from the cloudy relationship between three-dimensional and four-dimensional quantities one finds in the many variations of the Landau-Lifshitz splitting of the electromagnetic 2-form and in other circumstances as well. Precisely because of the power of four-dimensional geometrical methods, it is important to understand how three-dimensional concepts are imposed on spacetime and how these relate to physical measurements. In order to describe these ideas unambiguously, a neutral terminology will be introduced which at the same time detaches the concepts from names that history has fairly or unfairly attached to them.

First the terminology "*nonlinear reference frame*" (nonlinear to distinguish it from the alternative connotation of "frame" as a linear frame of vector fields) will replace the somewhat vague notion of "reference frame" or "system of reference" or "reference system" in the literature, and will refer to a compatible pair consisting of a *slicing* (equivalently foliation) of spacetime together with a congruence or "*threading*" of spacetime, the compatibility condition being transversality of the two families of hypersurfaces and curves. The "*slicing point of view*" will denote the approach frequently referred to as the ADM approach in which the slicing is assumed to be spacelike and no causality condition is imposed on the threading, which is merely used to identify points on different hypersurfaces of the family of slices. The threading associated with the nonlinear reference frame established by the Boyer-Lindquist coordinate system in the ergosphere of a black hole [10], as well as the one associated with uniformly rotating cartesian or polar coordinates in flat spacetime outside the cylinder at which the velocity of rotation exceeds the velocity of light, are in fact both spacelike. Reviews of various aspects of the slicing point of view, also referred to as the $3+1$ approach, have been given by Isenberg and Nester [19], York [20], Fischer and Marsden [21], and Gotay et al [22], among others.

The "*threading point of view*" will instead refer to the formalism based on a timelike threading with no causality condition imposed on the slicing, which will serve the role of synchronizing the time parameters along different curves in the threading congruence according to some arbitrary scheme. This apparently originated with Landau and Lifshitz in the first addition of their abovementioned text and the most well known and accessible exposition remains the latest version of this text [7] but its discussion is limited to the stationary case. The general case was dealt with in the early fifties by Møller in the first edition of his text [8] in a time-coordinate formalism and then refined in the late fifties by Zel'manov in the Soviet Union and independently by Cattaneo in Italy to yield a time-reparametrization independent formalism to be described here. The second edition of Møller's text [8] reviews the Zel'manov/Cattaneo formalism as well as his own approach. Only while writing this article did I become aware of Møller's work (following up comments after the lecture) and of the English language articles of Cattaneo in Nuovo Cimento [1]. Traces of this work in the form of references just do not seem to appear in the mainstream relativity literature of the past two and a half decades.

The phrase "*congruence point of view*" will be reserved for the threading formalism introduced by Ehlers [23] and reviewed by Hawking [24] and Ellis [25,26] based only on a congruence with no accompanying slicing. Here only a unit timelike vector field u is assumed, without explicit knowledge of its integral curves or its relationship to any special coordinates. The dyad formalism of Estabrook and Wahlquist [27] completes the unit timelike vector field u to an orthonormal frame by the addition of an orthonormal frame in the subspace of directions orthogonal to the congruence. These formalisms are easily translated into the threading point of view once a nonlinear reference frame is adapted to the unit timelike vector field by choosing u to be the unit tangent to the threading, but one can also translate them into the slicing point of view by choosing u to be the unit normal to the slicing. Maxwell's equations were written in the threading point of view by Benvenuti [28] a decade earlier than Ellis's closely related congruence version [26] but it was lost to the international relativity community since it appeared only in Italian.

In the special but very interesting case in which the slicing is spacelike and the threading is timelike, both points of view hold simultaneously. If the slicing and threading are orthogonal, then the two points of view coincide, but otherwise a well-defined nontrivial transformation procedure exists relating the two descriptions to each other. One also has another point of view which always holds independent of the causality conditions, although the use of "temporal" and "spatial" doesn't make any sense in this context unless at least one of those conditions holds. This is the "*reference point of view*" in which one simply decomposes tensor equations in terms of natural projections along the slicing and threading. Such an approach is used by Landau and Lifshitz in their treatment of Maxwell's equations. It is also used in classical mechanics when discussing the centrifugal and Coriolis forces which arise in rotating Cartesian coordinates. These fictitious forces are important for the interpretation of their natural generalization to spatial gravitational forces in general relativity.

The payoff for examining the slicing and threading points of view more carefully and especially their relationship to each other is a clarification of the buzzwords "gravitoelectric field" and "gravitomagnetic field" which have been introduced recently by Thorne [29–32] to describe these spatial gravitational forces. These concepts have not been unambiguously defined since they have occurred only in the linearized or stationary context in points of view which switch without warning between the slicing, threading and reference perspectives. This dates back to the discussion of Forward [33] of the analogy between electromagnetism and linearized general relativity, itself a slight variation of the earlier work of Møller. Spatial gravitational forces are discussed for the stationary threading case in the Landau-Lifshitz text [7] and in general by Møller [8], Cattaneo [1–3], Zel'manov [4,5] and Massa [15]. The gravitomagnetic field allows a simple threading description of the Sagnac effect and synchronization questions in the stationary case [34–38]. More importantly this closer look at formalism enables one to have a much cleaner discussion of the precession of the spin of a gyroscope in a gravitational field.

A convenient mix of concrete-index and index-free notation will be used in our discussion. The conventions of Misner, Thorne and Wheeler [10] will be followed unless stated otherwise or unless an ambiguity arises. The spacetime metric tensor will be denoted by ${}^{(4)}\mathrm{g}$ in an index-free notation (and the inverse or contravariant metric tensor by g^{-1}) so that the symbol ${}^{(4)}g \equiv |\det({}^{(4)}g_{\alpha\beta})|$ can denote (the absolute value of) its determinant as is customary in index notation. The "index lowering" and "index raising" maps associated with the spacetime metric will be denoted by $\flat$ and $\sharp$ respectively, and the symbols $S^\flat$ and $S^\sharp$ will refer to the fully covariant and fully contravariant forms respectively of a given tensor field S. Finally the spacetime or portion of spacetime under discussion will always be assumed to be orientable and time-orientable and local coordinates and frames will be assumed to be compatible with these orientations when appropriate.

2 The parametrized nonlinear reference frame

The nonlinear reference frame plays of fundamental role in the splitting formalism in two ways, first in terms of the *measurement* of spacetime quantities by a certain family of test observers associated with it, and second in terms of the description of the *evolution* of these quantities. Usually a nonlinear reference frame is introduced by means of an explicit choice of adapted spacetime coordinates $\{x^0 = t, x^a\} = \{x^\alpha\}_{\alpha=0,1,2,3}$ for which the "*time function*" t labels the hypersurfaces of the slicing and parametrizes the individual curves in the threading congruence while the three "spatial coordinates" $\{x^a\}_{a=1,2,3}$ parametrize the family of threading curves.

A choice of parametrization of the family of slices by itself provides the spacetime with a time function t and a natural parameter on each threading curve, leading to the more practical structure of a "*parametrized nonlinear reference frame*". The differential $dt \equiv \omega^0$ defines the "*slicing 1-form*" whose kernel is the distribution of subspaces tangent to the slicing. The parametrized threading curves define a tangent vector field e_0, the "*threading vector field*", whose 1-parameter group of diffeomorphisms has the threading congruence as its family of orbits. This group introduces a precise way of describing evolution, corresponding at a differential level to its associated Lie derivative. The dual role of t as a time function and as a parameter along the integral curves of e_0 is represented by the compatibility condition $\omega^0(e_0) = 1$.

The causality property of the nonlinear reference frame introduces the equally important concept of measurement. In each point of view, three-dimensional quantities must be interpreted in terms of the orthogonal decomposition of the tangent space induced by the 4-velocity of a family of *test observers* and their associated local rest spaces. When both points of view hold, the case of a spacelike slicing and a timelike threading, then a well-defined transformation between them exists.

In the slicing point of view, the slicing 1-form is timelike and can be normalized

$$N^{-2} = -{}^{(4)}\mathrm{g}^{-1}(\omega^0, \omega^0) \ , \qquad \omega^\perp = N\omega^0 \ , \qquad {}^{(4)}\mathrm{g}^{-1}(\omega^\perp, \omega^\perp) = -1 \ , \tag{2.1}$$

reversed in sign, and its index raised to obtain the future-pointing unit normal $n \equiv e_\perp$ to the slicing. This may be interpreted as the 4-velocity field of a family of test observers whose worldlines coincide with the normal congruence. The unit 1-form $\omega^\perp = -n^\flat$ has as its kernel the integrable distribution of local rest spaces of these observers, to be denoted by LRS_n. This 1-form also measures the differential of proper time along their individual worldlines. The "perp" index "$\perp$" indicates perpendicularity to the slicing.

In the threading point of view, the threading vector field is timelike and can be normalized

$$M^2 = -{}^{(4)}\mathrm{g}(e_0, e_0) \ , \qquad e_\top = M^{-1}e_0 \ , \qquad {}^{(4)}\mathrm{g}(e_\top, e_\top) = -1 \ , \tag{2.2}$$

to obtain a unit tangent vector field $e_\top \equiv m$ which may be interpreted as the 4-velocity of a family of test observers. The sign-reversed index-lowered 1-form $\omega^\top = -m^\flat$ has as its kernel the (in general) nonintegrable family of local rest spaces of this family of observers, to be denoted by LRS_m. This 1-form measures the differential of proper time along their worldlines. The index symbol"$\top$" is the reflection of "$\perp$" to a dual position, where it conveniently suggests the letter "T" for tangential to the threading and should probably be pronounced "tan".

The normalization factors N and M, expressable as

$$N = \omega^\perp(e_0) \ , \qquad M^{-1} = \omega^0(e_\top) \ , \tag{2.3}$$

or equivalently as

$$N^{-1} = \omega^0(e_\perp) \ , \qquad M = \omega^\top(e_0) \ , \tag{2.4}$$

will both be referred to as the *lapse function*, with the clarifier "slicing" or "threading" preceding this term when necessary to distinguish the two points of view. (The slicing point of view notation for the lapse and shift is that of Arnowit, Deser and Misner [9], while the terminology is due to Wheeler [39].) The lapse function in each case describes the rate of change of the observer proper time with respect to the coordinate time. Given any timelike curve parametrized by the time function t, one can rescale the tangent vector by the reciprocal lapse function to correspond to a new parametrization by the proper time τ_n and τ_m respectively measured by the appropriate test observers. Using a sloppy classical notation one has the coordinate velocity and the unit 4-velocity defined by

$$U^\alpha = dx^\alpha/dt \ , \qquad u^\alpha = dx^\alpha/d\tau_u = \Gamma U^\alpha \ , \qquad \Gamma = |U^\beta U_\beta|^{-1/2} = dt/d\tau_u \ , \tag{2.5}$$

which in turn may be rescaled to represent the rate of change of position with respect to observer proper time

$$\begin{aligned} \text{slicing:} \qquad & d\tau_n/dt = N \ , \qquad dx^\alpha/d\tau_n = N^{-1}U^\alpha \ , \\ \text{threading:} \qquad & d\tau_m/dt = M \ , \qquad dx^\alpha/d\tau_m = M^{-1}U^\alpha \ , \end{aligned} \tag{2.6}$$

where τ_u indicates the reparametrization of the curve by its own proper time and u is the unit tangent to the curve. The quantity Γ is the "coordinate gamma factor" introduced by Møller [8]. When both points of view hold, the case of a spacelike slicing and a timelike threading, then

$d\tau_n/d\tau_m = N/M = \gamma(m,n)$ relates the two observer proper times along this worldline and defines the Lorentz gamma factor of the boost in the plane of n and m which maps n onto m.

Given the family of test observers in each point of view, one can decompose all tensor fields on the spacetime using the orthogonal decomposition of the tangent space induced by their unit 4-velocity o, which will stand respectively for n and m in the two points of view. For any timelike unit vector field o, this orthogonal decomposition of the tangent space is accomplished by two projections

$$\begin{aligned} X &= T(o)X + P(o)X \ , \\ [T(o)X]^\alpha &= T(o)^\alpha{}_\beta X^\beta = (-o^\alpha o_\beta)X^\beta \ , \\ [P(o)X]^\alpha &= P(o)^\alpha{}_\beta X^\beta = (\delta^\alpha{}_\beta + o^\alpha o_\beta)X^\beta \ , \end{aligned} \tag{2.7}$$

where X is an arbitrary vector field. The first "*time projection*" $T(o)$ ("T" for tangential or time) projects along the local time direction of the observer with 4-velocity o, while the second "*spatial projection*" $P(o)$ ("P" for perpendicular or projection, to conform with the symbol conventionally used in these discussions) projects into the local rest space LRS_o associated with that observer. A single vector field X can either be represented as a sum of the vector fields $T(o)X$ and $P(o)X$, or as a pair consisting of the scalar $X^{(o)} = -o_\beta X^\beta$ and the vector $P(o)X$. The two projections can be identified with $\binom{1}{1}$-tensor fields acting on vector fields and 1-forms by contraction on the right or on the left, and by repeated contractions on higher rank tensor fields. In the same way one can extend the orthogonal decomposition to any tensor field and either represent that field as a sum of terms, each of which corresponds to a piece of the tensor product of the orthogonal decomposition on single indices, or as a collection of tensor fields of different ranks, corresponding to all possible distinct contractions of the original tensor field with the sign-reversed unit 4-velocity $-o$ rather than projection by $T(o)$. These latter tensor fields are "*spatial tensor fields*" with respect to this decomposition, in the sense that they give zero upon evaluation of any argument by o or $o^\flat$ as appropriate. The details are much easier to describe in terms of components with respect to a frame adapted to the nonlinear reference frame.

Once a given tensor field has been decomposed into a collection of spatial tensor fields in each point of view, corresponding to the notion of measurement, one can discuss the evolution of these "measured quantities", namely how they change along the threading congruence relative to the structure imposed on spacetime by the nonlinear reference frame. Given a parametrization of this nonlinear reference frame, the 1-parameter group of diffeomorphisms of the threading vector field e_0 provides a natural way of comparing fields at different "times" relative to that reference frame itself, namely by Lie dragging. "No evolution" corresponds to Lie invariance. However, if one evolves spatial fields rather than spacetime fields, one must compose the Lie dragging with the spatial projection in order to compare spatial fields since in general, spatial fields will not remain spatial under Lie dragging. At the differential level, one therefore needs a *spatial Lie derivative* [19] along e_0 rather than a Lie derivative along e_0. This derivative $\pounds(o)_{e_0}$ is defined for a spatial tensor S by spatially projecting the Lie derivative by e_0 on all indices

$$\pounds(o)_{e_0} S = P(o)[\pounds_{e_0} S] \ , \tag{2.8}$$

using the convention that for an arbitrary tensor S, the notation $P(o)S$ denotes its spatial projection on all indices. This projected Lie derivative has the property

$$\pounds(o)_{fo} S = f[\pounds(o)]_o S \ , \tag{2.9}$$

for any spatial tensor S and arbitrary function f; with the value $f = M$ in the threading point of view, this enables one to rewrite the e_0 Lie derivative as an m Lie derivative and vice versa.

Two approaches may be taken in describing the measurement and evolution of spacetime fields associated with a given nonlinear reference frame and a given choice of either the slicing or threading point of view. The first option is to work on the spacetime itself with the collection of spatial spacetime fields that come from each spacetime tensor field. The second option is to use a time-dependent isomorphism with the tensor algebra on some fixed 3-dimensional manifold, the "*Space*" of the chosen point of view. (A third variation instead makes use of an isomorphism between the orthogonal splitting on spacetime and another based on a nonorthogonal "reference decomposition" adapted to the nonlinear reference frame to be described below and which is in turn closely related to the "Space" representation by another isomorphism.)

In the slicing point of view, one makes use of a 1-parameter family I_t of imbeddings of the abstract slice manifold Σ into the spacetime ${}^{(4)}M$ yielding the 1-parameter family of slices $\Sigma_t = I_t(\Sigma)$. Σ is the "Space" manifold in this point of view. In the threading point of view one instead uses a 1-parameter family of "projections" π_t from the individual slices Σ_t down to the quotient space ${}^{(4)}M/\,\mathrm{Flow}(e_0)$ of the spacetime by the flow of e_0. This quotient space is the threading "Space" manifold. The "projection" π_t results from restricting the quotient space projection $\pi : {}^{(4)}M \to {}^{(4)}M/\,\mathrm{Flow}(e_0)$ to the slice Σ_t. Note that in both points of view a "fixed point of Space" corresponds to a threading curve. Figure 1 illustrates these ideas. Using the slicing point of view in the black hole context Thorne [13] originally referred to the slicing Space as "absolute space", but later decided to use this term for the space of observer worldlines rather than threading curves, leading to motion of the grid of an adapted spatial coordinate system relative to it [31]. Gerosh [40] explored the threading Space in the case of stationary spacetimes while studying solution generation techniques.

In the slicing point of view the fully covariant form of all the members of the collection of spatial tensor fields obtained by splitting a given spacetime tensor field may be *pulled back* to the slicing Space Σ using the 1-parameter family of imbeddings I_t to obtain a family of time-dependent covariant fields on that manifold. Applying this process to the spacetime metric itself only yields the constant scalar -1, the zero 1-form, and the second rank spatial metric tensor whose inverse may be used to raise indices at will on all such time-dependent fields. The remaining pieces of the spacetime metric must be obtained by applying this process to the threading vector field, the choice of which represents differentially the spatial gauge freedom in the slicing point of view. Fischer and Marsden and collaborators [21,22] use this pullback approach.

In the threading point of view the fully contravariant form of all the members of the collection of spatial tensor fields obtained by splitting a given spacetime tensor field may be *pushed forward* (downward) to the threading Space ${}^{(4)}M/\,\mathrm{Flow}(e_0)$ using the 1-parameter family of projections π_t to obtain a family of time-dependent contravariant fields on that manifold. Applying this process to the spacetime metric itself only yields the constant scalar -1, the zero vector, and the second rank inverse spatial metric tensor whose inverse may be used to lower indices at will on other time-dependent fields. The remaining pieces of the spacetime metric must be obtained by applying this process to the slicing 1-form, the choice of which represents differentially the spatial gauge freedom in the threading point of view.

Since the metric tensor itself determines the orthogonal decomposition, the splitting process loses information which can be regained only by splitting the threading vector field in the slicing point of view and the slicing 1-form in the threading point of view. In this way one recovers the lapse function already discussed as the normalizing factor for the slicing 1-form and threading vector field respectively from the scalar part and one obtains the slicing *shift vector field* or threading *shift*

1-form respectively from the rank one part

$$
\begin{aligned}
e_0 &= T(n)e_0 + P(n)e_0 = Nn + \vec{N} \quad \mapsto \quad (N, \vec{N}) \ , \\
\omega^0 &= T(m)\omega^0 + P(m)\omega^0 = M^{-1}(-m^\flat) + \bar{\bar{M}} \quad \mapsto \quad (-M^{-1}, \bar{\bar{M}}) \ .
\end{aligned}
\tag{2.10}
$$

Again the modifier "slicing" or "threading" will precede the term "shift" when necessary to distinguish the two points of view.

In slicing point of view the shift is most naturally considered as a vector field $\vec{N}$, and it determines the tilting of the threading curves away from the normal direction n, and consequently the shifting of the "points of Space" away from the identification associated with the normal congruence. In the threading point of view the shift is most naturally considered as a 1-form $\bar{\bar{M}}$, and it determines the tilting of the threading local rest spaces LRS_m away from the directions tangent to the slicing. Let $\bar{\bar{N}} = \vec{N}^\flat$ and $\vec{M} = \bar{\bar{M}}^\sharp$ denote the slicing shift 1-form and the threading shift vector field respectively. The vector oversymbol notation (vector like an arrow in the tangent space) and the double overbar notation (1-form like a pair of parallel planes in the tangent space) is necessary when an index-free notation is used in order to conform with the identical kernel symbols for the lapse and shift established by the slicing point of view conventions.

Of course indices can be very useful and a particular choice of frame used to introduce them is wise if one is interested in viewing the spacetime from the point of view of the nonlinear reference frame, as indeed we are. Given an explicit parametrization of the nonlinear reference frame, it is natural to complete the threading vector field e_0 to a frame adapted to both the threading and slicing and which may be thought of as a linear extension of the given parametrized nonlinear reference frame. It suffices to choose a frame $\{e_a\}$ for the 3-dimensional subspace of the tangent space which is tangent to the slicing and which is Lie dragged along e_0

$$
[e_a, e_b] = C^c{}_{ab} e_c \ , \qquad [e_0, e_a] = 0 \ . \tag{2.11}
$$

An unambiguous term for such a frame already exists, named a "*computational frame*" by York, generalizing the coordinate frame of adapted coordinates $\{t, x^a\}$ which occurs as the special case $C^c{}_{ab} = 0$. Adapted coordinate frames often prove useful for local computations. Note that only the "spatial" structure functions of the computational frame are nonzero: $C^\gamma{}_{\alpha\beta} = \delta^\gamma{}_c \delta^a{}_\alpha \delta^b{}_\beta C^c{}_{ab}$.

The dual frame $\{\omega^0, \omega^a\}$ consists of the slicing 1-form (whose kernel is the slicing tangent subspace) and three 1-forms which annihilate the threading vector field. Expressing a tensor field in terms of components with respect to a computational frame leads to a "*reference decomposition*" of the field according to the time index 0 and the spatial indices 1,2,3, corresponding to the (in general) nonorthogonal decomposition of the tangent space into the direct sum of the slicing subspace and the threading subspace. For example, the reference decomposition of a vector field and a 1-form defines the "*reference time projection*" $\mathcal{T}$ and the "*reference spatial projection*" $\mathcal{P}$ for one-index tensors

$$
\begin{aligned}
X &= X^0 e_0 + X^a e_a = [\mathcal{T}X] + [\mathcal{P}X] \ , \\
X^\flat &= X_0 \omega^0 + X_a \omega^a = [\mathcal{T}X^\flat] + [\mathcal{P}X^\flat] \ .
\end{aligned}
\tag{2.12}
$$

One can even speak of a collection of "*reference spatial fields*" into which a tensor decomposes as in the orthogonal decomposition associated with the observers. For a given type of tensor field, there is in fact an isomorphism between the orthogonal decomposition and certain pieces of the reference decomposition of the family of tensor fields related to the original one by index raising and lowering.

To see this it is enough to project the computational frame as necessary to obtain a frame adapted to the observer orthogonal decomposition, which then establishes the appropriate isomorphism with the reference decomposition. This observer adapted frame will be called the *projected computational frame*. In the slicing point of view, the reference spatial vector fields e_a are already adapted to the observer local rest space LRS_n and it suffices to project e_0 along the normal direction, while the slicing 1-form ω^0 is proportional to the covariant unit normal $n^\flat$ and one need only project the reference spatial 1-forms ω^a orthogonally to the normal direction

$$\begin{aligned} \text{slicing definitions:} \quad & \epsilon_0 = T(n) e_0 \ , \qquad \theta^a = P(n)\omega^a \ , \\ \text{projected computational frame:} \quad & \{\epsilon_0, e_a\} \ , \\ \text{dual frame:} \quad & \{\omega^0, \theta^a\} \ . \end{aligned} \tag{2.13}$$

In the threading point of view, the threading vector field e_0 is already tangent to the threading subspace and it suffices to project the reference spatial vectors e_a into the local rest space LRS_m, while the reference spatial 1-forms ω^a are already to orthogonal to the local time direction and one need only project the slicing 1-form along it

$$\begin{aligned} \text{threading definitions:} \quad & \epsilon_a = P(m) e_a \ , \qquad \theta^0 = T(m)\omega^0 \ , \\ \text{projected computational frame:} \quad & \{e_0, \epsilon_a\} \ , \\ \text{dual frame:} \quad & \{\theta^0, \omega^a\} \ . \end{aligned} \tag{2.14}$$

In each case the isomorphisms from the reference projection eigenspaces to those of the observer orthogonal projection spaces are just the corresponding orthogonal projections themselves. The inverse isomorphisms are just the reference projections $\mathcal{T}$ and $\mathcal{P}$, whose nontrivial values on the projected computational frame and dual frame are

$$\begin{aligned} \text{slicing:} \quad & \mathcal{T}\epsilon_0 = e_0 \ , \qquad \mathcal{P}\theta^a = \omega^a \ , \\ \text{threading:} \quad & \mathcal{P}\epsilon_a = e_a \ , \qquad \mathcal{P}\theta^0 = \omega^0 \ . \end{aligned} \tag{2.15}$$

Thus one can continue to work with the computational frame using these isomorphisms rather than explicitly introducing new components with respect to the projected computational frame, although for certain purposes like differentiation the latter is essential.

The orthogonal projections of the computational frame vectors and 1-forms define the shift components in each point of view; alternatively the matrix of the linear transformation between the two frames determines the shift components

$$\begin{aligned} \text{slicing:} \quad \epsilon_0 =& T(n) e_0 = e_0 - N^a e_a \ , \qquad \theta^a = P(n)\omega^a = \omega^a + N^a \omega^0 \ , \\ & P(n) e_0 = N^a e_a = \vec{N} \ , \\ \text{threading:} \quad \epsilon_a =& P(m) e_0 = e_a + M_a e_0 \ , \qquad \theta^0 = T(m)\omega^0 = \omega^0 - M_a \omega^a \ , \\ & P(m)\omega^0 = M_a \omega^a = \overset{=}{M} \ . \end{aligned} \tag{2.16}$$

The two choices of shift correspond to the two possible ways of completing the square in the metric to reduce it to block-diagonal form, which is the form it takes in the projected computational frame. The lapse then determines the metric on the 1-dimensional space along the time direction and the spatial metric on the 3-dimensional spatial subspace. Indices on components taken with respect to the projected computational frame can then be shifted with the appropriate subblock of the metric.

Given these definitions, the spacetime metric and inverse metric in the computational and projected computational frame in the slicing point of view are

$$\begin{aligned} {}^{(4)}\mathrm{g} &= {}^{(4)}g_{\alpha\beta}\omega^\alpha \otimes \omega^\beta = -N^2\omega^0 \otimes \omega^0 + g_{ab}(\omega^a + N^a\omega^0) \otimes (\omega^b + N^b\omega^0) \\ &\equiv -N^2\omega^0 \otimes \omega^0 + g_{ab}\theta^a \otimes \theta^b \ , \\ {}^{(4)}\mathrm{g}^{-1} &= {}^{(4)}g^{\alpha\beta}e_\alpha \otimes e_\beta = -N^{-2}(e_0 - N^a e_a) \otimes (e_0 - N^b e_b) + g^{ab}e_a \otimes e_b \\ &\equiv -N^{-2}\epsilon_0 \otimes \epsilon_0 + g^{ab}e_a \otimes e_b \ , \end{aligned} \tag{2.17}$$

i.e., in components

$$\begin{aligned} &{}^{(4)}g_{00} = -(N^2 - N_c N^c) \ , \quad {}^{(4)}g_{0a} = N_a \ , \quad {}^{(4)}g_{ab} = g_{ab} \ , \\ &{}^{(4)}g^{00} = -N^{-2} \ , \quad {}^{(4)}g^{0a} = N^{-2}N^a \ , \quad {}^{(4)}g^{ab} = g^{ab} - N^{-2}N^a N^b \ , \end{aligned} \tag{2.18}$$

where (g^{ab}) is the matrix inverse of the positive-definite matrix (g_{ab}) and the indices on the shift vector field $\vec{N} = N^a e_a$ are lowered and raised using g_{ab} and g^{ab}. As explained above the kernel symbol "g" is used for the un-indexed metric tensor to avoid confusion with the kernel symbol "g" of the indexed components, a symbol conventionally reserved for the determinant of the matrix of metric components, here taken to be the absolute value of that determinant. In this notation one has ${}^{(4)}g^{1/2} = N g^{1/2}$. Misner, Thorne and Wheeler [10] avoid this problem by referring instead to the line element ds^2.

In the threading point of view they are instead given by

$$\begin{aligned} {}^{(4)}\mathrm{g} &= -M^2(\omega^0 - M_a\omega^a) \otimes (\omega^0 - M_b\omega^b) + \gamma_{ab}\omega^a \otimes \omega^b \\ &\equiv -M^2\theta^0 \otimes \theta^0 + \gamma_{ab}\omega^a \otimes \omega^b \ , \\ {}^{(4)}\mathrm{g}^{-1} &= -M^{-2}e_0 \otimes e_0 + \gamma^{ab}(e_a + M_a e_0) \otimes (e_b + M_b e_0) \\ &\equiv -M^{-2}e_0 \otimes e_0 + \gamma^{ab}\epsilon_a \otimes \epsilon_b \ , \end{aligned} \tag{2.19}$$

i.e., in components

$$\begin{aligned} &{}^{(4)}g_{00} = -M^2 \ , \quad {}^{(4)}g_{0a} = M^2 M_a \ , \quad {}^{(4)}g_{ab} = \gamma_{ab} - M^2 M_a M_b \ , \\ &{}^{(4)}g^{00} = -(M^{-2} - M_c M^c) \ , \quad {}^{(4)}g^{0a} = M^a \ , \quad {}^{(4)}g^{ab} = \gamma^{ab} \ . \end{aligned} \tag{2.20}$$

Here the spatial metric matrix (γ_{ab}) is positive-definite, with inverse (γ^{ab}). These are used to lower and raise indices on the shift 1-form. Letting $\gamma = \det(\gamma_{ab}) > 0$, one has ${}^{(4)}g^{1/2} = M\gamma^{1/2}$.

For the slicing and threading parametrizations of the spacetime metric, it is precisely the two explicit terms in the final representation of ${}^{(4)}\mathrm{g}$ and ${}^{(4)}\mathrm{g}^{-1}$ above which correspond to the covariant and contravariant form of the orthogonal projections along the local time and space directions. The spatial metric in each case is just the covariant form of the projection. In the slicing point of view, its restriction to a slice yields the induced metric on the slice submanifold, but in the threading point of view no such analog exists in general. However, each point of view does have a time-dependent Riemannian metric on its corresponding Space which corresponds to the spatial metric on spacetime. In each point of view this metric describes the relative distances of the worldlines of nearby observers at a given time t.

Spatial tensor fields in each approach have only spatially indexed components nonzero when expressed in the projected computational frame. These spatial indices may be raised and lowered with the respective spatial metric matrix to yield the same result as index shifting with the spacetime metric on the computational frame indices. This equivalence follows from the relations

$$
\begin{aligned}
e_a{}^\flat &= g_{ab}\theta^b \ , \qquad & \theta^{a\,\sharp} &= g^{ab} e_b \ , \\
\epsilon_a{}^\flat &= \gamma_{ab}\omega^b \ , \qquad & \omega^{a\,\sharp} &= \gamma^{ab}\epsilon_b \ ,
\end{aligned}
\tag{2.21}
$$

which hold for the bases of spatial vector fields and 1-forms in each point of view. In fact on the respective spatial subspace of the tangent space, the bases $\{\theta^{a\,\sharp}\}$ and $\{\omega^{a\,\sharp}\}$ are respectively *reciprocal* to the bases $\{e_a\}$ and $\{\epsilon_a\}$ in the classical terminology [41]. The spatial metric matrices are projected computational frame components of the fully covariant and fully contravariant forms of the spatial projection tensors, themselves spatial tensors in each point of view

$$
\begin{aligned}
P(n)^\flat &= g_{ab}\theta^a \otimes \theta^b \ , \qquad & P(n)^\sharp &= g^{ab} e_a \otimes e_b \ , \\
P(m)^\flat &= \gamma_{ab}\omega^a \otimes \omega^b \ , \qquad & P(m)^\sharp &= \gamma^{ab} \epsilon_a \otimes \epsilon_b \ .
\end{aligned}
\tag{2.22}
$$

Similar remarks apply to index raising and lowering of purely temporal fields using the lapse function.

The explicit isomorphisms between the reference decomposition subspaces and the orthogonal decomposition subspaces can easily be seen from the following explicit form of both projections themselves and their effect on a vector field X and its corresponding 1-form

$$
\begin{aligned}
Id = \mathcal{T} + \mathcal{P} \qquad &= T(n) + P(n) &&= T(m) + P(m) \ , \\
\delta^\alpha{}_\beta e_\alpha \otimes \omega^\beta = e_0 \otimes \omega^0 + e_a \otimes \omega^a &= \underbrace{\epsilon_0 \otimes \omega^0}_{e_\perp \otimes \omega^\perp} + e_a \otimes \theta^a &&= \underbrace{e_0 \otimes \theta^0}_{e_\top \otimes \theta^\top} + \epsilon_a \otimes \omega^a \ , \\
X = X^0 e_0 + X^a e_a \qquad &= \underbrace{X^0 \epsilon_0}_{X^\perp e_\perp} + X_b g^{ba} e_a &&= \underbrace{-M^{-2} X_0 e_0}_{X^\top e_\top} + X^a \epsilon_a \ , \\
X^\flat = X_0 \omega^0 + X_a \omega^a \qquad &= \underbrace{-N^2 X^0 \omega^0}_{X_\perp \omega^\perp} + X_a \theta^a &&= \underbrace{X_0 \theta^0}_{X_\top \theta^\top} + X^b \gamma_{ba} \omega^a \ .
\end{aligned}
\tag{2.23}
$$

where

$$
\begin{aligned}
X^\perp &= -X^\alpha n_\alpha = N X^0 \ , \qquad & X^\top &= -X^\alpha m_\alpha = M(X^0 - M_a X^a) \ , \\
X_\perp &= X_\alpha n^\alpha = N^{-1}(X_0 - N^a X_a) \ , \qquad & X_\top &= X_\alpha m^\alpha = M^{-1} X_0 \ .
\end{aligned}
\tag{2.24}
$$

Figure 2 illustrates the three decompositions of a vector field X in a suggestive way.

These generalize in an obvious way to higher rank tensors. The orthogonal projection via computational frame indices is accomplished in the slicing point of view by choosing all possible sets of components with the zero indices up and the spatial indices down, while these positions are reversed in the threading point of view [4,42]. The lapse in each case can be used to rescale the zero-indexed components to obtain partially orthonormal components along the given time direction. For the sake of an explicit example, the reference splitting, slicing splitting, and threading splitting of a $\binom{1}{1}$-tensor field S are parametrized by the original computational frame components in the following way

$$
\begin{aligned}
S &= S^0{}_0 e_0 \otimes \omega^0 + S^0{}_a e_0 \otimes \omega^a + S^a{}_0 e_a \otimes \omega^0 + S^a{}_b e_a \otimes \omega^b \\
&\quad \leftrightarrow (S^0{}_0, S^0{}_a \omega^a, S^a{}_0 e_a, S^a{}_b e_a \otimes \omega^b) \ , \\
S^\flat &= (-N^2)^2 S^{00} \omega^0 \otimes \omega^0 + (-N^2)[S^0{}_a \omega^0 \otimes \theta^a + S_a{}^0 \theta^a \otimes \omega^0] + S_{ab} \theta^a \otimes \theta^b \\
&= S^{\perp\perp} \omega^\perp \otimes \omega^\perp + (-1) S^\perp{}_a \omega^\perp \otimes \theta^a + (-1) S_a{}^\perp \theta^a \otimes \omega^\perp + S_{ab} \theta^a \otimes \theta^b \\
&\quad \leftrightarrow (S^{\perp\perp}, S^\perp{}_a \theta^a, S_a{}^\perp \theta^a, S_{ab} \theta^a \otimes \theta^b) \ , \\
S^\sharp &= (-M^{-2})^2 S_{00} e_0 \otimes e_0 + (-M^{-2})[S_0{}^a e_0 \otimes \epsilon_a + S^a{}_0 \epsilon_a \otimes e_0] + S^{ab} \epsilon_a \otimes \epsilon_b \\
&= S_{\top\top} \epsilon_\top \otimes \epsilon_\top + (-1) S_\top{}^a e_\top \otimes \epsilon_a + (-1) S^a{}_\top \epsilon_a \otimes e_\top + S^{ab} \epsilon_a \otimes \epsilon_b \\
&\quad \leftrightarrow (S^{\top\top}, S^{\top a} \epsilon_a, S^{a\top} \epsilon_a, S^{ab} \epsilon_a \otimes \epsilon_b) \ .
\end{aligned}
\tag{2.25}
$$

Once the tensor is split into a collection of spatial fields in the slicing and threading points of view, their "spatial" indices may be shifted using the spatial metric as follows from (2.21).

Note that in the special case of a tensor which is already a spatial field in a given point of view, the projected computational frame components are directly equal to the reference spatial components. The remaining computational frame components are parametrized by these reference spatial components in such a way that index shifting the reference spatial components with the spatial metric matrices exactly reproduces index shifting of the tensor with respect to the spacetime metric taking into account its remaining components. For a tensor which is not spatial, the projected computational components differ from the reference spatial components by terms involving the temporal components and the shift.

One can also use the reference projection on the collection of spatial fields into which a given spacetime tensor is decomposed in order to represent the orthogonal splitting of this tensor in terms of reference spatial fields. This is equivalent to another isomorphism with the representation of these fields on the Space manifold and suggested by the sloppy but convenient identifications frequently made when working in local coordinate systems. For example, in adapted local coordinates, one often sees the reference spatial field $g_{ab} dx^a \otimes dx^b$ rather than $g_{ab}(dx^a + N^a dt) \otimes (dx^b + N^b dt)$ referred to as the slicing spatial metric, and indeed if one interprets $\{x^a\}$ as the restrictions of the spatial coordinates to the slices, then the induced metric on those slices is the former expression, which may also be identified with the metric on the Space manifold if one instead interprets these coordinates as their pullbacks $I_t{}^* x^a$ to that manifold. Durrer and Straumann [43] have chosen this option for the slicing point of view, referring to the reference spatial fields as "horizontal fields". Note that the restriction to a slice (or pullback to "Space") of either the spacetime metric, or the reference spatial metric, or the spatial metric yields the same induced metric on the slice (spatial metric on "Space") in the slicing point of view. Similarly in the threading point of view, the pushdowns to "Space" of the spacetime inverse metric, the reference inverse metric, or the inverse spatial metric all yield the spatial metric on "Space".

One can introduce a "*reference point of view*" analogous to the slicing and threading points of view based on the reference decomposition for the measurement process in place of the orthogonal decomposition associated with an observer congruence. The evolution is simpler, defined by comparison with Lie dragging along by e_0. This is represented differentially by the Lie derivative by e_0, which commutes with the reference projections. However, unless at least one of the two causality conditions on the nonlinear reference frame is satisfied, the use of the terms "temporal" and "spatial" in this context has no meaning. This point of view corresponds to what usually occurs when one works in an adapted coordinate system in the old-fashioned component style of discussion.

In many applications both n and m are timelike on some region of spacetime, in which case the two local rest spaces orthogonal to them are related by a boost $B(m,n)$ (a pure Lorentz transformation acting in the plane of n and m actively mapping n onto m) whose parameters are defined by the relations

$$\begin{aligned}
&m = B(m,n)n = \gamma_L(n + v^a e_a) \ , \quad \text{or} \quad n = B(n,m)m = \gamma_L(m - V^a \epsilon_a) \ , \\
&v^a = N^a/N \ , \quad V^a = \gamma_L^{-1} v^a = M M^a \ , \quad V_a = \gamma_L v_a \ , \\
&v^2 = g_{ab} v^a v^b = N^{-2} N^a N_a = M^2 M^a M_a = \gamma_{ab} V^a V^b = V^2 \ , \\
&\gamma_L = N/M = \gamma^{1/2}/g^{1/2} = (1 - v^2)^{-1/2} \ .
\end{aligned} \tag{2.26}$$

If one uses the notation $v(m,n)$ for the relative velocity of m with respect to n and $\gamma(m,n) = [1 - {}^{(4)}\mathrm{g}(v(m,n), v(m,n))]^{-1/2}$ for the associated gamma factor, then

$$\vec{v} = v^a e_a = N^{-1}\vec{N} = v(m,n) \tag{2.27}$$

is the spatial velocity associated with this boost as seen from the local rest space associated with the unit normal, while

$$\vec{V} = V^a \epsilon_a = M\vec{M} = -v(n,m) \tag{2.28}$$

is the one as seen from the local rest space orthogonal to the time lines, and

$$\gamma_L = \gamma(m,n) = \gamma(n,m) \tag{2.29}$$

is the Lorentz gamma factor associated with the proper time dilation and the spatial volume contraction. If either m or n (but not both) changes causal character by becoming null, this boost becomes singular, with $v \to 1$. Any remarks in what follows involving relationships between the slicing and threading perspectives will be understood to apply only when both n and m exist as timelike unit vectors, the case of a spacelike slicing and a timelike threading.

To summarize, the pair of fields (e_0, ω^0) with the compatibility condition $\omega^0(e_0) = 1$ represent the parametrized nonlinear reference frame at the linear level. In each point of view the causal assumption about the nonlinear reference frame leads one of these fields to introduce a local notion of time which is represented by a unit timelike vector field o, interpretable as the 4-velocity of a field of test observers, while the other field represents the gauge freedom remaining in the nonlinear reference frame holding that family of observers fixed, a freedom which is natural to call the spatial gauge freedom. This observer 4-velocity field o is the unit normal $n = e_\perp$ to the slicing in the slicing point of view, obtained by normalizing and index-lowering the sign-reversed slicing 1-form $-\omega^0$, and it is the unit tangent $m = e_\top$ to the threading in the threading point of view, obtained by normalizing the threading vector field e_0. The orthogonal decomposition of the tangent space into the local time direction of the observer and the local rest space LRS_o may be used to "measure" a spacetime field, leading to a collection of "spatial" fields of different ranks obtained by contraction with $-o$ rather than projection along o in the tensor product of the orthogonal decomposition. The "measurement" of the spacetime metric gives only one nontrivial piece, the "spatial metric" which is the metric on the local rest space LRS_o, while the decomposition of the spatial gauge field, respectively e_0 and ω^0, leads to the remaining pieces of the spacetime metric. The spatial projection gives the shift vector field and 1-form respectively, while the temporal contraction yields the lapse and the negative reciprocal of the lapse respectively. In each case the lapse function is the normalizing function for the parametrized nonlinear reference frame field which determines the local time direction, and determines the rate of change of the observer proper time with respect to the time parameter of the parametrized nonlinear reference frame.

All of the spatial fields may be interpreted as time-dependent fields on the Space associated with the point of view. The reference projections also allow one to represent these fields on the spacetime in terms of fields which instead are "spatial" with respect to the reference decomposition of the tangent space associated with the nonlinear reference frame, and these reference fields lead to the same time-dependent fields on the Space as the original spatial fields. Finally, in the event that both points of view hold, one has a unique Lorentz boost which transforms between the two points of view. The principal difference between the two points of view is that the test observers are fixed in Space in the threading point of view, but move with respect to Space in the slicing point of view, the position of a given observer at time t determined by an integral curve of the time-dependent sign-reversed shift vector field. This makes the slicing point of view a hybrid type of formalism based on two distinct congruences, while the threading point of view is entirely determined by a single congruence.

3 Decomposing derivatives

The evolution of a spacetime in the context of a given parametrized nonlinear reference frame describes how the collections of spatial fields obtained from spacetime fields by the "measurement process" change along the threading congruence. On the Space of each point of view, these fields are just time-dependent fields depending on the time parameter t, and one may directly compare corresponding fields at different values of the time parameter, as well as differentiate them with respect to that time parameter, for which one may use the sloppy notation $\partial/\partial t$. This corresponds directly to the *spatial Lie derivative* $\pounds(o)_{e_0}$ along e_0 of the corresponding spatial field on spacetime, which has been defined simply by projecting the ordinary Lie derivative $\pounds_{e_0}$ on all indices into the local rest space of the observer. At a finite level, the evolution corresponds to the change of spatial fields along the threading congruence with respect to spatial Lie transport, namely Lie transport along e_0 followed by projection into the local rest space LRS_o. This in turn may be related to ordinary Lie transport of the reference spatial fields of the appropriate valence that one may use instead to represent the spatial fields. Note that $\{e_a\}$ and $\{\omega^a\}$ have vanishing Lie derivative along e_0 by choice, but their respective projections ϵ_a in the threading point of view and θ^a in the slicing point of view do not. Instead they have vanishing spatial Lie derivatives along e_0, so that when expressed in the spatial projected computational frame, the components of this Lie derivative of a spatial tensor reduce to the e_0 derivative of that tensor's components

$$\pounds(o)_{e_0} S^{a\cdots}{}_{b\cdots} = e_0 S^{a\cdots}{}_{b\cdots} \ , \tag{3.1}$$

which in turn reduces to the partial derivative $\partial/\partial t$ in an adapted coordinate system. (The fields ϵ_0 and θ^0 have vanishing time projected Lie derivatives along e_0, appropriate since they themselves are defined by time projection.)

Thus a *noncovariant time derivative* is crucial to the nonlinear reference frame approach to splitting spacetime. On the other hand, one needs a covariant spatial derivative, covariant in the sense that it is independent of the spatial gauge freedom which relates the orthogonal decomposition of the test observers to the nonlinear reference frame. In the slicing point of view one wants a covariant spatial derivative which is independent of the choice of threading, while in the threading point of view it must be independent of the choice of slicing.

An obvious way of defining such a *spatial covariant derivative* exists. One merely projects the spacetime covariant derivative on all indices into the local rest space of the test observer

$$\nabla(o)_X S = P(o)[{}^{(4)}\nabla_{P(o)X} S] \ , \tag{3.2}$$

where S is an arbitrary tensor field. For spatial differential forms, the antisymmetrized spatial covariant derivative reduces to the *spatial exterior derivative*, similarly obtained by spatially projecting the spacetime exterior derivative on all indices

$$d(o)\sigma = P(o)[d\sigma] \ . \tag{3.3}$$

One can also introduce in an obvious way the spatial inner product "$\cdot_o$", the cross product "$\times_o$", and spatial duality operator $^{*(o)}$ and the spatial derivative operators grad_o, div_o, and curl_o in order to mimic Euclidean vector analysis. This is described in Appendix A.

An important covariant time derivative exists as well, namely the *spatial Fermi-Walker derivative* along the observer congruence, obtained by spatially projecting on all indices the spacetime covariant derivative by the test observer 4-velocity o

$$\nabla_{(\mathrm{fw})}(o)S = P(o)[{}^{(4)}\nabla_O S] \ , \tag{3.4}$$

where again S is an arbitrary tensor. When S is spatial, this derivative coincides with the Fermi-Walker derivative. For such spatial fields, Fermi-Walker transport along the observer congruence is described differentially by vanishing spatial Fermi-Walker derivative. This *covariant time derivative* is used in the congruence point of view, as discussed below.

Finally one needs to "measure" the covariant derivative of spacetime fields, expressing it in terms of the noncovariant time derivative and the spatially covariant spatial derivative of the collections of spatial fields which represent them. The leftover pieces of the spacetime covariant derivative then introduce the effects of "spatial gravitational fields". This decomposition can be applied to the geodesic equation to define the spatial gravitational fields in analogy with Newtonian and electromagnetic theory. It can also be applied to the Fermi-Walker derivative along the worldline of a test gyroscope in order to describe the effect of these spatial gravitational fields on the spin of the gyro. In each case one needs a hybrid operator mixing the noncovariant time derivative and the spatially covariant spatial covariant derivative to describe the derivative along an arbitrary worldline. Such an operator is the total spatial covariant derivative along a parametrized curve, which corresponds to decomposing the tangent into a component along the threading congruence and a remaining spatial component, and then using the noncovariant time derivative along the threading component and the spatial covariant derivative along the remaining spatial component. The missing pieces in the measurement of the spacetime total covariant derivative again involve the spatial gravitational fields. It is also important to remark that the distinction between the local time direction and the evolution direction makes the total spatial covariant derivative in the slicing point of view a hybrid object which depends both on the slicing and threading, and it is not invariant under a change of threading.

The spatial covariant derivative $\nabla(o)$ contains two distinct kinds of information, first the "transverse" covariant derivative of o itself, involving the rotation and expansion tensors of o which together describe the relative motion of neighboring worldlines of o, and second the "transverse" covariant derivative of fields which are spatial with respect to o. All of this information is contained in certain projections of the computational frame components of the spacetime connection.

In order to discuss these objects, it is convenient to introduce some component notation. Introduce first the anticyclic sum notation

$$A_{\{\alpha\beta\gamma\}_-} = A_{\alpha\beta\gamma} - A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} \ . \tag{3.5}$$

Then let

$$\partial_\alpha f = f_{,\alpha} = e_\alpha f \ , \qquad f_{,,a} = \epsilon_a f \tag{3.6}$$

denote the ordinary derivatives along the computational frame vectors and along their spatial projections in the threading point of view. Similarly let

$$ {}^{(4)}\nabla_\beta X^\alpha = X^\alpha{}_{;\beta} \ , \qquad \nabla(n)_\beta X^\alpha = X^\alpha{}_{|\beta} \ , \qquad \nabla(m)_\beta X^\alpha = X^\alpha{}_{||\beta} \ , \tag{3.7} $$

denote the components of the spacetime covariant derivative and of the spatial covariant derivative in the slicing and threading points of view respectively.

The components of the spacetime connection in the computational frame are defined by either of the following relations

$$ {}^{(4)}\nabla_{e_\alpha} e_\beta = {}^{(4)}\Gamma^\gamma{}_{\alpha\beta} e_\gamma \ , \qquad {}^{(4)}\nabla_{e_\alpha} \omega^\beta = -{}^{(4)}\Gamma^\beta{}_{\alpha\gamma}\omega^\gamma \ , \tag{3.8} $$

and are explicitly given by the formulas

$$ {}^{(4)}\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} {}^{(4)}g^{\alpha\delta}[{}^{(4)}g_{\{\delta\beta,\gamma\}_-} + C_{\{\delta\beta\gamma\}_-}] \ , \tag{3.9} $$

where the usual index shifting convention is used for the structure function indices. Note that the covariant component indices of a connection are interchanged with respect to the convention of Misner, Thorne and Wheeler [10]. The purely spatial components of the slicing and threading point of view spatial connections taken with respect to their respective projected spatial frame are defined in a similar way

$$ \begin{aligned} \nabla(n)_{e_a} e_b &= \Gamma(n)^c{}_{ab} e_c \ , & \nabla(m)_{\epsilon_a} \epsilon_b &= \Gamma(m)^c{}_{ab}\epsilon_c \ , \\ \nabla(n)_{e_a} \theta^b &= -\Gamma(n)^b{}_{ac}\theta^c \ , & \nabla(m)_{\epsilon_a} \omega^b &= -\Gamma(m)^b{}_{ac}\omega^c \ . \end{aligned} \tag{3.10} $$

and have a similar explicit expressions

$$ \Gamma(n)^a{}_{bc} = \tfrac{1}{2} g^{ad}[g_{\{db,c\}_-} + C(n)_{\{dbc\}_-}] \ , \qquad \Gamma(m)^a{}_{bc} = \tfrac{1}{2}\gamma^{ad}[\gamma_{\{db,,c\}_-} + C(m)_{\{dbc\}_-}] \ , \tag{3.11} $$

where the kernel symbol notation $C(o)$ indicates that the indices on the spatial structure functions $C^a{}_{bc}$ of both the slicing spatial frame $\{e_a\}$ and of the threading spatial frame $\{\epsilon_a\}$ are shifted with the spatial metric of the respective observer.

These spatial components of each connection describe the composition of the connection with its projection (applied first, i.e., $\nabla(o) \circ P(o)$) and are related to the computational components of the spacetime connection in the same way any field is spatially projected

$$ \begin{aligned} \Gamma(n)_{abc} &= {}^{(4)}\Gamma_{abc} \ , \\ \Gamma(m)^{abc} &= {}^{(4)}\Gamma^{abc} \ , \end{aligned} \tag{3.12} $$

index-shifting then being performed with the corresponding spatial metric. The formula for the spatial covariant derivative of a spatial tensor $S(o)$ expressed in terms of the projected computational frame is exactly what one would expect using the usual 3-dimensional formula

$$ \begin{aligned} \nabla(n)_c S(n)^{a\cdots}_{b\cdots} &\equiv [\nabla(n)S(n)]^{a\cdots}_{b\cdots c} \equiv S(n)^{a\cdots}_{b\cdots|c} \\ &= S(n)^{a\cdots}_{b\cdots,c} + \Gamma(n)^a{}_{cd} S(n)^{d\cdots}_{b\cdots} + \cdots - \Gamma(n)^d{}_{cb} S(n)^{a\cdots}_{d\cdots} - \cdots \ , \\ \nabla(m)_c S(m)^{a\cdots}_{b\cdots} &\equiv [\nabla(m)S(m)]^{a\cdots}_{b\cdots c} \equiv S(m)^{a\cdots}_{b\cdots||c} \\ &= S(m)^{a\cdots}_{b\cdots,,c} + \Gamma(m)^a{}_{cd} S(m)^{d\cdots}_{b\cdots} + \cdots - \Gamma(m)^d{}_{cb} S(m)^{a\cdots}_{d\cdots} - \cdots \ . \end{aligned} \tag{3.13} $$

Recall that a spatial tensor has its nonzero projected computational frame components equal to the reference spatial components.

The rather widespread notational conventions of Misner, Thorne and Wheeler [10] for the slicing point of view suppress the qualifier "(n)" which is necessary here to distinguish the two points of view. The spatial part of the slicing spatially projected connection is equivalent to the unique symmetric connection of the induced metric on the slices. In the threading case such an induced connection (on some family of slices) exists only in the special case that the threading is hypersurface-orthogonal. Independent of the choice of observer it is easily shown that the three valence forms of the projection $P(o)$ are all covariant constant with respect to the associated projected connection $\nabla(o)$. Cattaneo [1–3] refers to this connection as the *transverse covariant derivative* with respect to the observer congruence.

The remaining part of the spatial connection is determined by the rotation and expansion of the observer congruence

$$\nabla(n)_{e_a} n = -k(n)^b{}_a e_b \ , \qquad \nabla(m)_{\epsilon_a} m = -k(m)^b{}_a \epsilon_b \ , \tag{3.14}$$

where the mixed spatial tensor

$$k(o) = -\nabla(o) o = -\theta(o) + \omega(o) = k(o)^\alpha{}_\beta e_\alpha \otimes \omega^\beta \tag{3.15}$$

is a kinematical factor describing the transverse covariant derivative of the observer 4-velocity. The mixed rotation tensor $\omega(o)$ and the mixed expansion tensor $\theta(o)$ (often decomposed into the shear tensor $\sigma(o) = \theta(o) - \frac{1}{3}[\operatorname{Tr}\theta(o)]P(o)$ and the shear scalar $\operatorname{Tr}\theta(o) = o^\alpha{}_{;\alpha}$) appear in the well known decomposition of the covariant derivative of a timelike unit vector o [10,23–26,44]

$$\begin{aligned} o_{\alpha;\beta} = \theta(o)_{\alpha\beta} - \omega(o)_{\alpha\beta} - a(o)_\alpha o_\beta \ , & \\ \theta(o)_{\alpha\beta} = P(o)^\gamma{}_\alpha P(o)^\delta{}_\beta o_{(\gamma;\delta)} = \tfrac{1}{2} \pounds_O P(o)_{\alpha\beta} \ , & \\ \omega(o)_{\alpha\beta} = -P(o)^\gamma{}_\alpha P(o)^\delta{}_\beta o_{[\gamma;\delta]} = \tfrac{1}{2}[d(o) o^\flat]_{\alpha\beta} \ , & \\ a(o)^\alpha = o^\alpha{}_{;\beta} o^\beta \ . & \end{aligned} \tag{3.16}$$

The only nonspatial part of this decomposition involves the spatial acceleration vector field $a(o) = \nabla_O o = \nabla_{(\mathrm{fw})}(o) o$ which is associated with the "longitudinal" covariant derivative of o. All of these kinematical quantities may be evaluated by projecting certain computational frame components of the spacetime connection as discussed in Appendix A.

The geometrical interpretation of $k(o)$ is the following. If X (connecting vector field) is a vector field which is Lie dragged along o, then its spatial projection $Y = P(o)X$ (relative position vector) is by definition transported along o by the spatial Lie transport and satisfies

$$\nabla_{(\mathrm{fw})}(o) Y = -k(o) Y \ . \tag{3.17}$$

Thus $k(o)$ measures the difference between Fermi-Walker transport and spatially projected Lie transport of spatial vector fields along the observer congruence. In the slicing point of view the observer congruence is hypersurface-orthogonal and therefore rotation-free $\omega(n) = 0$, so that $k(n) = -\theta(n) \equiv K$ reduces to the *extrinsic curvature tensor* K of the slicing. This same discussion also applies to the case of a tangent vector dragged along a single curve in the observer congruence.

Having defined both the noncovariant time derivative and the spatially covariant spatial covariant derivative for spatial fields in each point of view, one can consider mixing them to describe the related parts of the total covariant derivative

$${}^{(4)}\nabla_u = u^{\alpha\,(4)}\nabla_\alpha = {}^{(4)}D/d\tau_u \tag{3.18}$$

along an arbitrary timelike worldline with unit tangent u and parametrized by the proper time τ_u. The sloppy but convenient practice of ignoring the distinction between the differential operator ${}^{(4)}D/d\tau_u$ which acts only on tensors defined along the parametrized worldline and the operator ${}^{(4)}\nabla_u$ which acts instead on tensor fields defined in a neighborhood of the worldline will be followed here, and all equations involving these operators must be understood in this light.

First one can split the 4-velocity u in both the slicing and threading points of view as suggestively illustrated in Figure 3

$$u = u^0 e_0 + u^a e_a = \gamma(u,o)[o + \vec{\nu}(o)] = E(o)o + \vec{p}(o) \ , \tag{3.19}$$

where $\gamma(u,o) = E(o)$ is the relative gamma factor and energy per unit (rest) mass as seen by the observer, while $\vec{\nu}(o)$ is the relative spatial velocity and $\vec{p}(o)$ the relative spatial momentum per unit mass as seen by the observer.

Next one can introduce a *total spatial covariant derivative* for both the slicing and threading points of view as well as for the equivalent nontilted slicing point of view based on the normal congruence threading of the given slicing

$$\begin{aligned}
D_{(\mathrm{sl})}(u)/d\tau_u &= u^0 \pounds(n)_{e_0} + u^a \nabla(n)_{e_a} &&= u^0 \pounds(o)_{e_0} + \nabla(n)_{\mathcal{P}u} \ , \\
D_{(\mathrm{th})}(u)/d\tau_u &= -M^{-2} u_0 \pounds(m)_{e_0} + u^a \nabla(m)_{\epsilon_a} &&= \pounds(m)_{T(m)u} + \nabla(m)_{P(m)u} \ , \\
D_{(\perp)}(u)/d\tau_u &= u^0 \pounds(n)_{\epsilon_0} + u_b g^{ba} \nabla(n)_{e_a} &&= \pounds(n)_{T(n)u} + \nabla(n)_{P(n)u} \ .
\end{aligned} \tag{3.20}$$

These generalize respectively the reference splitting, the threading splitting and the slicing splitting of u when acting as a directional derivative of a function, the case in which all three coincide and the qualifying subscript notation is unnecessary. However, when acting on tensor fields, the first and last involve the slicing spatial covariant derivative and the remaining one the threading spatial covariant derivative, while all three use the relevant spatial Lie derivative along the nonspatial component of u. The threading definition $D_{(\mathrm{th})}/d\tau_u$ is independent of the slicing, depending only on the threading congruence, while $D_{(\perp)}/d\tau_u$ depends only on the slicing, or equivalently on the normal congruence, and is the same operator as the equivalent threading operator defined for the normal congruence. Each of these two operators has a natural interpretation in terms of a single observer measuring the rate of change of a spatial field in a spatially covariant way. On the other hand the slicing definition [31] $D_{(\mathrm{sl})}/d\tau_u$ is instead a hybrid operator depending on both the slicing and threading, incorporating the tangential motion of the slice as it moves through spacetime along the threading congruence. In this important respect the slicing point of view differs from the equivalent threading point of view associated with the normal congruence.

The total spatial covariant derivatives introduced above correspond to the proper time parametrization along the curve; they may be rescaled to correspond to the proper time of the appropriate observers. Using the relative gamma factor, one can transform the proper time differential to represent the derivative with respect to the proper time of the appropriate reference frame observer, leading to the three derivatives

$$\begin{aligned}
D_{(\mathrm{sl})}(u)/d\tau_n &= \gamma(u,n)^{-1} D_{(\mathrm{sl})}(u)/d\tau_u = N^{-1} \pounds(n)_{e_0} + [\nu(n)^a - v^a] \nabla(n)_{e_a} \ , \\
D_{(\mathrm{th})}(u)/d\tau_m &= \gamma(u,m)^{-1} D_{(\mathrm{th})}(u)/d\tau_u = \pounds(m)_m + \nu(m)^a \nabla(m)_{\epsilon_a} \ , \\
D_{(\perp)}(u)/d\tau_n &= \gamma(u,n)^{-1} D_{(\perp)}(u)/d\tau_u = \pounds(n)_n + \nu(n)^a \nabla(n)_{e_a}
\end{aligned} \tag{3.21}$$

suggestively illustrated in Figure 3. These total derivatives along the curve use the metric-independent time derivative (Lie derivative) along the appropriate congruence and incorporate only the "intrinsic" spatial geometry of the spatial projection (through the spatial projection of the Lie derivative and connection), leaving the remaining extrinsic parts of the spacetime total covariant derivative involving the lapse and shift and Lie derivative of the projection to appear as apparent spatial gravitational forces. Shifting indices of spatial quantities inside such a derivative leads to additional Lie derivative terms involving the spatial projection,

$$
\begin{aligned}
&D_{(\mathrm{sl})}(u)g_{ab}/d\tau_n = N^{-1}\pounds(n)_{e_0}g_{ab}\ ,\\
&D_{(\mathrm{th})}(u)\gamma_{ab}/d\tau_m = \pounds(m)_m\gamma_{ab} = 2\theta(m)_{ab}\ ,\\
&D_{(\perp)}(u)g_{ab}/d\tau_n = \pounds(n)_n g_{ab} = 2\theta(n)_{ab}\ ,
\end{aligned}
\tag{3.22}
$$

so one must be careful about choosing the appropriate valence of indices when identifying spatial quantities like the force, except for a stationary nonlinear reference frame in the the stationary case where the Lie derivative terms vanish in both the slicing and threading points of view.

4 Spatial gravitational forces

Consider decomposing the geodesic equation

$$
0 = {}^{(4)}\nabla_u u = u^\alpha{}_{;\beta}u^\beta e_\alpha = {}^{(4)}Du/d\tau_u\ . \tag{4.1}
$$

The spatial part determines the rate of change of the spatial momentum, while the time part determines the rate of change of the energy.

A short calculation using the projection technique discussed in Appendix A yields the following results for the spatial projection of the geodesic equations with respect to n and m respectively

$$
\begin{aligned}
0 = \gamma(u,n)^{-1}u_{a;\beta}u^\beta \quad &= D_{(\mathrm{sl})}(u)p(n)_a/d\tau_n - F(n)^{(\mathrm{G})}_a\ ,\\
0 = \gamma(u,m)^{-1}\gamma_{ab}u^b{}_{;\beta}u^\beta &= D_{(\mathrm{th})}(u)p(m)_a/d\tau_m - F(m)^{(\mathrm{G})}_a\ ,
\end{aligned}
\tag{4.2}
$$

where the *spatial gravitational forces* (per unit mass) have been defined in terms of the total spatial covariant derivatives of the covariant spatial momentum (per unit mass) in both points of view. With this convention one finds

$$
\begin{aligned}
F(n)^{(\mathrm{G})} &= \gamma(u,n)[-\,\mathrm{grad}_n(\ln N) + N^{-1}N^{b|a}\nu(n)_b e_a]\ ,\\
F(m)^{(\mathrm{G})} &= \gamma(u,m)[-\,\mathrm{grad}_m(\ln M) - (\pounds(m)_{e_0}M_b)\gamma^{ba}\epsilon_a + 2MM^{[b||a]}\nu(m)_b\epsilon_a]\ ,
\end{aligned}
\tag{4.3}
$$

Following Thorne [31], one can introduce the *gravitoelectric* vector field and the *gravitomagnetic* tensor field and vector field in the slicing point of view and extend them in a symmetrical way to the threading point of view, using the qualifiers "slicing" and "threading" when it is necessary to distinguish them

$$
\begin{aligned}
\vec g(n) &= -\,\mathrm{grad}_n(\ln N)\ , & \vec g(m) &= -\,\mathrm{grad}_m(\ln M) - [\pounds(m)_{e_0}M_b]\gamma^{ba}\epsilon_a\ ,\\
\overleftrightarrow{H}(n) &= N^{-1}N^{b|a}e_a\otimes e_b\ , & \overleftrightarrow{H}(m) &= MM^{b||a}\epsilon_a\otimes\epsilon_b\ ,\\
\vec H(n) &= N^{-1}\eta(n)^{abc}N_{c|b}e_a\ , & \vec H(m) &= M\eta(m)^{abc}M_{c||b}\epsilon_a\ ,\\
&= N^{-1}\,\mathrm{curl}_n\,\vec N\ , & &= M\,\mathrm{curl}_m\,\vec M\ ,
\end{aligned}
\tag{4.4}
$$

in terms of which the gravitational force (per unit mass) takes the form

$$
\begin{aligned}
F(n)^{(\mathrm{G})} &= \gamma(u,n)[\vec{g}(n) + \overleftrightarrow{H}(n) \cdot_n \vec{\nu}(n)] \\
&= \gamma(u,n)[\vec{g}(n) + \tfrac{1}{2}\vec{\nu}(n) \times_n \vec{H}(n) + (\mathrm{SYM}\,\overleftrightarrow{H}(n)) \cdot_n \vec{\nu}(n)] \ , \\
F(m)^{(\mathrm{G})} &= \gamma(u,m)[\vec{g}(m) + 2(\mathrm{ALT}\,\overleftrightarrow{H}(m)) \cdot_m \vec{\nu}(m)] \\
&= \gamma(u,m)[\vec{g}(m) + \vec{\nu}(m) \times_m \vec{H}(m)] \ .
\end{aligned}
\tag{4.5}
$$

In the slicing point of view, Thorne [31] has called the first term in the force the "gravitational acceleration" or "gravitoelectric force", and the second term which is linear in the spatial velocity the "gravitomagnetic force", in analogy with the Lorentz force acting on a charged particle. Analogously one can use these same names to describe the corresponding two terms in the force in the threading point of view. These threading forces have been discussed by Landau and Lifshitz [7] in the stationary case and by Zel'manov [4,5], Cattaneo [1–3], Møller [8] and Massa [15] in the general case.

Notice that the gravitational force gives rise to two force fields almost exactly as in the electromagnetic case, except that in the gravitational case one must first remove the gamma factor depending on the four-velocity. In the threading point of view, these force fields are the gravitoelectric and gravitomagnetic vector fields, exactly analogous to the electromagnetic case except for the extra gamma factor of the four-velocity which enters the formula for the force itself, but in the slicing point of view there is also the tensor force field which enters the force through the symmetric part of the gravitomagnetic tensor field. Had the threading spatial gravitational force been defined in terms of the contravariant spatial momentum respecting the dual symmetry between the two points of view, the threading gravitomagnetic field term in (4.3) would have been instead

$$
\gamma(u,m)[2\overleftrightarrow{k}(m)] \cdot_m \vec{\nu}(m) = \gamma(u,m)[2\,\mathrm{ALT}\,\overleftrightarrow{H}(m) - 2\overleftrightarrow{\theta}(m)] \cdot_m \vec{\nu}(m) \tag{4.6}
$$

so that it too would involve a symmetric tensor force field, namely the expansion tensor of the threading congruence. Conversely, if the slicing spatial gravitational force had been defined in terms of the rate of change of the contravariant spatial momentum, an additional symmetric tensor part would have appeared

$$
\begin{aligned}
&\gamma(u,m)[\overleftrightarrow{H}(n) + N^{-1}\pounds(n)_{e_0}\overleftrightarrow{\mathrm{g}}^{-1}] \cdot_n \vec{\nu}(n) \\
&\qquad = \gamma(u,m)[\overleftrightarrow{H}(n) - 2\,\mathrm{SYM}\,\overleftrightarrow{H}(n) - 2\overleftrightarrow{\theta}(n)] \cdot_n \vec{\nu}(n) \ .
\end{aligned}
\tag{4.7}
$$

The lapse and shift in each approach act respectively as scalar and vector potentials for the spatial gravitational forces. The gravitoelectric force is just the sign reversal of the acceleration of the corresponding observers, while the gravitomagnetic force arises from the spatial gradients of the threading congruence. In the threading point of view, Appendix A shows that the latter force field is just the rotation tensor of the threading congruence, with $\mathrm{ALT}\,\overleftrightarrow{H}(m) = \omega(m)^\sharp$, and it corresponds exactly to a Coriolis force associated with an angular velocity of rotation $\vec{\Omega}_{(\mathrm{th})} = \frac{1}{2}\vec{H}(m) = \vec{\omega}(m)$, which is just the rotation vector field of the unit vector field m. In the slicing point of view where the rotation of the observer congruence is instead zero, it is a hybrid object associated with the sliding of the points of "Space" and it vanishes if one adopts the equivalent slicing/threading point of view associated with the normal congruence. In a Gaussian normal coordinate system, both points of view coincide and both spatial gravitational forces vanish as is appropriate to describe the free motion of the corresponding geodesic observers. The term *gravitomagnetic vector potential* has also been used by Wheeler [45] to refer to a related but distinct vector field used in the decomposition of the extrinsic curvature tensor in the resolution of the initial value problem in the slicing point of view, and care should be taken not to confuse this usage of the term gravitomagnetic with that of Thorne.

The nonlinear reference frame associated with a geodesically parallel slicing (acceleration of n vanishes) leads to a zero slicing gravitoelectric field, while a geodesic threading (acceleration of m vanishes) leads to zero threading gravitoelectric field. A rotationfree threading leads to zero threading gravitomagnetic field, while a zero slicing gravitomagnetic field arises from an orthogonal threading or more generally, from a curlfree shift vector field. The slicing gravitomagnetic tensor vanishes as well only if the shift vector field is spatially covariant constant.

The gravitomagnetic field in each case governs the Fermi-Walker transport of a spatial vector along the corresponding congruence of observers, modulo Lie derivative terms. Consider a spatial vector $\vec{S}(o)$ which might represent the spin of a gyro carried by one of the observers. Since it is Fermi-Walker transported it remains spatial and its magnitude remains constant but its direction may change corresponding to precession of the spin vector

$$\nabla_{(\mathrm{fw})}(o)\vec{S}(o) = 0 \ . \tag{4.8}$$

One finds for the respective spatial projections of this equation

$$\begin{aligned} D_{(\mathrm{sl})}(n)\vec{S}(n)/d\tau_n &= -\tfrac{1}{2}\vec{H}(n) \times_n \vec{S}(n) + \tfrac{1}{2}N^{-1}(\pounds(n)_{e_0} g^{ba})S(n)_b e_a \\ &= [\overleftrightarrow{H}(n) - \overleftrightarrow{\theta}(n)] \cdot_n \vec{S}(n) \ , \\ D_{(\mathrm{th})}(m)\vec{S}(m)/d\tau_m &= -\tfrac{1}{2}\vec{H}(m) \times_m \vec{S}(m) - \overleftrightarrow{\theta}(m) \cdot_m \vec{S}(m) \\ &= \overleftrightarrow{k}(m) \cdot_m \vec{S}(m) = [\mathrm{ALT}\, \overleftrightarrow{H}(m) - \overleftrightarrow{\theta}(m)] \cdot_m \vec{S}(m) \ , \end{aligned} \tag{4.9}$$

where these total spatial covariant derivatives are

$$\begin{aligned} D_{(\mathrm{sl})}(n)/d\tau_n &= N^{-1}[\pounds(n)_{e_0} - N^a \nabla_{e_a}] \ , \\ D_{(\mathrm{th})}(m)/d\tau_m &= M^{-1}\pounds(m)_{e_0} \ . \end{aligned} \tag{4.10}$$

In each case the angular velocity of precession, ignoring expansion effects, is minus half the gyromagnetic vector field.

The physical interpretation of these equations in the threading point of view is the following. In the stationary case where the Lie derivative term vanishes, the nonlinear reference frame is locally rotating relative to the gyro with an angular velocity equal to half the gravitomagnetic vector field, so the gyro spin vector must rotate in the opposite direction with respect to the nonlinear reference frame to compensate, explaining the minus sign in these formulas. This is entirely analogous to evaluating the time derivative with respect to body fixed axes of a vector which is time independent with respect to space-fixed axes in a rotating frame of reference in flat spacetime (and not vice versa which leads to a positive sign). The spatially projected computational frame is analogous to space-fixed axes (space is here determined by the nonlinear reference frame), while a Fermi-Walker transported frame is analogous to body-fixed axes. In the nonstationary case, the expansion tensor of the observer congruence makes its own contribution in the threading point of view according to the interpretation of the kinematical quantities of the threading congruence following from equation (3.17), again with a minus sign as above. In the slicing point of view neither the gravitomagnetic or Lie derivative term is directly connected to the kinematical quantities of either congruence. Instead hybrid quantities appear which are related to the relative motion of the two congruences. When the spatial Fermi-Walker transport equation is expressed in terms of the expansion tensor, one sees that the symmetric part of the gravitomagnetic tensor also contributes to the change in the spin vector.

The congruence point of view associated with a 4-velocity field u can be adapted either to the normal congruence ($u = n$) in the slicing point of view or to the threading congruence ($u = m$) in the threading point of view. One can then translate the congruence point of view representation of the splitting of spacetime tensor equations into either of those points of view. The congruence point of view makes use of the same spatial covariant derivative but uses the spatial Fermi-Walker derivative as a covariant time derivative in place of the noncovariant total spatial covariant derivative. These are simply related. For example, for a single index spatial field $S(o)$ one has

$$\begin{aligned} \nabla_{(\mathrm{fw})}(n)\vec{S}(n) &= D_{(\mathrm{sl})}(n)\vec{S}(n)/d\tau_n - \overleftrightarrow{W}(n) \cdot_n \vec{S}(n) \ , \\ \nabla_{(\mathrm{fw})}(m)\vec{S}(m) &= D_{(\mathrm{th})}(m)\vec{S}(m)/d\tau_m - \overleftrightarrow{W}(m) \cdot_m \vec{S}(m) \ , \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \overleftrightarrow{W}(n) &= \overleftrightarrow{H}(n) - \overleftrightarrow{\theta}(n) \ , \\ \overleftrightarrow{W}(m) &= \mathrm{ALT}\,\overleftrightarrow{H}(m) - \overleftrightarrow{\theta}(m) = \overleftrightarrow{k}(m) \ . \end{aligned} \tag{4.12}$$

This extends in an obvious way to multiple index spatial tensor fields. The spatial tensor $\overleftrightarrow{W}(o)$ describes the linear transformation of each spatial index necessary to convert the noncovariant time derivative along the observer congruence to the covariant one. In the threading point of view, this noncovariant time derivative is just the spatial Lie derivative $\pounds(m)_{e_0}$, but in the slicing point of view the spatial connection enters the picture. For equations like Maxwell's equations which may be expressed entirely in terms of exterior derivatives, it is much simpler to avoid the connection entirely, splitting only the exterior derivatives.

All of the above results hold independently for the slicing and threading points of view but in the case of a spacelike slicing and a timelike threading, both points of view hold simultaneously and one can consider transforming between the two. The unique boost discussed above relates the two orthogonal decompositions and may be used to transform between the two families of spatial tensors which represent a given spacetime field. In the case of the spatial gravitational fields, their definitions in each point of view determine the transformation which relates them to each other.

In comparing the two points of view it is helpful to recall that $\vec{v} = N^{-1}\vec{N}$ and $\vec{V} = M\vec{M}$ define the relative velocity of the threading observers with respect to the normal observers as measured by the normal and threading observers respectively. This explains the reciprocally related factors of the lapse and dual lapse which appear in the gravitomagnetic fields and in many other contexts. A simple calculation yields the following formulas for the transformation of the the gravitomagnetic vector field and the gravitoelectric 1-form

$$\begin{aligned} P(m)\tfrac{1}{2}\vec{H}(n) &= \tfrac{1}{2}\vec{H}(m) + \vec{V} \times_m \vec{g}(m) + \vec{V} \times_m [\tfrac{1}{2}M^2 \pounds(m)_{e_0}(M^{-2}\bar{\bar{M}})]^\sharp \ , \\ \mathcal{P}\ \bar{\bar{g}}(n) &= {\gamma_L}^2[\vec{g}(m) - \overleftrightarrow{H}(m)\cdot\vec{V}]^\flat + \vec{V}^{\,\flat}[\pounds_m \ln(M\gamma_L)] + {\gamma_L}^2 \pounds(m)_{e_0}\bar{\bar{M}} \\ &= {\gamma_L}^2[\vec{g}(m) - \vec{V} \times_m \tfrac{1}{2}\vec{H}(m) - \mathrm{SYM}\,\overleftrightarrow{H}(m)\cdot\vec{V}]^\flat + \dots \ . \end{aligned} \tag{4.13}$$

King and Ellis [46] represent the threading kinematical quantities in terms of those of the slicing, which is relevant to the inverse transformation

$$\begin{aligned} \mathcal{P}\tfrac{1}{2}\vec{H}(m) &= {\gamma_L}^2[\tfrac{1}{2}\vec{H}(n) + (\tfrac{1}{2}\vec{H}(n)\cdot_n\vec{v})\vec{v} + \vec{v}\times_n(\vec{g}(n) + (\mathrm{SYM}\,\overleftrightarrow{H}(n))\cdot_n\vec{v})] \\ &\quad + \vec{v}\times_n[\tfrac{1}{2}\pounds(n)_{e_0}(M^{-2}\bar{\bar{N}})]^\sharp \ , \\ P(n)\ \bar{\bar{g}}(m) &= {\gamma_L}^2[\vec{g}(n) + \overleftrightarrow{H}(n)\cdot\vec{v}]^\flat - \gamma_L\vec{v}^{\,\flat}\pounds_{e_0}\ln M - \pounds(n)_{e_0}[M^{-2}\bar{\bar{N}}] \\ &= {\gamma_L}^2[\vec{g}(n) + \vec{v}\times_n\tfrac{1}{2}\vec{H}(n) + \mathrm{SYM}\,\overleftrightarrow{H}(n)\cdot_n\vec{v}]^\flat + \dots \ . \end{aligned} \tag{4.14}$$

The relations

$$\begin{aligned} P(m)(Y^a e_a) &= Y^a \epsilon_a \ , \qquad & \mathcal{P}(\sigma_a \theta^a) &= \sigma_a \omega^a \ , \\ \mathcal{P}(Y^a \epsilon_a) &= Y^a e_a \ , \qquad & P(n)(\sigma_a \omega^a) &= \sigma_a \theta^a \ , \end{aligned} \tag{4.15}$$

enable one to convert the above index-free equations for the transformation of the gravitomagnetic vector field and the gravitoelectric 1-form back into component form

$$\begin{aligned} \tfrac{1}{2} H(n)^a &= \tfrac{1}{2} H(m)^a + \dots \ , \qquad & g(n)_a &= \gamma_L{}^2 [g(m)_a + \dots \ , \\ \tfrac{1}{2} H(m)^a &= \gamma_L{}^2 [\tfrac{1}{2} H(n)^a + \dots \ , \qquad & g(m)_a &= \gamma_L{}^2 [g(n)_a + \dots \ , \end{aligned} \tag{4.16}$$

Since the gravitomagnetic vector field and the gravitoelectric 1-form only involve the ordinary derivative and the exterior derivative respectively, their transformation laws are relatively simple and closely analogous to the electromagnetic case, especially in the stationary case when all the Lie derivative terms vanish. The transformation formulas for the gravitomagnetic vector field may be directly compared with the transformation of the magnetic field under the boost between the two points of view, but the formulas for the gravitoelectric 1-form must be compared with the boost of the electric 1-form field. The symmetric part of the gravitomagnetic tensor on the other hand involves the spatial connection and has a much more complicated transformation law. In order to compare the above equations with the familiar orthonormal component form of the Lorentz transformation of the electric and magnetic fields, one must bridge the gap between the four-dimensional spatial notation and the usual orthonormal frame component notation which is so familiar. This is done in appendix B.

In the weak field slow motion approximation, no distinction is made between the slicing lapse and shift and the threading lapse and shift in the lowest approximation (they differ in second order), but there is still a (first order) Galilean velocity transformation between the two points of view. On the other hand the reference and threading points of view differ by second order terms and so are not distinguished. In this limit in adapted local coordinates, the expression $dx^a/dt = \nu(m)^a$ defines the threading spatial velocity of a worldline, and the second time derivative defines the threading spatial Lie derivative of the (nonrelativistic) contravariant spatial momentum (per unit mass). When Thorne et al [31] express their slicing spatial force equation (3.20) for a stationary spacetime in this limit, they implicitly abandon their slicing point of view for the threading/reference point of view. Their equation (3.32) describes the time rate of change of the spatial momentum per unit mass in that latter point of view. This accounts for the loss of the symmetric part of the slicing gravitomagnetic tensor field and the increase by a factor of two of the antisymmetric part. This same reference/threading point of view is used in all discussions of weak field slow motion effects [29–32].

The original discussion by Forward [33] for the linearized theory splits the geodesic equation explicitly using the reference decomposition of the covariant spacetime metric and the contravariant 4-velocity, thus mixing the two points of view (slicing and threading), but the linearization erases the distinction in the metric at lowest order, leading to the threading point of view and the appearance of the "gravitomagnetic induction" term involving the time derivative of the threading shift 1-form, exactly analogous to the relation of the electric field to the scalar and vector potentials in electromagnetism. In fact this reference point of view more closely corresponds to the point of view of classical mechanics taken when discussing the centrifugal and Coriolis forces which arise in a rotating Cartesian coordinate system. This is treated below.

The next level of the splitting discussion would focus on the curvature, enabling one to consider geodesic deviation from each point of view. Relevant formulas for the splitting of the Einstein tensor can be found in Misner, Thorne and Wheeler [10] and Zel'manov [4,5], while the splitting of the

Riemann tensor in the threading point of view is dealt with by Cattaneo-Gasperini [47], Ferrarese [48] and Massa [16]. In the weak field slow motion approximation, one finds that the electric and magnetic parts of the Weyl curvature are directly related to the corresponding gravitoelectric and gravitomagnetic fields [31]. The Einstein equations in this approximation are closely analogous to Maxwell's equations [27–33]. The initial value problem in the threading point of view, unlike in the slicing point of view, is not very well known or understood. It has been discussed by Zel'manov [4] and Ferrarese [49].

5 Rotating coordinates in flat spacetime

No discussion of spatial gravitational forces would be complete without relating them to the familiar notions of centrifugal and Coriolis forces which arise in everyday experience. It is these forces which have been generalized by the splitting discussion of spacetime geodesics, so it is important to understand exactly how they fit into the general picture as a special case. The point of view in which they are discussed in classical mechanics is neither the slicing or threading point of view but rather the reference point of view.

Consider adapted coordinates $\{t, x^a\}$ for a nonlinear reference frame in an arbitrary spacetime. The reference decomposition of the geodesic equation considered as a (contravariant) vector equation is just its coordinate decomposition. The calculation of the spatial reference projection of that equation may be found in Forward's discussion [33] of linearized general relativity based on Møller's first edition discussion [8] of spatial gravitational forces using the coordinate time as the parameter for the geodesic, but modified to use the spatial coordinate decomposition of the geodesic equation rather than the threading decomposition. Linearization allows one to sidestep the some of the complications which arise in the general case, but the logarithmic time derivative of the coordinate gamma factor Γ defined by (2.5) leads to an additional acceleration-dependent term proportional to the velocity (compare with equation (6.25) of Misner, Thorne and Wheeler [10]).

However, in flat spacetime with a coordinate system $\{t, x^a\}$ consisting of a rigidly rotating system of orthonormal Cartesian spatial coordinates and the usual time coordinate t of the related nonrotating inertial coordinates, the complications wash out and lead to the usual formulas

$$\begin{aligned} \ddot{t} &= 0 \ , \\ \ddot{\vec{x}} &= \vec{g}_{(ref)} + \dot{\vec{x}} \times \vec{H}_{(ref)} \ . \end{aligned} \tag{5.1}$$

Here the dot is the usual time coordinate derivative and the usual vector notation is employed. The spatial forces are

$$\begin{aligned} \vec{g}_{(ref)} &= -\vec{\Omega} \times (\vec{\Omega} \times \vec{x}) \ , \\ \vec{H}_{(ref)} &= 2\vec{\Omega} \ , \end{aligned} \tag{5.2}$$

where $\vec{\Omega}$ is the constant vector describing the angular velocity of the rotating system, identified with a constant vector field. The reference gravitoelectric field is exactly the centrifugal force (per unit mass) and the reference gravitomagnetic field vector is just twice the global rotation of the coordinate system, leading to the Coriolis force (per unit mass).

To see how these relate to the slicing and threading fields, one must evaluate them for the nonlinear reference frame for which these coordinates are adapted. The slicing is a flat time slicing of Minkowski spacetime but the threading is an inhomogeneous tilted threading relative to that slicing, and it is timelike only within a certain cylinder in space called the light cylinder inside of which the velocity of rotation is less than the speed of light. Thus the slicing point of view holds everywhere, while the threading point of view is limited to the interior of the light cylinder.

Since the spatial metric is Euclidean in the slicing point of view, one can use the customary vector notation unambiguously in expressing the line element (instead of metric to allow the dot product notation)

$$\begin{aligned} ds^2 &= -dt^2 + (d\vec{x} + \vec{\Omega} \times \vec{x}dt) \cdot (d\vec{x} + \vec{\Omega} \times \vec{x}dt) \ , \\ N &= 1 \ , \qquad \vec{N} = \vec{\Omega} \times \vec{x} \ , \qquad g_{ab} = \delta_{ab} \ . \end{aligned} \tag{5.3}$$

The differentials $d\vec{x}+\vec{\Omega}\times\vec{x}dt$ are the rotated differentials of the nonrotating Cartesian coordinates. The slicing spatial gravitational fields in this notation are

$$\vec{g}(n) = 0 = \text{SYM}\,\overleftrightarrow{H}(n) \ , \qquad \vec{H}(n) = 2\vec{\Omega} = \vec{H}_{(ref)} \ . \tag{5.4}$$

The Lorentz boost parameters are

$$\vec{v} = \vec{N} \ , \qquad v = ||\vec{v}|| \ , \qquad \gamma_L = (1 - v^2)^{-1/2} \ , \tag{5.5}$$

leading to the threading quantities

$$\begin{aligned} &M = \gamma_L{}^{-1} \ , \qquad \bar{\bar{M}} = \gamma_L{}^2 \delta_{ab} v^a dx^b \ , \qquad \gamma_{ab} = \delta_{ab} + \gamma_L{}^{-2} M_a M_b \ , \\ &g(m)_a = \gamma_L{}^2 \delta_{ab} g(m)^b_{(ref)} \ , \qquad H(m)^a = \gamma_L{}^2 H^a_{(ref)} \ . \end{aligned} \tag{5.6}$$

Thus it is the gravitoelectric force in the threading point of view, apart from a time reparametrization and a projection, which corresponds to the centrifugal force. The slicing gravitomagnetic vector field directly equals the reference gravitomagnetic vector field, which in turn is twice the global angular velocity of the coordinate system, and leads to the Coriolis force. The threading gravitomagnetic vector field differs from twice the constant global rotation vector by the time reparametrization and a projection in such a way that it equals the inhomogeneous local rotation of the threading congruence.

6 Gyro precession?

An obvious question to ask is, what does all of this formalism have to do with the classic gyro precession formula? This formula, describing the precession of a gyroscope in the field of a rotating body, seems to have been first obtained in its present form by Schiff [50] within linearized GR and was later extended to the PPN theory, as reviewed for example by Misner, Thorne and Wheeler [10], and more recently discussed within linearized GR in a very elegant way by Thorne [32]. Physically gyros define operationally what it means to be locally nonrotating, so if there is rotation of the spin of a gyro, it has to be a relative rotation with respect to something which is locally rotating. This is exactly what the splitting formalism is set up to measure and which leads to the introduction of the concept of the gravitomagnetic field, the measurement of which is the goal of the long awaited Stanford gyroscopic precession experiment [51]. It is this problem which has provided much of the motivation for talking about "gravitomagnetism."

However, the famous precession formula is not just a splitting version of the equation for Fermi-Walker transport along a worldline of a vector which is spatial with respect to the worldline's 4-velocity u

$$\nabla_{(\text{fw})}(u) S = 0 \ , \qquad S \in LRS_u \ . \tag{6.1}$$

One can easily split this equation as done above for the observer spatial Fermi-Walker transport. For example, in the threading point of view, where the spacelike spin vector has the decomposition

$$\begin{aligned} S &= S^\top m + S(m)^a \epsilon_a \ , \\ S^\top &= \gamma_{ab} \nu(m)^a S(m)^b \end{aligned} \tag{6.2}$$

due to the orthogonality condition, the spatial projection of the Fermi-Walker transport equation is

$$\begin{aligned} D_{(\mathrm{th})}(m)S(m)^a/d\tau_m = &-S^\top a(m)^a + [S^\top a(u)^\top + S(m)^b \gamma_{bc} a(u)^c]\nu(m)^a \\ &+ k(m)^a{}_b[S(m)^b + S^\top \nu(m)^b] \ , \end{aligned} \tag{6.3}$$

where the magnitude $||S|| = S^\alpha S_\alpha$ is constant. This reduces to the previous result when the relative velocity $\nu(m)$ vanishes so that $u = m$ and $S^\top = 0$. For a geodesic, the acceleration term involving $a(u)$ vanishes. This formula describes how the spin vector S changes with respect to the threading point of view nonlinear reference frame. It does not directly describe a rotation, since it must take into account the nonorthonormality of the spatial projected computational frame and changes in the magnitude of the spatial spin vector. What then is the classic precession formula?

Consider a black hole spacetime in the Boyer-Lindquist coordinate system, with its associated nonlinear reference frame. The threading point of view holds outside the ergosphere where the Killing observers follow the timelike time lines, while the slicing point of view holds outside of the event horizon where the slicing is spacelike. The stationary threading observers have the interpretation of being nonrotating with respect to the asymptotically flat region of spacetime, while the nonstationary slicing observers have the interpretation of being locally nonrotating with respect to the spacetime geometry.

The Boyer-Lindquist spatial coordinates $\{r, \theta, \phi\}$ are orthogonal so both the coordinate derivatives $\{e_a\}$ and coordinate differentials $\{\omega^a\}$ are orthogonal and can be normalized and then completed uniquely to an (axially symmetric stationary) orthonormal spacetime frame or dual frame. Normalizing the spatial coordinate derivatives leads to the slicing orthonormal frame $\{n, e_{\hat{a}}\}$ with dual frame $\{\omega^\perp, \theta^{\hat{a}}\}$ while normalizing the spatial coordinate differentials leads to the threading orthonormal frame $\{m, \epsilon_{\hat{a}}\}$ with dual frame $\{\omega^\top, \omega^{\hat{a}}\}$.

One can boost each of these two orthonormal frames uniquely to align them with the 4-velocity of an arbitrary gyro worldline

$$\begin{aligned} &B(u,n)\{n, e_{\hat{a}}\} = \{u, E_{(sl)a}\} \ , \\ &B(u,m)\{m, \epsilon_{\hat{a}}\} = B(u,m)B(m,n)\{n, e_{\hat{a}}\} = \{u, E_{(th)a}\} \ . \end{aligned} \tag{6.4}$$

The two orthonormal frames so obtained are related to each other by the time-dependent Thomas rotation determined by the composition of the two boosts $B(u,m)$ and $B(m,n)$

$$E_{(th)a} = B(u,m)B(m,n)B(n,u)E_{(sl)a} = RE_{(sl)a} \ , \tag{6.5}$$

which may in some sense be interpreted as the relative rotation of the spatial axes of the slicing and threading observers. The boosted frame in each point of view is the spatial frame that an observer following the worldline of the gyro would reconstruct as the frame he would see if that frame were not moving relative to him. Since an orthonormal spatial frame in relative motion does not appear orthonormal, actively boosting it to one's rest frame is the only way one can determine the orientation of a vector with respect to the moving axes.

One can then calculate the relative rotation of the gyro spin vector in its own local rest space relative to the boosted spatial frame in each point of view. This is equivalent to boosting the gyro spin vector back to the local rest frame of the test observers and calculating its rotation relative to the original frame, which is easier to evaluate. In the slicing point of view, this would measure in some sense the rotation of the spin relative to the locally nonrotating observers, while in the threading point of view, it would instead be relative to the Killing observers which in some sense reflect the properties of the nonrotating frame of the "distant stars". Once aberration of starlight is taken into account, this latter effect may be used to measure the precession of the gyro in its own rest frame relative to the "distant stars".

A similar situation exists in the PPN theory, where the PPN spatial coordinates are orthogonal to the lowest nontrivial order and hence the previous discussion holds. The derivation of the gyro precession within the PPN theory is described in detail in Misner, Thorne and Wheeler [10] using the threading point of view choice of orthonormal frame tied to the PPN coordinate grid. (Schiff [50] omits this discussion for a rotating body, quoting only the result.) Thorne [32] has given a very physical explanation for each of the terms in the expression for the precession angular velocity within linearized GR, showing how each is associated with either the gyro acceleration, the "spatial curvature", an "induced gravitomagnetic field" effect, and the actual gravitomagnetic field.

7 The co-rotating Fermi-Walker derivative

It remains to be seen how the covariant result of Massa and Zordan [17] for the spin precession (reviewed in this volume) relates to the previous noncovariant discussion. The precession problem is one which is closely related to the relative rotation of orthonormal spatial frames, but spatial Lie transport is incompatible with orthonormality except along Killing vector fields. This is reflected in the noncommutivity of index shifting and differentiating by a total spatial covariant derivative.

The Fermi-Walker derivative along the observer congruence does commute with index shifting of spatial fields, but its transport of spatial fields exhibits a rotation relative to the observer congruence. By removing this relative rotation from the spatial Fermi-Walker transport, one obtains a compromise between the spatial Lie and Fermi-Walker time derivatives: the *spatial co-rotating Fermi-Walker derivative*. An orthonormal spatial frame transported along the observer congruence by its corresponding transport co-rotates with the observer congruence but does not undergo the expansion and shear of the spatial Lie transported spatial frames.

For a spatial vector field X this new derivative is related to the spatial Fermi-Walker derivative and the spatial Lie derivative by

$$\begin{aligned} \nabla_{(\mathrm{cfw})}(o)X^a &= \nabla_{(\mathrm{fw})}(o)X^a + \omega(o)^a{}_b X^b \\ &= \pounds(o)_O X^a + \theta(o)^a{}_b X^b \end{aligned} \tag{7.1}$$

and is extended to arbitrary rank spatial tensor fields in the usual way. Like the spatial Lie and Fermi-Walker derivatives, it too is obtained by spatially projecting a corresponding operator on spacetime. These operators are the spacetime Fermi-Walker derivative, the spacetime co-rotating Fermi-Walker derivative and the Lie derivative along o

$$\begin{aligned} {}^{(4)}\nabla_{(\mathrm{cfw})}(o)X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(o)X^\alpha + \omega(o)^\alpha{}_\beta X^\beta \ , \\ {}^{(4)}\nabla_{(\mathrm{fw})}(o)X^\alpha &= X^\alpha{}_{;\beta} - [o^\alpha a(o)_\beta - a(o)^\alpha o_\beta] X^\beta \ , \\ \pounds(o)_O X^\alpha &= X^\alpha{}_{;\beta} - o^\alpha{}_{;\beta} X^\beta \ . \end{aligned} \tag{7.2}$$

For a worldline with 4-velocity u as above, one may define total spatial covariant derivatives using any of these temporal derivatives

$$\begin{aligned} D_{(\mathrm{lie})}(u,o)/d\tau_o &= \pounds(o)_O + \nabla(o)_{\vec{\nu}} \ , \\ D_{(\mathrm{fw})}(u,o)/d\tau_o &= \nabla_{(\mathrm{fw})}(o) + \nabla(o)_{\vec{\nu}} \ , \\ D_{(\mathrm{lie})}(u,o)/d\tau_o &= \nabla_{(\mathrm{cfw})}(o) + \nabla(o)_{\vec{\nu}} \ , \end{aligned} \tag{7.3}$$

related to the previous operators by

$$D_{(\perp)}(u)/d\tau_n = D_{(\mathrm{lie})}(u,n)/d\tau_n \ , \qquad D_{(\mathrm{th})}(u)/d\tau_m = D_{(\mathrm{lie})}(u,m)/d\tau_m \ , \tag{7.4}$$

and finally

$$\begin{aligned} D_{(\mathrm{sl})}(u)X^a/d\tau_n &= D_{(\mathrm{lie})}(u,n)X^a/d\tau_n + N^{-1}[\pounds(o)_O - \nabla(o)_{\vec{n}}]X^a \ , \\ &= D_{(\mathrm{lie})}(u,n)X^a/d\tau_n - N^{-1}N^a{}_{|b}X^b \ . \end{aligned} \tag{7.5}$$

The threading operator $D_{(\mathrm{cfw})}(u,m)/d\tau_m$ is the operator $\delta^*/\delta T$ of Massa [15] as defined in this volume. Note that when acting on functions, all of the above operators reduce to the ordinary derivative by o so one may drop the qualifying subscript notation.

8 Spatial forces and torques revisited

To simplify the notation temporarily, the qualifiers (o) and (u,o) will be selectively suppressed. All symbols will be understood to hold with reference to a given worldline and observer congruence. Let the worldline belong to a test particle or test gyro under a given nongravitational force

$$f = \gamma[\wp o + \vec{F}] \ , \qquad f_\alpha u^\alpha = 0 \rightarrow \wp = \vec{F} \cdot_o \vec{\nu} \ , \tag{8.1}$$

which must be orthogonal to the unit 4-velocity. For either the ordinary or co-rotating total spatial covariant derivative, the force equation $a(u) = f$ can then be rewritten in the form

$$\begin{aligned} D\vec{p}/d\tau_o &= \vec{F}^{(\mathrm{G})} + \vec{F} \ , \\ DE/d\tau_o &= [\vec{F}^{(\mathrm{G})} + \vec{F}] \cdot_o \vec{\nu} \ , \end{aligned} \tag{8.2}$$

where the spatial gravitational force terms differ from the corresponding spatial Lie forces by the same terms as the corresponding total spatial covariant derivative operators, reversed in sign

$$\begin{aligned} \vec{F}^{(\mathrm{G})}_{(\mathrm{fw})} &= \gamma[\vec{g}(o) + \vec{\nu} \times_o \vec{H}(o) - \vec{\nu} \cdot_o \overleftrightarrow{\theta}(o)] \ , \\ \vec{F}^{(\mathrm{G})}_{(\mathrm{cfw})} &= \gamma[\vec{g}(o) + \tfrac{1}{2}\vec{\nu} \times_o \vec{H}(o) - \vec{\nu} \cdot_o \overleftrightarrow{\theta}(o)] \ . \end{aligned} \tag{8.3}$$

Similarly the Fermi-Walker spin transport equation has the form

$$\begin{aligned} D_{(\mathrm{cfw})}\vec{S}/d\tau_o = &-\tfrac{1}{2}\vec{H}(o) \times_o \vec{S} + (\vec{\nu} \cdot_o \vec{S})\vec{F}^{(\mathrm{G})}_{(\mathrm{fw})} \\ &+ \gamma[(\vec{\nu} \cdot_o \vec{F})(\vec{\nu} \cdot_o \vec{S}) - \vec{F} \cdot_o \vec{S}]\vec{\nu} \ . \end{aligned} \tag{8.4}$$

From these equations it is relatively straightforward to calculate the co-rotating Fermi-Walker total spatial covariant derivative of the boosted spin vector $\vec{\mathcal{S}} = B(o,u)S$, once an expression for the boost $B(u,o)$ from LRS_u to LRS_o is obtained. For this one need only decompose the spatial projection perpendicular and parallel to the relative velocity in the observer local rest space

$$P(o) = P^{(\perp)}(o) + P^{(||)}(o) = [P(o) - \hat{\nu} \otimes \hat{\nu}^\flat] + [\hat{\nu} \otimes \hat{\nu}^\flat] \ . \tag{8.5}$$

The boost differs only by Lorentz contraction along the unit relative velocity $\hat{\nu} = \nu^{-1}\vec{\nu}$

$$\begin{aligned} P(o) \circ B(o,u) \circ P(u) &= [P(o) - \hat{\nu} \otimes \hat{\nu}^\flat] + \gamma^{-1}[\hat{\nu} \otimes \hat{\nu}^\flat] = \cdots \\ &= P(o) - \gamma^{-1}(\gamma+1)^{-1}\vec{p} \otimes \overline{\overline{p}} \ , \end{aligned} \tag{8.6}$$

so that the boosted spin vector is

$$\vec{\mathcal{S}} = \vec{S} - \gamma^{-1}(\gamma+1)^{-1}(\vec{p}\cdot_o \vec{S})\vec{p} \; . \tag{8.7}$$

Its derivative is then calculated from the derivatives of $\vec{S}$, $\vec{p}$ and $\gamma = E$, leading to the result

$$D_{\rm (cfw)}\vec{\mathcal{S}}/d\tau_o = \vec{\zeta}_{\rm (cfw)} \times_o \vec{\mathcal{S}} \; , \tag{8.8}$$

where the angular velocity of $\vec{\mathcal{S}}$ relative to o is given by

$$\begin{aligned}\vec{\zeta}_{\rm (cfw)} &= -\tfrac{1}{2}\vec{H}(o) - \gamma^{-1}(\gamma+1)^{-1}\vec{\nu}\times_o \vec{F} + (\gamma+1)^{-1}\vec{\nu}\times_o \vec{F}^{\rm (G)}_{\rm (fw)} \\ &= \vec{\zeta}_{\rm (gm)} + \vec{\zeta}_{\rm (thom)} + \vec{\zeta}_{\rm (fok)}\end{aligned} \tag{8.9}$$

or equivalently by the Massa-Zordan formula

$$\begin{aligned}\vec{\zeta}_{\rm (cfw)} = &-\tfrac{1}{2}\vec{H}(o) - \gamma^{2}(\gamma+1)^{-1}\vec{\nu}\times_o \vec{a}_{\rm (cfw)}(u) \\ &+ \gamma\vec{\nu}\times_o [\vec{F}^{\rm (G)}_{\rm (cfw)} - \tfrac{1}{2}(\gamma+1)^{-1}\vec{\nu}\times_o \vec{H}(o)]\end{aligned} \tag{8.10}$$

when expressed instead in terms of the relative acceleration of u

$$\begin{aligned}\vec{a}_{\rm (cfw)}(u) &= D_{\rm (cfw)}\vec{\nu}/d\tau_o = D_{\rm (cfw)}[\gamma^{-1}\vec{p}]/d\tau_o = \cdots \\ &= \gamma^{-1}[\vec{F}^{\rm (G)} + \vec{F}] - \gamma^{-1}[\vec{F}^{\rm (G)} + \vec{F}]\cdot_o \vec{\nu}\;\vec{\nu} \; .\end{aligned} \tag{8.11}$$

However, this is still not the classical precession formula. Suppose one has a stationary nonlinear reference frame in a stationary spacetime and one chooses a stationary orthonormal spatial frame in the threading point of view. Since the expansion tensor vanishes by stationarity, the spatial co-rotating Fermi-Walker derivative reduces to the spatial Lie derivative and the same is true of the corresponding total spatial covariant derivatives. The spatial frame thus undergoes co-rotating Fermi-Walker transport along the observer congruence. Since the co-rotating Fermi-Walker total spatial covariant derivative contains the spatial connection, co-rotating Fermi-Walker transport of a spatial vector around a curve which returns to the same point in Space, i.e., to the same threading curve, will lead to a nontrivial rotation relative to the stationary frame. To eliminate this rotation due to the space curvature, one must add a compensating *space curvature* precession term to the relative angular velocity $\vec{\zeta}_{\rm (cfw)}$.

In a general spacetime assume that the orthonormal spatial frame $\{e_a\}$ is transported along the observer congruence by co-rotating Fermi-Walker transport. Since the connection 1-form in an orthonormal frame is antisymmetric, one can define a space curvature angular velocity by

$$\begin{aligned}D_{\rm (cfw)}(u,o)\mathcal{S}^a/d\tau_o &= d\mathcal{S}^a/d\tau_o + \Gamma(o)^a{}_{bc}\nu^b\mathcal{S}^c \\ &= d\mathcal{S}^a/d\tau_o - \eta(o)^a{}_{bc}\zeta_{\rm (sc)}{}^b\mathcal{S}^c \; ,\end{aligned} \tag{8.12}$$

where

$$\zeta_{\rm (sc)}{}^a = -\tfrac{1}{2}\eta(o)^{abc}\Gamma(o)_{bdc}\nu^d \; . \tag{8.13}$$

One then has the absolute angular velocity of the boosted spin relative to this frame as the relative angular velocity plus this space curvature term

$$d\mathcal{S}^a/d\tau_o = \eta(o)^a{}_{bc}[\vec{\zeta}_{\rm (cfw)} + \vec{\zeta}_{\rm (sc)}]^b\mathcal{S}^c \; . \tag{8.14}$$

Of course now that ordinary derivatives are being used one has to choose the spatial frame carefully in order that this make some physical sense. For example, if one introduces an axially symmetric stationary frame in an stationary axially symmetric spacetime, then this will introduce an extra rotation relative to spacelike infinity due to the rotation of the frame as one revolves around the symmetry axis. However, this would not effect the total rotation of a spin vector along a orbit which returns to the same point of Space. In fact in the linearized discussions one always assumes a cartesian-like choice of spatial frame to avoid this problem.

In order to see how this space curvature term appears in the classical precession formula, consider the Schwarzschild line element in the isotropic cartesian coordinate form

$$ds^2 = -(1-2U)dt^2 + (1+2U)\delta_{ab}dx^a dx^b \ , \tag{8.15}$$

where $U = M/r$ and $r = (\delta_{ab}x^a x^b)^{1/2}$. Let $o = m = n = (1-2U)^{-1/2}\partial/\partial t$ be the usual Schwarzschild observer 4-velocity, and let $e_a = (1+2U)^{-1/2}\partial/\partial x^a$ be the preferred spatial orthonormal frame discussed above. For the corresponding nonlinear reference frame, the slicing and threading points of view coincide.

Then the spatial structure functions for this observer-adapted orthonormal frame, the spatial connection components, and the space curvature precession, in the limit $U \ll 1$ are

$$\begin{aligned} C^a{}_{bc} &= 2\delta^a{}_{[b} g(o)_{c]} \ , \\ \Gamma(o)_{abc} &= 2\delta_{b[a} g(o)_{c]} \ , \\ \zeta_{(\mathrm{sc})}{}^a &= \eta(o)^{abc}\nu_b g(o)_c = [\vec{\nu} \times_o \vec{g}(o)]^a \ , \end{aligned} \tag{8.16}$$

where the frame components of the gravitoelectric field are

$$g(o)_a = e_a U \ . \tag{8.17}$$

Adding this term to the nonrelativistic limit of the Fokker precession term, namely the *spin-orbit* precession [32]

$$\vec{\zeta}_{(\mathrm{fok})} \to \vec{\zeta}_{(\mathrm{so})} = \tfrac{1}{2}\vec{\nu} \times_o \vec{g}(o) \ , \tag{8.18}$$

one obtains the famous factor $\frac{3}{2}$ of the Schiff precession formula. The same discussion holds for the PPN calculation of Misner, Thorne and Wheeler [10] for an isolated rotation body in general relativity using the threading point of view.

9 Discussion

The threading point of view seems to have been initially favored in the fifties but was eclipsed in the sixties by the slicing point of view and has remained slighted in comparison ever since. Certainly the explanation is due to the principal motivation for using the latter point of view, quantum gravity. However, for better understanding certain classical problems like stationary spacetimes and the PPN theory in astrophysical problems, the two points of view together help provide better insight into the geometry than either one alone. The common mathematical framework introduced here facilitates this comparison. Indeed for stationary spacetimes, the threading point of view seems to be better adapted to the symmetry and certain physical problems of interest than the slicing point of view, and when one looks closer at the analyses of spatial gravitational forces and spin precession in linearized treatments, one sees that the threading point of view is frequently used instead of the slicing point of view.

The precise definition of the possible total spatial covariant derivatives gives a clean formulation of the spatial gravitational forces in the slicing point of view, and relates them to the other points of view which have a much longer history. Similarly the co-rotating Fermi-Walker derivative makes the discussion of spin precession clearer.

The machinery developed here can also clarify greatly the relationship between the various splittings of Maxwell's equations (slicing, threading, reference, and congruence splittings) and explain neatly the spacetime geometry of the Landau-Lifshitz coordinate decomposition of the electromagnetic field, which is based on the reference decomposition. The notation also leads to a simple discussion of the Sagnac effect and synchronization questions in stationary spacetimes. The Taub [52,53] approach to isoentropic perfect fluids fits nicely into threading point of view and allows an elegant geometrization in terms of a principal fiber bundle, as does the stationary case with any source. The extension of Taub's work to the slicing point of view by Bao, Marsden and Walton [54] might also benefit somewhat from a comparison with the present framework; the alternative fluid gauge variables of Walton are in fact related to a reference decomposition of the fluid circulation vector. The common framework for Newtonian gravitational theory and general relativity recently introduced by Ehlers [55] (and presented in this volume) in order to clarify the Newtonian limit of general relativity also fits nicely into the present scheme. It also allows one to relate recent work of Ellis and Bruni [56] on the perturbations of Friedmann-Robertson-Walker spacetimes from the congruence point of view to more classic treatments from the slicing point of view [43,58].

The present article has tried to formulate a bit more carefully ideas which are scattered about in the literature and put them all into a single coherent picture. The great advance of relativity may have been the unification of space and time into the single entity of spacetime, but splittings enable one to perceive this object in terms of our spatial intuition and prove useful in many circumstances. By clearly understanding how these spatial perspectives are imposed on spacetime and how different choices relate to each other, one retains some of the power of the concept of invariant spacetime geometry.

A Splitting spacetime derivatives

Extending index raising to the symbol ∂_α, which is just the computational frame component symbol for the exterior derivative operator on functions ($\partial_\alpha \equiv e_\alpha f$), leads to the following relations which describe the orthogonal splitting of the differential of a function

$$df = \omega^\alpha \partial_\alpha f \tag{A.1}$$

in the two points of view via the isomorphisms with the reference decompositions of the covariant and contravariant forms of that differential

$$\begin{array}{llll} \text{slicing:} & \partial^0 f = -N^{-2}\epsilon_0 f \ , & \partial^\perp f = -nf \ , & \partial_a f = e_a f \ , \\ \text{threading:} & \partial_0 f = e_0 f \ , & \partial_\top f = mf \ , & \partial^a f = \gamma^{ab}\epsilon_b f \ . \end{array} \tag{A.2}$$

The spatial reference components respectively determine via the above isomophisms the projected computational components of the the spatial exterior derivative $d(o)$ of the function, while the time reference components are instead related to the two Lie derivatives $\pounds_{\epsilon_0} = \pounds_{e_0} - \pounds_{\vec{N}}$ and $\pounds_{e_0}$ which respectively describe the rate of change of the function with respect to the observer congruence in the two points of view.

In order to decompose the spacetime covariant derivative ${}^{(4)}\nabla_X S$ of a spacetime tensor field S with respect to a spacetime vector field X, using the orthogonal decomposition associated with a unit timelike vector field o, one can consider decomposing S itself, the result ${}^{(4)}\nabla_X S$, and also the vector field X doing the differentiation. The end result has many different terms, a few of which have a special significance. Since the covariant derivative is linear

$$ {}^{(4)}\nabla_X = (-X^\alpha o_\alpha)^{(4)}\nabla_o + {}^{(4)}\nabla_{P(o)X} \ , \tag{A.3} $$

one is led to individually consider covariant derivatives along o itself and along directions which are orthogonal to o, i.e., spatial vector fields with respect to o. The spatial projection of each of these on all indices yields the interesting operators

$$ \begin{aligned} &\nabla_{(\mathrm{fw})}(o)S \equiv P(u)^{(4)}\nabla_o S \ , \\ &\nabla(o)_X S \equiv P(u)^{(4)}\nabla_{P(o)X} S \end{aligned} \tag{A.4} $$

already described above. The same reference decomposition isomorphisms may be used to obtain expressions for these operators in terms of the projections of the computational frame components of the tensor S and of the spacetime connection.

First one may consider decomposing the $\binom{1}{2}$-tensor ${}^{(4)}\Gamma^\alpha{}_{\beta\gamma} e_\alpha \otimes \omega^\beta \otimes \omega^\gamma$ representing the connection components with respect to the computational frame as one would any other tensor field. (It is of course a different tensor field for each choice of frame.) The purely spatial part of this decomposition gives the spatial part of the spatial covariant derivative, while some of the remaining mixed parts represent the kinematical quantities of the observer 4-velocity. Since the covariant slicing observer 4-velocity and the contravariant threading observer 4-velocity are "reference temporal" fields (no nonzero spatially indexed computational components), these kinematical quantities are directly equal to certain projections of the computational frame connection component tensor. As an example, consider the slicing acceleration $a(n) = a(n)_a g^{ab} e_b$, which is a spatial field parametrized by its covariant spatial computational components

$$ a(n)_a = n_{a;\beta} n^\beta = n_{a,\beta} n^\beta - {}^{(4)}\Gamma^\beta{}_{a\gamma} n^\gamma n_\beta = {}^{(4)}\Gamma^\perp{}_{a\perp} = -{}^{(4)}\Gamma^{\perp\perp}{}_a \ , \tag{A.5} $$

where the last equality follows from the relation ${}^{(4)}\Gamma^\gamma{}_{[\alpha\beta]} = \frac{1}{2} C^\gamma{}_{\alpha\beta}$ and the fact that the zero-indexed structure functions vanish. Similar calculations may be used to evaluate the remaining kinematical quantities in each point of view. (Note that the index $\perp$ or $\top$ on ${}^{(4)}\Gamma$ is here intended only as a rescaling of the index 0.)

The acceleration of both congruences can be expressed in terms of an acceleration potential [23,26], which is just the natural logarithm of the respective lapse function, and an additional shift term in the threading case

$$ \begin{aligned} a(n)_a &= e_a(\ln N) = -{}^{(4)}\Gamma^{\perp\perp}{}_a \\ &\qquad \to \quad a(n)_\alpha = \nabla_\alpha \ln N \ , \\ a(m)^a &= \gamma^{ab}[\epsilon_b(\ln M) + e_0 M_b] = {}^{(4)}\Gamma^a{}_{\top\top} \\ &\qquad \to \quad a(m)_\alpha = \nabla(m)_\alpha \ln M + [\pounds(m)_{e_0} \bar{\bar{M}}]_\alpha \ . \end{aligned} \tag{A.6} $$

For a perfect fluid spacetime in the Taub nonlinear reference frame [18,43,52], the Lie derivative of the dual shift vanishes and the acceleration of the fluid is entirely determined by the acceleration potential.

The expansion is parametrized by the following computational components

$$ \begin{aligned} \theta(n)_{ab} &= n_{(a;b)} = {}^{(4)}\Gamma^\perp{}_{(ba)} \equiv -K_{ab} = -N^{-1}[-\tfrac{1}{2} e_0 g_{ab} + N_{(a|b)}] \ , \\ \theta(m)^{ab} &= m^{(a;b)} = {}^{(4)}\Gamma^{(ab)}{}_\top = -M^{-1}[\tfrac{1}{2} e_0 \gamma^{ab}] \ , \end{aligned} \tag{A.7} $$

while the rotation is parametrized by the following computational components

$$
\begin{aligned}
-\,\omega(n)_{ab} &= n_{[a;b]} = {}^{(4)}\Gamma^{\perp}{}_{[ba]} \equiv 0 \ , \\
-\,\omega(m)^{ab} &= m^{[a;b]} = {}^{(4)}\Gamma^{[ab]}{}_{\top} = M M^{[a||b]} = \tfrac{1}{2}\gamma^{ac}\gamma^{bd} M d\bar{\bar{M}}[\epsilon_d, \epsilon_c] \ .
\end{aligned} \tag{A.8}
$$

Note that both covariant expansion tensors may be represented as Lie derivatives or spatial Lie derivatives (the projection is unnecessary) by the expressions

$$
\begin{aligned}
\theta(n)^\flat &= \tfrac{1}{2} N^{-1} \pounds_{\epsilon_0} P(n)^\flat \ , \\
\theta(m)^\flat &= \tfrac{1}{2} M^{-1} \pounds_{e_0} P(m)^\flat \ ,
\end{aligned} \tag{A.9}
$$

which in turn are one step from the general formula

$$
\theta(o)^\flat = \tfrac{1}{2} \pounds_O P(o)^\flat = \tfrac{1}{2} \pounds(o)_O P(o)^\flat \ , \tag{A.10}
$$

recalling the property (2.9) of the spatial Lie derivative.

The curvature tensor $R(o)$ of the projected connection $\nabla(o)$ may be defined invariantly by a generalization of the usual formula for a nondegenerate connection including a term involving the rotation tensor. For spatial vector fields X, Y, and Z, this identity takes the form

$$
([\nabla(o)_X, \nabla(o)_Y] - \nabla(o)_{[X,Y]})Z = R(o)(X,Y)Z + 2\omega(o)^\flat(X,Y)\pounds(o)_o Z \ , \tag{A.11}
$$

which may be found in a somewhat disguised form in the work of Zel'manov [4,5] and more explicitly in Ferrarese [48] and Massa [16] for the threading point of view where the rotation tensor is nonzero.

The spatial structural functions in both spatial projected computational frames are identical

$$
\begin{aligned}
&[\epsilon_a, \epsilon_b] = C^c{}_{ab}\epsilon_c + D^0{}_{ab} e_0 \ , \\
&D^0{}_{ab} = \epsilon_a M_b - \epsilon_b M_a - M_c C^c{}_{ab} = -d\theta^0(\epsilon_a, \epsilon_b) = [d(m)\bar{\bar{M}}](\epsilon_a, \epsilon_b) \ ,
\end{aligned} \tag{A.12}
$$

which explains the similarity in the expressions for the spatial part of the spatial connection components. Only when $D^0{}_{ab} = 0$, i.e., when the spatial exterior derivative $d(m)$ of the threading shift 1-form vanishes (vanishing gravitomagnetic vector field), do the vector fields $\{\epsilon_a\}$ form an integrable distribution, equivalent to the condition that m is hypersurface-forming. The only other nonzero structure functions are

$$
\begin{aligned}
&[e_0, \epsilon_a] = D^0{}_{0a} e_0 \ , \\
&D^0{}_{0a} = e_0 M_a = [\pounds(m)_{e_0} \bar{\bar{M}}](\epsilon_a) \ ,
\end{aligned} \tag{A.13}
$$

which vanish only if the spatial frame $\{\epsilon_a\}$ is Lie dragged along the threading, equivalent to the vanishing of the projection orthogonal to m of the Lie derivative of the threading shift 1-form. This occurs in the Taub [52,53] perfect fluid comoving system as a consequence of the fluid equations of motion and in a stationary nonlinear reference frame in a stationary spacetime, in each case permitting a principal bundle interpretation of the splitting with a connection related to the dual shift 1-form [18].

The Cattaneo formalism [1–3] makes use of the partially normalized frame $\{e_\top, \epsilon_a\}$ which has the following structure functions

$$
\begin{aligned}
&[\epsilon_a, \epsilon_b] = C^c{}_{ab}\epsilon_c + E^\top{}_{ab} e_\top \ , \qquad [e_\top, \epsilon_a] = E^\top{}_{\top a} e_\top \ , \\
&E^\top{}_{ab} = 2\omega(m)_{ab} = 2H(m)_{[ab]} \equiv \tilde{\Omega}_{ab} \ , \\
&E^\top{}_{\top a} = a(m)_a = -g(m)_a \equiv C_a \ .
\end{aligned} \tag{A.14}
$$

Cattaneo's symbols $\frac{1}{2}\tilde{\Omega}_{ab}$ and C_a are just the spatially projected computational frame components of the rotation and acceleration of the threading congruence, or respectively the negative gravitoelectric field and the antisymmetric part of the gravitomagnetic tensor field. The partially normalized frame $\{e_\perp, e_a\}$ in the slicing point of view has similar structure functions, but the gravitomagnetic field does not appear cleanly

$$\begin{aligned} &[e_a, e_b] = C^c{}_{ab} e_c \ , \qquad [e_\perp, e_a] = E^\perp{}_{\perp a} e_\perp + E^b{}_{\perp a} e_b \ , \\ &\quad E^\perp{}_{\perp a} = a(n)_a = -g(n)_a \ , \\ &\quad E^b{}_{\perp a} = -N^{-1} \theta^b (\pounds_{\vec{N}} e_a) = H(n)^b{}_a + [C^b{}_{ca} - \Gamma(n)^b{}_{ca}] N^c \ . \end{aligned} \tag{A.15}$$

In order to mimic Euclidean vector analysis, one may introduce covariant versions of the usual spatial operations. Working entirely with spatial vector fields with respect to a given orthogonal decomposition associated with a unit timelike vector field o, one can define the dot and cross products and the gradient, divergence and curl operators

$$\begin{aligned} X \cdot_o Y &= P(o)_{\alpha\beta} X^\alpha Y^\beta \ , & X \times_o Y &= \eta(o)^\alpha{}_{\beta\gamma} X^\beta Y^\gamma e_\alpha \ , \\ \vec{\nabla}(o) X &= \nabla(o)^\alpha X \otimes e_\alpha \ , & \mathrm{grad}_o\, f &= \vec{\nabla}(o) f \ , \\ \mathrm{div}_o\, X &= \vec{\nabla}(o) \cdot_o X \ , & \mathrm{curl}_o\, X &= \vec{\nabla}(o) \times_o X \ . \end{aligned} \tag{A.16}$$

One can also introduce the spatial duality operation ${}^{*(o)}$ for antisymmetric spatial fields defined by contraction with the spatial unit volume element 3-form $\eta(o)$ which is obtained from the spacetime unit volume element 4-form

$${}^{(4)}\eta = {}^{(4)}\eta_{0123} \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \ , \qquad {}^{(4)}\eta_{\alpha\beta\gamma\delta} = {}^{(4)}g^{1/2} \epsilon_{\alpha\beta\gamma\delta} \ , \qquad \epsilon_{0123} = 1 \tag{A.17}$$

by a single contraction with o

$$\begin{aligned} \eta(o)_{\alpha\beta\gamma} &= o^\delta\, {}^{(4)}\eta_{\delta\alpha\beta\gamma} \ , \\ \eta(n) &= g^{1/2} \theta^1 \wedge \theta^2 \wedge \theta^3 \ , \\ \eta(m) &= \gamma^{1/2} \omega^1 \wedge \omega^2 \wedge \omega^3 \ . \end{aligned} \tag{A.18}$$

The gradient of a function is just the contravariant form of the spatial exterior derivative, while the curl of a vector field is just the contravariant form of the spatial dual of the spatial exterior derivative of its corresponding 1-form

$$\begin{aligned} [\mathrm{grad}_o\, f]^\alpha &= P(o)^{\alpha\beta} (df)_\beta = [(d(o) f)^\sharp]^\alpha \ , \\ [\mathrm{curl}_o\, X]^\alpha &= [({}^{*(o)}(dX^\flat))^\sharp]^\alpha = \tfrac{1}{2} \eta(o)^{\alpha\beta\gamma} [d(o) X^\flat]_{\beta\gamma} \ . \end{aligned} \tag{A.19}$$

Since many second rank tensors also enter into vector analysis in the form of linear transformations of vectors, one can follow the notation of Thorne [31] for contravariant second rank tensors which are spatial with respect to o

$$\begin{aligned} \overleftrightarrow{S} &= S^{\alpha\beta} e_\alpha \otimes e_\beta \ , \\ \overleftrightarrow{S} \cdot_o \vec{X} &= S^{\alpha\beta} P(o)_{\beta\gamma} X^\gamma e_\alpha \ . \end{aligned} \tag{A.20}$$

These operations can be defined for both n and m and expressed in terms of the appropriate computational frame spatial components

$$\begin{aligned}
&(X^a e_a)\cdot_n (X^b e_b) = g_{ab}X^a X^b\ ,\\
&(X^a e_a)\times_n (X^b e_b) = g^{1/2} g^{ad}\epsilon_{dbc}X^b X^c\, e_a\ ,\\
&\text{div}\ X^a e_a = \vec{\nabla}\cdot_n (X^a e_a) = g^{-1/2}\partial\!\!\!/_a (g^{1/2}X^a)\ ,\\
&\text{curl}\ X^a e_a = \vec{\nabla}\times_n (X^a e_a) = g^{-1/2}\epsilon^{abc}(X_{c,b} - \tfrac{1}{2}X_d C^d{}_{bc})\, e_a\\
&\qquad\qquad = g^{-1/2}\epsilon^{abc}X_{c|b}e_a\ ,\\
&(X^a\epsilon_a)\cdot_m (X^b\epsilon_b) = \gamma_{ab}X^a X^b\ ,\\
&(X^a\epsilon_a)\times_m (X^b\epsilon_b) = \gamma^{1/2}\gamma^{ad}\epsilon_{dbc}X^b X^c\,\epsilon_a\ ,\\
&\text{div}_m\ X^a\epsilon_a = \vec{\nabla}(m)\cdot_m (X^a\epsilon_a) = \gamma^{-1/2}(\epsilon_a - C^b{}_{ab})(\gamma^{1/2}X^a)\ ,\\
&\text{curl}_m\ X^a\epsilon_a = \vec{\nabla}(m)\times_m (X^a\epsilon_a) = \gamma^{-1/2}\epsilon^{abc}(X_{c,,b} - \tfrac{1}{2}X_d C^d{}_{bc})\,\epsilon_a\\
&\qquad\qquad = \gamma^{-1/2}\epsilon^{abc}X_{c||b}\,\epsilon_a\ ,
\end{aligned}\tag{A.21}$$

where the slashed partial notation stands for

$$\partial\!\!\!/_a f \equiv (e_a - C^b{}_{ab})f\ . \tag{A.22}$$

The divergence operation applied to the contravariant components and the curl operation applied to the covariant components (spatial exterior derivative) only involve the spatial metric through the volume form, and hence are closely related to the usual Euclidean formulas apart from scaling factors. Using this notation, the rotation vector field for the unit vector m, which is the spatial dual of the rotation tensor, can be expressed as

$$\vec{\omega}(m) = \tfrac{1}{2}\eta(m)^{\alpha\beta\gamma\,(4)}\nabla_\beta m_\gamma e_\alpha = \tfrac{1}{2}{}^{(4)}\eta^{\alpha\beta\gamma\delta}m_\delta m_{\beta;\gamma}e_\alpha = \tfrac{1}{2}M\,\text{curl}_m\,\vec{M}\ . \tag{A.23}$$

In each point of view the time-dependent Riemannian metric on the Space manifold has its own connection and one can ask how it is related to the spatial part $\nabla(o)\circ P(o)$ of the spatial connection on spacetime at corresponding values of the time coordinate t. In the slicing point of view, the isomorphism between the spatial tensor algebra on spacetime and the algebra of time-dependent tensors on the Space manifold extends to the respective covariant differentiations. However, the expansion tensor of the threading congruence and nonzero time derivatives of fields being covariantly differentiated break the isomorphism in the threading point of view.

In the slicing point of view there is a natural identification between the spatial frame $\{e_a\}$ on spacetime and the frame on Space whose dual frame is $I_t{}^*\theta^a = I_t{}^*\omega^a$. If $\{x^a\}$ are adapted spatial coordinates, then this frame and its dual have the same expressions as $\{e_a\}$ and $\{\omega^a\}$ in terms of the pullback coordinates $I_t{}^*x^a$ on Space. Agreeing to use the same symbols on spacetime and on Space, then the connection components of the metric connection on Space are just the pullbacks $I_t{}^*\Gamma^a{}_{bc}$ of the spatial components of the spatial covariant derivative connection on spacetime. The ordinary derivatives of frame components along the frame vectors are related in exactly the same way, so the covariant derivative of a spatial field on spacetime is related in the same way to the corresponding covariant derivative on Space.

Since $\{e_a\}$, $\{\omega^a\}$ and the structure functions $C^a{}_{bc}$ are Lie dragged along the threading, they project to well-defined fields on the quotient space ${}^{(4)}M/\operatorname{Flow}(e_0)$ which may be denoted by the same symbols without too much confusion, the context clarifying their meaning. Thus in the threading point of view, the spatial frame $\{\epsilon_a\}$ projects down to $\{e_a\}$ in this notation, and the spatial metric to the time-dependent metric $\gamma_{ab}\omega^a \otimes \omega^b$, interpreting the components γ_{ab} as time-dependent functions on Space in a natural way. Its connection is defined by

$$\begin{aligned} &\nabla(\gamma)_{e_a} e_b = \lambda^c{}_{ab} e_c \ , \\ &\lambda^a{}_{bc} = \tfrac{1}{2}\gamma^{ad}[\gamma_{\{db,c\}_-} + C(m)_{\{dbc\}_-}] \ , \end{aligned} \tag{A.24}$$

where all indices are shifted with γ_{ab} and γ^{ab}. These components are related to those of the connection associated with m in the following way

$$\Gamma(m)_{abc} - \lambda_{abc} = \tfrac{1}{2}[\pounds_{e_0}\gamma]_{\{ab}M_{c\}} = M\theta(m)_{\{ab}M_{c\}} \ , \tag{A.25}$$

where all components and fields are understood to be projected down to the quotient space. These connection components agree only for a threading congruence with vanishing expansion tensor as occurs in the stationary case when e_0 is taken to be a Killing vector field. However, in this case the covariant derivative with respect to $\nabla(\gamma)$ of a spatial tensor projected to the quotient space will be the projection of the spatial covariant derivative with respect to $\nabla(m)$ only if the tensor is itself also stationary, so that no difference exists between the e_a and ϵ_a derivatives of its components, which in general differ by time derivatives. Thus only in the stationary case does the connection $\nabla(\gamma)$ on the quotient space have a real geometric meaning.

Working in classical coordinate notation, Landau and Lifshitz [7] follow the threading point of view in their text except for their treatment of Maxwell's equations, where it is abandoned in favor of a spacetime splitting based on the reference decomposition. This slight of hand is not made explicit in their presentation and is allowed by the fact that if one uses only classical component notation, then given only a set of spatial components X^a or X_a, one does not have a spacetime field until the time component is specified. By never discussing the time components of their electric and magnetic fields, spacetime fields are not defined but must be inferred from Maxwell's equations.

The "reference spatial fields" of the form

$$S = S^{a\cdots}{}_{b\cdots} e_a \otimes \cdots \otimes \omega^b \otimes \cdots \tag{A.26}$$

may be identified with the time-dependent fields onto which they project on the quotient space of the spacetime by the threading congruence. This identification enables one to pull the time-dependent Riemannian geometry of that quotient back up onto spacetime. Indices on such fields may be shifted using the covariant reference spatial metric $\gamma_{ab}\omega^a \otimes \omega^b = P(m)^\flat$ and the contravariant reference spatial metric $\gamma^{ab} e_a \otimes e_b = \mathcal{P}P(m)^\sharp$; the contraction of these two on adjacent indices yields the reference projection $\mathcal{P}$, the reference spatial identity tensor. The reference spatial covariant derivative corresponds to the Riemannian covariant derivative on the quotient space and yields a reference spatial tensor whose components are

$$\nabla(LL)_c S^{a\cdots}{}_{b\cdots} = S^{a\cdots}{}_{b\cdots,c} + \lambda^a{}_{cd} S^{d\cdots}{}_{b\cdots} + \cdots - \lambda^d{}_{cb} S^{a\cdots}{}_{d\cdots} - \cdots \ . \tag{A.27}$$

This defines the Landau-Lifshitz connection, which they use to define their spatial divergence and curl operations.

The Landau-Lifshitz divergence operator for a reference spatial vector field $Y = Y^a e_a$ is a slight rescaling of the slicing divergence operator

$$\begin{aligned} \mathrm{div}_{LL}\, Y &= \nabla(LL)_a Y^a = \gamma^{-1/2} \partial\!\!\!/_a(\gamma^{1/2} Y^a) \\ &= M^{(4)}\mathrm{div}\ M^{-1} Y = \gamma_L\, \mathrm{div}(n)\, \gamma_L^{-1} Y \\ &= \mathrm{div}_m\, P(m) Y - \gamma^{-1/2} \bar{\bar{M}}\, \pounds(m)_{e_0}(\gamma^{1/2} Y) \ . \end{aligned} \tag{A.28}$$

Similarly one can introduce the Landau-Lifshitz curl operator for a reference spatial 1-form $W = W_a\omega^a$

$$\begin{aligned}\mathrm{curl}_{LL}\, W &= \gamma^{-1/2}\epsilon^{abc}\nabla(LL)_b W_c e_a = \gamma^{-1/2}\epsilon^{abc}[dW]_{bc} e_a \\ &= \mathcal{P}[\mathrm{curl}_m\, W^\sharp - \vec{M}\times_m (\pounds(m)_{e_0} W)^\sharp]^\flat \ .\end{aligned} \tag{A.29}$$

One can also introduce the Landau-Lifshitz spatial gradient of a function

$$\mathrm{grad}_{LL}\, f = \gamma^{ab}(e_a f)e_b = \mathcal{P}[\mathrm{grad}_m\, f - (\pounds_{e_0} f]\vec{M}) \tag{A.30}$$

which is obtained from the Landau-Lifshitz spatial exterior derivative of the function

$$d_{LL} f = (e_a f)\omega^a \tag{A.31}$$

by contraction with $\gamma^{ab} e_a \otimes e_b$.

One can develop a parallel reference decomposition formalism departing from the slicing point of view, as noted by Hanni [59]. One need only use instead $g_{ab}\omega^a \otimes \omega^b$ as the reference spatial metric, leading to a reference covariant derivative $\nabla(H)$ in which the connection components $\Gamma^a(n)_{bc}$ replace $\lambda^a{}_{bc}$ in $\nabla(LL)$. The associated divergence and curl are just rescalings of the same differential operators used in the Landau-Lifshitz reference frame approach.

B The reference frame boost in four-dimensional form

In the case that both n and m are timelike, the boost which relates them may first be decomposed in terms of its action on the orthogonal subspaces relative to the two orthogonal decompositions of the full tangent space and then on each of the spatial subspaces it may be further decomposed in terms of the orthogonal decomposition of that subspace with respect to the relative velocity lying in it. Identifying the boost and its inverse with $\binom{1}{1}$-tensor fields leads to the following representation of the boost $B(m,n)$ taking n to m and of its inverse $B(n,m)$ taking m to n

$$\begin{aligned} B(n,m) &= {\gamma_L}^{-1}\epsilon_0 \otimes \theta^0 + [\gamma_L(\hat{V}^a\hat{V}_b) + (\delta^a{}_b - \hat{V}^a\hat{V}_b)]e_a \otimes \omega^b \ , \\ B(m,n) &= \gamma_L e_0 \otimes \omega^0 + [{\gamma_L}^{-1}(\hat{v}^a\hat{v}_b) + (\delta^a{}_b - \hat{v}^a\hat{v}_b)]\epsilon_a \otimes \theta^b \ , \end{aligned} \tag{B.1}$$

where it is helpful to note the relations

$$\begin{aligned} &V^a = {\gamma_L}^{-1} v^a \ , \qquad V_a = \gamma_L v_a \ , \\ &\hat{V}^a = {\gamma_L}^{-1}\hat{v}^a \ , \qquad \hat{V}_a = \gamma_L \hat{v}_a \ , \\ &\hat{V}^a\hat{V}_b = \hat{v}^a\hat{v}_b \end{aligned} \tag{B.2}$$

satisfied by the relative velocity vectors and the unit relative velocity vectors

$$\hat{v} = \hat{v}^a e_a \equiv v^{-1} v^a e_a \ , \qquad \hat{V} = \hat{V}^a \epsilon_a \equiv v^{-1} V^a \epsilon_a \ . \tag{B.3}$$

Indices on v^a and V^a are raised and lowered with the slicing and threading spatial metrics respectively.

On the other hand it is useful to consider the piece of the Lorentz boost which relates the local rest spaces alone, in order to compare it with the orthogonal projections between those local rest spaces as well as with the reference projection

$$\begin{aligned} &P(n) \llcorner B(n,m) \llcorner P(m) = [\gamma_L(\hat{V}^a\hat{V}_b) + (\delta^a{}_b - \hat{V}^a\hat{V}_b)]e_a \otimes \omega^b \ , \\ &P(m) \llcorner B(m,n) \llcorner P(n) = [{\gamma_L}^{-1}(\hat{v}^a\hat{v}_b) + (\delta^a{}_b - \hat{v}^a\hat{v}_b)]\epsilon_a \otimes \theta^b \ , \\ &\mathcal{P} = [(\hat{v}^a\hat{v}_b) + (\delta^a{}_b - \hat{v}^a\hat{v}_b)]e_a \otimes \omega^b \ . \end{aligned} \tag{B.4}$$

The two terms in parentheses project respectively along and perpendicular to the relative velocity within the local rest space. Let the superscripts "$^{(||)}$" and "$^{(\perp)}$" be used to indicate these individual spatial projections as in equation (8.6). The symbol $\llcorner$ between two tensors indicates the contraction of adjacent indices.

The action of the boost and the natural projection $\mathcal{P}$ on the slicing spatial subspace differs only by the explicit γ_L factor in these formulas. The transformation formula [18] for the magnetic vector field for example,

$$B(n) = \mathcal{P}\gamma_L[B(m) + \vec{V} \times_m E(m)] \tag{B.5}$$

then implies

$$\begin{aligned} B(n)^{(||)} &= B(n,m) \llcorner B(m)^{(||)} \ , \\ B(n)^{(\perp)} &= \gamma_L B(n,m) \llcorner [B(m)^{(\perp)} + \vec{V} \times_m E(m)] \ . \end{aligned} \tag{B.6}$$

The boost is necessary to avoid indices. These formulas translate in an obvious way directly into the usual ones relating the components between two orthonormal spatial frames related by this boost. Similar formulas with the velocity reversed in sign and the fields interchanged hold for the electric field.

For comparison with the gravitoelectric 1-form, consider the transformation of the electric 1-form field, for which the inverse transformation is simpler to discuss with the present choice of spatial frames

$$E(m)^\flat = \mathcal{P}\gamma_L[E(n) + \vec{v} \times_n B(n)]^\flat \ . \tag{B.7}$$

Comparing $\mathcal{P}\gamma_L$ with the boost $P(n)B(n,m)P(m)$ required to transform a slicing spatial 1-form field to the threading local rest space, one finds

$$\begin{aligned} E(m)^{(||)\flat} &= B(n)^{(||)\flat} \llcorner B(n,m) \ , \\ E(m)^{(\perp)\flat} &= \gamma_L[E(n)^{(\perp)} - \vec{v} \times_n B(n)]^\flat \llcorner B(n,m) \ . \end{aligned} \tag{B.8}$$

Now consider the spatial gravitational force fields in the stationary case with a stationary nonlinear reference frame. The reference projection from the local rest space of m to the local rest space of n and its transpose from the dual space back transform the gravitomagnetic vector field and the gravitoelectric 1-form field in the following way

$$\begin{aligned} \tfrac{1}{2}\vec{H}(n) &= \mathcal{P}[\tfrac{1}{2}H(m) + \vec{V} \times_m \vec{g}(m)] \ , \\ \bar{\bar{g}}(m) &= \mathcal{P}{\gamma_L}^2[\vec{g}(n) + \vec{v} \times_n \tfrac{1}{2}\vec{H}(n) + \vec{v} \cdot_n \mathrm{SYM}\, \overleftrightarrow{H}(n)]^\flat \ . \end{aligned} \tag{B.9}$$

Comparison with the corresponding transformations of the magnetic vector field and the electric 1-form field immediately gives the results

$$\begin{aligned} \tfrac{1}{2}\vec{H}(n)^{(||)} &= {\gamma_L}^{-1} B(n,m) \llcorner \tfrac{1}{2}\vec{H}(m)^{(||)} \ , \\ \tfrac{1}{2}\vec{H}(n)^{(\perp)} &= B(n,m) \llcorner [\tfrac{1}{2}\vec{H}(m)^{(\perp)} + \vec{V} \times_m \vec{g}(m)] \ , \end{aligned} \tag{B.10}$$

Notice that half the gravitomagnetic vector field transforms exactly like the magnetic field except for an overall missing gamma factor. A similar comparison for the gravitoelectric 1-form field yields

$$\begin{aligned} \bar{\bar{g}}(m)^{(||)} &= \gamma_L[\vec{g}(n)^{(||)} + (\vec{v} \cdot_n \mathrm{SYM}\, \overleftrightarrow{H}(n))^{(||)}]^\flat \llcorner B(n,m) \ , \\ \bar{\bar{g}}(m)^{(\perp)} &= \gamma_L^2[\vec{g}(n)^{(\perp)} + \vec{v} \times_n \vec{H}(n) + (\vec{v} \cdot_n \mathrm{SYM}\, \overleftrightarrow{H}(n))^{(\perp)}]^\flat \llcorner B(n,m) \ , \end{aligned} \tag{B.11}$$

which is analogous to the transformation of the electric 1-form field except for an overall extra gamma factor and the symmetric part of the gravitomagnetic tensor field.

The inverses of the above transformations are more complicated algebraically to discuss because of the asymmetry between the two spatial frames, the threading frame arising as the threading projection of the slicing spatial frame. The formulas become more complicated since the composed projections come into play

$$\begin{aligned} P(n) \llcorner P(m) &= [{\gamma_L}^2 (\hat{V}^a \hat{V}_b) + (\delta^a{}_b - \hat{V}^a \hat{V}_b)] e_a \otimes \omega^b \ , \\ P(m) \llcorner P(n) &= [(\hat{v}^a \hat{v}_b) + (\delta^a{}_b - \hat{v}^a \hat{v}_b)] \epsilon_a \otimes \theta^b \ , \end{aligned} \tag{B.12}$$

and these are not as simple as the single reference projection.

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Figure Captions

Figure 1. The two dual realizations of "Space" associated with a nonlinear reference frame, illustrated with adapted coordinates. In the slicing point of view, valid when the slicing is spacelike, one has a one-parameter family of imbeddings of the slice manifold Σ into spacetime and the pullback of the spacetime metric defines a time-dependent Riemannian metric on Σ. In the threading point of view, valid when the threading is timelike, one has a projection π from spacetime down onto its quotient space by the flow of the threading tangent vector field e_0, leading to a one-parameter family of maps π_t by restriction to the family of slices, and the pushdown of the contravariant spacetime metric defines a time-dependent contravariant Riemannian metric on that quotient space.

Figure 2. The slicing, reference and threading decompositions of a vector field X suggestively represented by a two-dimensional diagram.

Figure 3. A suggestive two-dimensional representation of the decomposition of the three possible total spatial covariant derivatives associated wtih a parametrized nonlinear reference frame, corresponding respectively to the equivalent normal threading point of view, the slicing point of view

and the threading point of view. These are normalized to correspond to the proper time of the appropriate observers.