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Higher Dimensional Cosmological Models: the View from Above

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Abstract

A broader perspective is suggested for the study of higher dimensional cosmological models.

1 Introduction

In this context one might imagine “the view from above” to refer to the vantage point of the extra dimensions of spacetime somewhere “up there” above us, as in the fiber bundle diagrams one often sees in discussions of higher dimensional theories. However, in using this phrase here I have in mind something quite different.

No matter how many extra dimensions we consider for spacetime, they all live in a 2-dimensional world of paper, blackboards, computer screens, overhead transparencies, . . . In fact an incredible quantity of 2-dimensional space has been filled with discussions of these higher dimensional models. It is often easier to keep grinding out the next small step in this production line, but sometimes one has to step back and take a more global look at the situation, to rise up above this 2-dimensional detail (in our 3-dimensional world) and think about broader questions. How do things fit together? Are there properties which are general for the many special cases considered? What properties of these special cases will continue to have validity in a more general setting?

In these brief remarks I cannot say many things which deserve being said. Being somewhat of an outsider in the field, I must also be careful not to say things which reveal my own ignorance of and lack of familiarity with much of the current work in higher dimensional models. Having recently studied some properties of classical symmetric cosmological models in higher dimensions,¹ this invited talk pushed me to try to make some sense of some of the details scattered in the literature.

Since time has been short, I cannot explain exactly how everything fits together, but I hope to convey a general picture which if pursued may put many of these results into perspective. In so doing I would suggest that perhaps a little more attention should be paid to how specific calculations fit into the larger scheme of things. Certainly it is much more satisfying when many little things fit together as particular cases of a single bigger thing. This idea characterizes present day theoretical physics and the importance it gives to unifying symmetries.

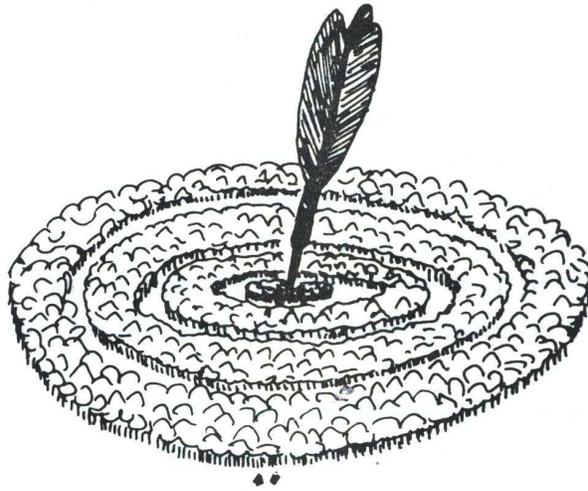
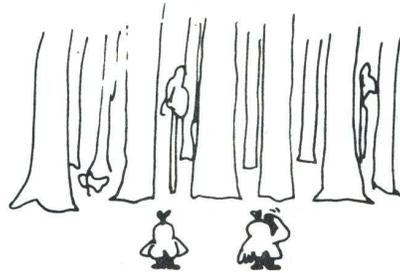
Unfortunately the search for exact solutions of 4-dimensional gravitational theories has more often than not focused on special cases without putting them into perspective. In higher dimensions there is much more room to play games. I think the situation demands some restraint. Some effort should be made toward “looking at the forest rather than the trees.” The game of looking for exact or even qualitative solutions of gravitational field equations all too often finds itself at the tree level when an aerial view of the forest is really what is needed.

Unification of the fundamental forces has led people to adopt the Kaluza-Klein idea²⁻⁵ of formulating theories on higher dimensional spacetimes. At first the extra dimensions were introduced as a mathematical device for obtaining dimensionally reduced unified theories on 4-dimensional spacetime. These additional dimensions, tied to symmetries of the spacetime, allowed the collapse of the theory to an equivalent 4-dimensional one. Eventually ideas progressed to the point where the extra dimensions were taken seriously. The “fiber” symmetry of these fiber bundle formulations of 4-dimensional theories was abandoned and such theories became truly higher dimensional ones.^{6,7}

This immediately created a problem. If the fundamental theory of matter is really higher dimensional in such a nontrivial way, one must explain the present effective 4-dimensional spacetime and the apparently 4-dimensional theory which describes nature. Some mechanism is required to collapse the additional dimensions to a point where they are not observable. This could be accomplished if they are associated with spatially compact dimensions with very small circumferences, but one needs some way for this “compactification” to be a natural consequence of the theory itself.

An interesting possibility for this compactification mechanism is the dynamical behavior of the gravitational field itself within these theories. The present 4-dimensional phase of the universe is very anisotropic, given the necessarily large difference in length scales associated with the “ordinary” and “extra” dimensions. Chodos and Detweiler⁸ first suggested this anisotropy might be a consequence of the evolution of the gravitational field. Examining purely gravitational 5-dimensional theories with flat homogeneous space sections charac-

the trees



the forest

Figure 1: "...not seeing the forest for the trees..." [English language saying]

terized entirely by “free” anisotropic expansion,⁹ they noted that classical cosmological solutions are *allowed* in which preferential expansion of three spatial directions occurs. However, such solutions are not *selected* by the theory. One needs matter fields to obtain such a selection effect.¹⁰ Such fields may be added in ordinary Kaluza-Klein theories, while supersymmetry dictates what matter fields exist in locally supersymmetric theories (supergravity, superstrings).^{11–19}

With this in mind classical solutions of the Bose sector of higher dimensional theories have been studied to throw some light on the nature of the ground state of such theories. Such studies fall into two categories, characterized by the behavior of the gravitational field. In the first, solutions are considered in which the spacetime is the product of two static symmetric component manifolds representing the ordinary spacetime and the internal spaces, the discussion being more particle physics in flavor (e.g., reference 13). In the second, evolving models are considered, typically from the viewpoint of a relativist somewhat less concerned with the particle physics motivation for considering such models.

As a classical relativist I wish to address the second kind of investigation in the spirit of Freund,¹⁰ Demaret et al^{20,21} and others.^{22–27} Many particular exact or qualitative cosmological solutions of various higher dimensional theories have been discussed. Although only for very special conditions can one obtain closed form solutions, such solutions may describe a single isolated dynamical effect of the full equations which when accompanied by other degrees of freedom no longer allows exact solutions but perhaps exhibits a similar qualitative behavior, or at least one which can be understood in the context of the coupled behavior. Properties exhibited by special solutions are probably only interesting to the extent that they relate to the general case in this way.

Certain exact solutions in fact exhibit a “typical behavior” during certain phases of the evolution. Such solutions can be understood in terms of additional properties related to a qualitative formulation of the field equations. From a purely mathematical viewpoint, the existence of closed form exact solutions often is accompanied by underlying properties of the differential equations that are rarely understood in first obtaining them and which help to relate various particular solutions to each other. Understanding how exact solutions fit into a bigger picture is perhaps more important than any particular exact solution found by chance or “cleverness”.

These ideas are also useful in understanding the idea of “gravitational chaos” both in the general cosmological context and in the context of the homogeneous models.^{28–31,23–26} This question is studied as a purely gravitational phenomenon relevant to the classical initial singularity independent of particle theory questions. Although quantum effects at the Planck length probably make the question irrelevant, it is an interesting mathematical problem.

Limitations of time and space permit only a hint of a more global point of view illustrated by a few key examples. Hopefully such a discussion, necessarily abbreviated, can still be valuable in establishing the importance of such a point of view for this branch of cosmology.

2 Spatially Homogeneous Models

The kinds of cosmological solutions considered in higher dimensional theories in connection with possible properties of the “ground state” all share one feature: they are spatially homogeneous.³² The d -dimensional spacetime contains a 1-parameter family of $(d - 1)$ -dimensional spacelike hypersurfaces (let $D = d - 1$) which are orbits of an isometry group G . The space sections are therefore “homogeneous spaces” diffeomorphic to the coset space G/H , where H is the isotropy group at some point of spacetime, namely the subgroup of isometries which leave that point fixed.

When H is the identity subgroup or a discrete invariant (i.e., normal) subgroup, “space” is isometric to a group manifold (namely the group G/H) with a Riemannian metric invariant under either the left or right translations of the group into itself, the choice being a matter of convention (left in this article). If H is discrete but not invariant, this is still true locally. All of these cases may be referred to as the simply transitive case, where G acts as a D -dimensional translation group of a D -dimensional Riemannian manifold.

In the multiply transitive case, H is continuous and G acts as a group of translations and certain rotations of the space sections, each of which is isometric to the left coset space G/H with a Riemannian metric invariant under the natural left action of G . If H is an invariant subgroup, G/H is a group and hence this occurs as a special case of the simply transitive case for which additional symmetry exists; the case where H is not invariant will be referred to as nontrivial.

In either case the metric on a given spatial section is completely determined by its value at a single point and hence the field equations reduce to ordinary differential equations in the parameter labeling the spatial slices, easily interpreted as a time function on the spacetime. (This assumes that the matter fields either share the symmetry of the metric or at least have enough symmetry to be compatible with the field equations.) By choosing the coordinate lines orthogonal to the spatial sections, the spacetime metric can be expressed at least locally in the form³³

$${}^{(d)}g = -N(t)^2 dt \otimes dt + g_{ab}(t)\omega^a \otimes \omega^b \quad (2.1)$$

on the manifold $R \times (G/H)$, where $\{\omega^a\}$ are 1-forms on G/H dual to the frame $\{e_a\}_{a=1,\dots,D}$ characterized by structure functions $C^a{}_{bc} = \omega^a([e_b, e_c])$ on G/H . These are constants in the simply transitive case where $\{e_a\}$ is a left invariant frame (i.e., a basis of the Lie algebra \mathfrak{g} of G), but functions on G/H in the nontrivial multiply transitive case. In the first case the matrix $\mathbf{g} = (g_{ab})$ is an arbitrary positive definite symmetric matrix, while in the second case it must satisfy additional linear constraints.^{18,19} Similar constraints in the first case may lead to additional spatial symmetries.

The choice of lapse function $N(t)$ determines the parametrization of the family of homogeneous spatial sections. The proper time τ , defined by $d\tau = N(t)dt = \omega^\perp$ corresponds to unit lapse. In expanding models with an initial singularity, τ is usually chosen so that this singularity occurs at $\tau = 0$. Defining

BIANCHI TYPES	nonsemisimple		semisimple	
	noncompact			compact
	abelian	nonabelian		
class <i>A</i> (unimodular)	I	II, VI ₀ , VII ₀	VIII	IX
class <i>B</i> (nonunimodular)		III, IV, V, VI _h , VII _h		

Table 1: The isomorphism classes of 3-dimensional Lie groups/algebras. The compact/noncompact division refers to the simply connected covering groups of each type, while unimodularity refers to the matrix representation of the adjoint group of each type.

$\omega^0 = dt$ leads to the spacetime 1-forms ω^α dual to the frame $\{e_\alpha\}_{\alpha=0,1,\dots,D}$, where $e_0 = \partial/\partial t$.

In practice one is usually interested in spatial geometries which decompose into an orthogonal product of an ordinary 3-dimensional homogeneous space and a compact internal homogeneous space

$$G/H \sim M_3 \times (G^{int}/H^{int}) . \quad (2.2)$$

M_3 is a Bianchi manifold³⁴ in the simply transitive (group manifold) case and the Kantowski-Sachs manifold³⁵ in the nontrivial multiply transitive case, where M_3 is of the form G/H with $G \sim G_1 \times SO(3, R)$, G_1 the abelian group R or S_1 , and $H \sim SO(2, R)$. (One might refer to M_3 as a BKS space.) A classification of symmetry types in the Bianchi case amounts to a classification of 3-dimensional Lie groups/algebras;^{34,36} such a classification is summarized in Table 1.

3 Field Equations

Since we wish to focus on the gravitational properties of higher dimensional models, we consider the Einstein equations with all other fields lumped into the energy-momentum tensor. This assumes that the gravitational part of the classical field equations can be represented in this way, thus excluding “higher derivative” theories. These equations

$$0 = M_{\alpha\beta} \equiv |^{(d)}g|^{1/2} ({}^{(d)}G_{\alpha\beta} - T_{\alpha\beta}) , \quad |^{(d)}g|^{1/2} = Ng^{1/2} \quad (3.1)$$

(where metric quantities without the leading superscript ${}^{(d)}$, refer to the spatial metric) naturally split into evolution equations $0 = M_{ab}$ which evolve the spatial metric and constraints on the solutions of those equations

$$\begin{aligned} 0 = \mathcal{H} &\equiv 2N^{-1}M^\perp{}_\perp \equiv 2M^0{}_\perp && \text{(super-Hamiltonian constraint)} \\ 0 = \mathcal{H}_a &\equiv 2N^{-1}M^\perp{}_a \equiv 2M^0{}_a && \text{(supermomentum constraint)} . \end{aligned} \quad (3.2)$$

The Einstein equations may also be written in “Ricci form”

$$0 = P_{\alpha\beta} \equiv {}^{(d)}R_{\alpha\beta} - E_{\alpha\beta} , \quad E_{\alpha\beta} \equiv T_{\alpha\beta} - (d-2)^{-1}T^\gamma{}_\gamma g_{\alpha\beta} , \quad (3.3)$$

a form very convenient for static solutions in discussions of spontaneous compactification. However, for dynamical solutions a compromise is preferable, namely the Ricci evolution equations $0 = P_{ab}$, supplemented by the Einstein constraints rather than more equations involving second time derivatives.

To understand the usefulness of this compromise for spatially homogeneous solutions, it is convenient to introduce the Hamiltonian formulation. The Einstein evolution equations written in first order form are the Hamiltonian equations for the spatial metric derived from the Hamiltonian $H = N\mathcal{H}$ with the lapse treated as an independent variable; here π^{ab} is the momentum canonically conjugate to the spatial metric g_{ab}

$$\begin{aligned} 0 &= \dot{\pi}^{ab} + \partial/\partial g_{ab}(N\mathcal{H}) - NQ^{ab} \\ &= \dot{\pi}^{ab} + N\partial H/\partial g_{ab} - NQ^{ab} + \partial/\partial g_{ab}(\ln N) = M^{ab} + H\partial/\partial g_{ab}(\ln N) . \end{aligned} \tag{3.4}$$

(The term Q^{ab} arises in the case of nonunimodular symmetry groups where the spatial Einstein tensor force is no longer derivable from the scalar curvature potential but has a nonpotential component or from similar problems with the source energy-momentum.^{37,38}) If N is treated as independent of g_{ab} , the final term vanishes and one obtains the Einstein evolution equations. Letting N depend explicitly on g_{ab} leads to evolution equations which differ by a term proportional to the super-Hamiltonian constraint. The choice of unit lapse $N = 1$ made in proper time gauge leads to the Einstein evolution equations, while the choice $N = (\text{const})g^{1/2}$ made in Taub's time gauge³² (Misner supertime³⁹ time gauge) leads to a set of equations which are equivalent to the Ricci evolution equations.

Why is the Taub time gauge useful, i.e., why is it useful to choose the lapse proportional to the spatial volume factor? For the answer, one must examine the differential geometry of the Einstein equations in Hamiltonian form. The metric configuration space is the space \mathcal{M} of positive definite symmetric matrices $\mathbf{g} = (g_{ab})$ or a submanifold of this space. The Hamiltonian

$$H = N\mathcal{H} = NT + NU \tag{3.5}$$

consists of a kinetic term associated with the metric variables and a potential energy term composed of a spatial curvature term and a source term

$$\begin{aligned} U &= U^{curv} + U^{source} , & -dU + Q^{ab}dg_{ab} &= -g^{1/2}(G^{ab} - T^{ab})dg_{ab} , \\ U^{curv} &= -g^{1/2}R , & -dU^{curv} + Q^{curv\ ab}dg_{ab} &= -g^{1/2}G^{ab}dg_{ab} , \\ U^{source} &= -2g^{1/2}T^\perp_\perp , & -dU^{source} + Q^{source\ ab}dg_{ab} &= g^{1/2}T^{ab}dg_{ab} . \end{aligned} \tag{3.6}$$

The exterior derivative d is the one on \mathcal{M} , while both the source and the curvature forces may involve nonpotential components. (One may also simply treat

the source force entirely as an external force.) The kinetic energy term

$$\begin{aligned}
NT &= \frac{1}{4}N^{-1}\mathcal{G}^{abcd}\dot{g}_{ab}\dot{g}_{cd} && \text{(velocity phase space) ,} \\
&= N\mathcal{G}_{abcd}^{-1}\pi^{ab}\pi^{cd} && \text{(momentum phase space) ,} \\
\pi^{ab} &\equiv \partial/\partial\dot{g}_{ab}(NT) && \text{(Legendre transformation) ,}
\end{aligned} \tag{3.7}$$

is just the one associated with the DeWitt⁴⁰ metric \mathcal{G} on \mathcal{M} , apart from the rescaling by the lapse function

$$N^{-1}\mathcal{G} = N^{-1}g^{1/2}(g^{a(c}g^{d)b} - g^{ab}g^{cd})dg_{ab} \otimes dg_{cd} . \tag{3.8}$$

Allowing the lapse to have a factor depending explicitly on g_{ab} is equivalent to conformally rescaling the DeWitt metric by that factor.³⁹ In particular the Taub time gauge choice eliminates the spatial volume factor from the DeWitt metric, thus simplifying the kinetic terms in the evolution equations as much as possible and also eliminating a term involving the total kinetic energy which appears in the Einstein form of those equations due to the derivative of the spatial volume factor. This explains the advantage of using the Ricci evolution equations accompanied by the Taub time gauge.

Neglecting the potential energy, equivalent to restricting one's attention to the vacuum abelian case where the spatial curvature vanishes, the evolution equations describe geodesics of the DeWitt metric \mathcal{G} in proper time gauge and geodesics of the rescaled DeWitt metric $g^{-1/2}\mathcal{G}$ in Taub time gauge. The Hamiltonian constraint in this case requires that the kinetic energy vanish and therefore be a null vector. (In fact the null geodesics are conformally invariant.)

This immediately raises the question of the signature of the DeWitt metric. It is a Lorentz metric which endows \mathcal{M} with a natural splitting into space and time which itself plays an important role in the geometry of the Einstein equations. Let

$$\mathbf{g} = e^{2\alpha}\tilde{\mathbf{g}} , \quad g = \det \mathbf{g} = e^{2D\alpha} , \quad \det \tilde{\mathbf{g}} = 1 \tag{3.9}$$

decompose \mathbf{g} into the ‘‘scale invariant’’ conformal metric $\tilde{\mathbf{g}}$ and the conformal scale factor e^α . Dynamically in the spatially homogeneous models e^α describes the overall expansion or contraction of the spatial sections while $\tilde{\mathbf{g}}$ describes the anisotropy. The DeWitt metric in these new variables is explicitly Lorentzian with α playing the role of a natural time variable

$$g^{-1/2}\mathcal{G} = -D(D-1)d\alpha \otimes d\alpha + \text{Tr}(\tilde{\mathbf{g}}^{-1}d\tilde{\mathbf{g}} \otimes \tilde{\mathbf{g}}^{-1}d\tilde{\mathbf{g}}) . \tag{3.10}$$

The second term is Riemannian since at $\tilde{\mathbf{g}} = \mathbf{1}$ it leads to the trace of the square of a symmetric matrix and such an inner product is positive definite on the space of symmetric matrices.

The diagonal submanifold $\mathcal{M}_D \subset \mathcal{M}$ is in fact a flat Lorentz spacetime with respect to the rescaled DeWitt metric. At diagonal points $\tilde{\mathbf{g}}_D$ one may define $\boldsymbol{\beta} \equiv \frac{1}{2} \ln \tilde{\mathbf{g}}_D$ and introduce a basis $\{\mathbf{e}_A\}_{A=1,\dots,D-1}$ of tracefree diagonal matrices normalized by the condition $\text{Tr} \mathbf{e}_A \mathbf{e}_B = D(D-1)\delta_{AB}$, and let $\boldsymbol{\beta} = \beta^A \mathbf{e}_A$, in

which case $\{\alpha, \beta^A\}$ are orthonormal cartesian coordinates with respect to the suitably rescaled DeWitt metric

$$[D(D-1)g^{1/2}]^{-1}\mathcal{G}|_{\mathcal{M}_D} = -d\alpha \otimes d\alpha + \delta_{AB}d\beta^A \otimes d\beta^B . \quad (3.11)$$

The vacuum abelian diagonal dynamics in Taub time gauge has as solutions straight null lines in α - $\boldsymbol{\beta}$ space (unit α -speed trajectories in $\boldsymbol{\beta}$ space) with these orthonormal coordinates linear in the Taub time t . (It is convenient to choose $N = 2D(D-1)g^{1/2}$ to obtain the usual factor of one half in the kinetic energy.) The diagonal components of \mathbf{g} are therefore exponential in the Taub time. Transforming to proper time gauge leads to power law dependence on the proper time τ

$$\mathbf{g}_D = e^{\mathbf{A}t + \mathbf{B}} = \mathbf{g}_{D(0)} \text{diag}(\tau^{2p_1}, \dots, \tau^{2p_D}) , \quad (3.12)$$

the latter form recognizable as the famous Kasner solution, the Kasner exponents satisfying the well known identities

$$\sum_{a=1}^D p_a = \sum_{a=1}^D p_a^2 = 1 . \quad (3.13)$$

The Ricci evolution equations in Taub time gauge for diagonal metrics are explicitly

$$(\ln \mathbf{g})'' = -[2D(D-1)]^2 \cdot 2g(\mathbf{R} - \mathbf{E}) \quad (3.14)$$

where $\mathbf{R} = (R^a_b)$ and $\mathbf{E} = (E^a_b)$. When the righthand side is absent, one has free motion with the metric exponential in the Taub time. This remains true for those degrees of freedom along which the force term on the righthand side has no component.

As an aside, it is worth mentioning how spatial curvature and nonzero source energy density change the causal character of the solution curves. The super-Hamiltonian constraint $\mathcal{T} = -U$ determines the square of the velocity (or momentum) of the system. In the case of positive source energy density, U may be negative and the motion spacelike only for sufficiently positive spatial curvature. In a contracting phase this can lead to a ‘‘bounce’’ rather than a cosmological singularity, while in an expanding phase it can lead to a point of maximum expansion and recollapse. Such ‘‘turnaround phases’’ are not allowed when the spatial curvature is negative and the source energy density is positive, where U is positive and the motion always timelike, with g or equivalently α monotonically increasing (expansion) or decreasing (contraction), respectively starting or ending at the ‘‘frontier’’ $g = 0$ ($\alpha \rightarrow -\infty$), where the spatial metric is singular.⁴⁰ The intersection of a trajectory with the frontier may or may not represent a physical cosmological singularity. The variable $\alpha \in (-\infty, \infty)$ is a natural time variable for \mathcal{M} ; during a phase of monotonic expansion or contraction, one can choose it as the time variable t on the underlying spacetime, a gauge introduced by Misner⁴¹ (α -time gauge).

4 Exact Solutions

A useful example to illustrate some of these ideas and others are the Taub solutions³² in $d = 4$ spacetime dimensions. This class of metrics is defined by

$$\begin{aligned} C^1_{23} = C^2_{31} = n^{(1)} , \quad C^3_{12} = n^{(3)} , \quad \mathbf{g}_D = e^{2\alpha} e^{2(\beta^+ \mathbf{e}_+ + \beta^- \mathbf{e}_-)} , \\ \mathbf{e}_+ = \text{diag}(1, 1, -2) , \quad \mathbf{e}_- = \sqrt{3} \text{diag}(1, -1, 0) , \quad \beta^- = \delta(n^{(1)})\beta_0^- , \end{aligned} \quad (4.1)$$

and their dynamics in Taub time gauge is described by the Hamiltonian (expressed in velocity phase space)

$$H = \frac{1}{2}(-\dot{\alpha}^2 + \dot{\beta}^{+2} + \dot{\beta}^{-2}) + 6n^{(3)2} e^{4(\alpha - 2\beta^+)} - 24n^{(3)}n^{(1)} e^{-2(\beta^+ - 2\alpha)} . \quad (4.2)$$

The Lorentz transformation³⁹

$$(\bar{\alpha}, \bar{\beta}^+) = 3^{-1/2}(2\alpha - \beta^+, -\alpha + 2\beta^+) \quad (4.3)$$

leads to a completely decoupled Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}(-\dot{\bar{\alpha}}^2 + \dot{\bar{\beta}}^{+2} + \dot{\beta}^{-2}) + 6n^{(3)2} e^{-4\sqrt{3}\bar{\beta}^+} - 24n^{(3)}n^{(1)} e^{2\sqrt{3}\bar{\alpha}} \\ &= -H_{\bar{\alpha}} + H_{\bar{\beta}^+} + H_{\beta^-} = 0 \end{aligned} \quad (4.4)$$

consisting of three 1-dimensional scattering problems with exponential potentials and constant energies, restricted only by the constraint that the appropriately signed sum of the individual energies vanish. The solution of an exponential scattering problem leads to hyperbolic or trigonometric functions

$$\begin{aligned} H_x = \frac{1}{2}(\dot{x}^2 + \mu e^{\nu x}) = E_x \quad \rightarrow \quad t = \int dx (2E_x - \mu e^{\nu x})^{-1/2} , \\ \text{or defining } \gamma = (2E_x)^{1/2} , \quad (4.5) \\ e^{-\frac{1}{2}\nu x} = \begin{cases} \mu^{1/2} \gamma^{-1} \cosh(\frac{1}{2}\gamma \nu t) & \mu > 0 , \quad E_x > 0 \\ |\mu|^{1/2} \gamma^{-1} \sinh(\frac{1}{2}\gamma |\nu| t) & \mu < 0 , \quad E_x \in \mathfrak{R} . \end{cases} \end{aligned}$$

$(\bar{\alpha}, \bar{\beta}^+, \beta^-)$ are inertial coordinates of the rest frame of the $n^{(3)2}$ potential, which moves with speed $d\beta^+/d\alpha = \frac{1}{2}$ in $\boldsymbol{\beta}$ space. For the Bianchi type IX case, characterized by $n^{(3)}n^{(1)} > 0$, the hyperbolic cosine solution is relevant, interpolating between the asymptotic free positive and negative exponential solutions (for the metric components) at $t = \pm\infty$; the unit velocities in $\boldsymbol{\beta}$ space of the asymptotic solutions are related by a simple reflection in the rest frame of this potential. Letting $n^{(1)} \rightarrow 0$ contracts the group to Bianchi type II, eliminating the ‘‘tachyonic’’ $n^{(3)}n^{(1)}$ potential (it moves with speed $d\beta^+/d\alpha = 2$ in $\boldsymbol{\beta}$ space) and allows free motion parallel to the $n^{(3)2}$ potential (the $\bar{\alpha}$ and $\bar{\beta}^-$ directions). For the Bianchi type VIII case, characterized by $n^{(3)}n^{(1)} < 0$, only the positive energy solutions are relevant for the $\bar{\alpha}$ motion (the potential is negative) due to the super-Hamiltonian constraint.

evolution equation (set =0)	spatial curvature	2nd time derivative	quadratic decoupling	quadratic elimination	linear choice
${}^4R_A + {}^4G^\perp_\perp$	$n^{(3)2}$	A	$n = -1$	$m = 1$	(1, -1)
4R_B	$n^{(3)2}$	B	$m = 2$	$n = 1$	(2, 1)
${}^4R_B - 2({}^4G^\perp_\perp)$	$n^{(3)}n^{(1)}$	B	$m = 2$	$n = 1$	(2, 1)
${}^4R_B + {}^4R_A$	$n^{(3)}n^{(1)}$	AB	$m = n + 1$	$n = 1$	(2, 1)

Table 2:

independent variables	lapse choice	constant of motion
(B, AB)	(2, 1): Taub	$A^2 B ({}^4G^\perp_\perp)$
(A, B)	(2, -1): Brill	$A ({}^4G^\perp_\perp)$
(A, AB)	(0, -1): Misner	$A^2 ({}^4G^\perp_\perp)$

Table 3:

The same problem may be viewed directly from the Einstein equations.⁴² Defining $N^2 = A^m B^n$, $\mathbf{g}_D = \text{diag}(A, A, B)$, ${}^4R^1_1 = {}^4R^2_2 = {}^4R_A$, and ${}^4R^3_3 = {}^4R_B$ for the locally rotationally symmetric case, one has

$$\begin{aligned}
2N^2 ({}^4G^\perp_\perp) &= -(\ln A) \cdot (\ln B) \cdot - \frac{1}{2}(\ln A) \cdot^2 + N^2 A^{-2} (\frac{1}{2}n^{(3)2} B - 2n^{(3)}n^{(1)} A) , \\
2N^2 ({}^4R_A) &= (\ln A) \cdot - (\ln A) \cdot (\ln N A^{-1} B^{-1/2}) \cdot + N^2 A^{-2} (-n^{(3)2} B + 2n^{(3)}n^{(1)} A) , \\
2N^2 ({}^4R_B) &= (\ln B) \cdot - (\ln B) \cdot (\ln N A^{-1} B^{-1/2}) \cdot + N^2 A^{-2} (n^{(3)2} B) .
\end{aligned} \tag{4.6}$$

Different choices of (m, n) lead to decoupling of different products of A and B . Taub's³² choice (2, 1) decouples $(AB)^{-1}$ and B , Brill's⁴³ choice (2, -1) decouples $A^{-1/2}$ and B , while Misner's⁴⁴ choice (0, -1) decouples AB and A , the special exponents yielding the simplest form for the solutions. However, the decoupling of the evolution equations requires adding appropriate multiples of the super-Hamiltonian constraint to the Einstein evolution equations. Table 2 summarizes the decoupling possibilities for the evolution equations for the semisimple case.

The first column of Table 2 lists those linear combinations of the three independent spacetime curvatures 4R_A , 4R_B , and ${}^4G^\perp_\perp$ which contain only one of the two independent spatial curvatures $R^1_1 + R^3_3$ ($\sim n^{(3)}n^{(1)}$) and R^3_3 ($\sim n^{(3)2}$). The second column indicates which spatial curvature appears while the third column indicates the variable whose natural log appears in the second derivative term. The fourth column gives the condition that the quadratic derivative term only involve this variable and the fifth column gives the condition that the quadratic term be absent. The last column gives the values of (m, n) for satisfying both conditions.

Table 3 describes the choice of variables for which both evolution equations decouple in the semisimple case; the form of the constant of the motion may be derived from the evolution equations. In the nonsemisimple case of Bianchi

type II ($n^{(1)} = 0, n^{(3)} \neq 0$), the vanishing of the $n^{(3)}n^{(1)}$ curvature term leads to the additional lapse choice $(1, -1)$ of Bradley and Sviestens^{45,46} for which both evolution equations decouple. When $n^{(1)} = n^{(3)} = 0$, one obtains the abelian case of Bianchi type I and the solutions are equivalent to the Kasner solution expressed with different time functions.

Why give so much attention to a vacuum solution of $d = 4$ general relativity in an article devoted to higher dimensional cosmological models? The answer is that the Taub family of solutions are not content to remain in four spacetime dimensions but keep reappearing in higher dimensional models.

For solutions of the Ricci evolution equations, the Taub time gauge Hamiltonian $H \sim g^{(4)}G^{\perp}_{\perp}$ is a constant function of the parameters of those solutions. For negative values of H , one must add something positive to H to obtain zero. This corresponds to a source with $E_{ab} = 0$ ($= E^{\perp}_a$) but $gT^{\perp}_{\perp} = H_{source} > 0$. Such a source is a homogeneous stiff perfect fluid moving orthogonally to the space sections, which in turn is equivalent to a homogeneous massless scalar field. The latter is equivalent by conformal transformations to a solution of the Brans-Dicke field equations.⁹ In other words, by “varying the parameters”⁴² which occur in the solution of the Ricci evolution equations away from the vacuum super-Hamiltonian constraint values, one obtains new solutions which may be interpreted as containing either or both of these equivalent sources or as a solution of the Brans-Dicke theory. (Such a variation of parameters applied to other choices of evolution equations leads to the introduction of a locally rotationally symmetric electromagnetic field source and the Brill generalization of the Taub solutions,⁴³ to which stiff perfect fluids or scalar fields may be added,⁴² or a cosmological constant.⁴⁷)

These are still $d = 4$ solutions. However, the massless scalar field source or Brans-Dicke theory is equivalent to a $d = 5$ Kaluza-Klein theory with an additional flat dimension.⁹ More scalar fields lead to higher dimensional Kaluza-Klein models, always with flat extra dimensions.

In these examples, essentially only gravitational degrees of freedom have been considered. As an example of a model with nongravitational degrees of freedom, consider $d = 11$ supergravity with the Freund-Rubin ansatz.^{10,11} Although already out of fashion, it is a cute example of another appearance of the Taub solutions. Spatially homogeneous solutions of the bosonic sector of the theory have been considered by Freund,¹⁰ Demaret et al^{20,21} and Lorentz-Petzold.²² The spacetime is assumed to be of the form $M_4 \times M_7$, where $M_4 \sim R \times M_3$ and the spatial metric is decomposable, i.e., equivalent to independent metrics on the factor manifolds M_3 and M_7 which are orthogonal in the product. The spatial metric matrix \mathbf{g} is then in block diagonal form with matrix blocks \mathbf{g}_3 and \mathbf{g}_7 , each of which may be decomposed into scale invariant and scale factor parts

$$\mathbf{g}_3 \leftrightarrow (g_3, \tilde{\mathbf{g}}_3), \quad \mathbf{g}_7 \leftrightarrow (g_7, \tilde{\mathbf{g}}_7). \quad (4.7)$$

The Freund-Rubin ansatz which solves the field equations for the 3-form A of the theory assumes that the 4-form $F = dA$ has as its only nonvanishing

components (let $\bar{a}, \bar{b} = 1, 2, 3$)

$$F^{0\bar{a}\bar{b}\bar{c}} = f_D g_7^{-1/2} {}^{(4)}\eta^{0\bar{a}\bar{b}\bar{c}}, \quad {}^{(4)}\eta^{0\bar{a}\bar{b}\bar{c}} = -(N g_3^{1/2})^{-1/2} \epsilon^{0\bar{a}\bar{b}\bar{c}}, \quad \epsilon^{0123} = 1, \quad (4.8)$$

so that

$$F_{0\bar{a}\bar{b}\bar{c}} F^{0\bar{a}\bar{b}\bar{c}} = -4! f_D^2 g_7^{-1} = \frac{1}{4} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta}, \quad (4.9)$$

where f_D is a constant (as in Demaret et al²⁰).

This leaves an anisotropic source for the Einstein equations. The energy-momentum tensor or its Ricci equivalent form

$$\begin{aligned} T_{\alpha\beta} &= \frac{1}{4} 8(8F^2_{\alpha\beta} - g_{\alpha\beta} F^2), & E_{\alpha\beta} &= \frac{1}{6}(F^2_{\alpha\beta} - \frac{1}{1} 2g_{\alpha\beta} F^2), \\ F^2_{\alpha\beta} &\equiv F_{\alpha\gamma\delta\epsilon} F_{\beta}{}^{\gamma\delta\epsilon}, & F^2 &\equiv F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta}, \end{aligned} \quad (4.10)$$

reduces to

$$T^0_0 = -2f_D^2 g_7^{-1}, \quad T^{\bar{a}}_{\bar{b}} = -2f_D^2 g_7^{-1} \delta^{\bar{a}}_{\bar{b}}, \quad T^{\bar{a}}_{\bar{b}} = 2f_D^2 g_7^{-1} \delta^{\bar{a}}_{\bar{b}}, \quad (4.11)$$

while the source super-Hamiltonian is

$$U^{source} = 4f_D^2 (g_3/g_7)^{1/2}, \quad -dU^{source} = g^{1/2} T^{ab} dg_{ab}. \quad (4.12)$$

Note that by replacing F by its expression in terms of the metric leads to an effective potential which does not necessarily yield the correct equivalent energy-momentum upon variation. For the Freund-Rubin source no problem arises, but for a Freund-Rubin-Englert¹² source, one must introduce a nonpotential force to compensate.

In Taub time gauge the source potential is

$$NU^{source} = (2 \cdot 10 \cdot 9) g^{1/2} U^{source} = (2 \cdot 10 \cdot 9) \cdot 4f_D^2 g_3. \quad (4.13)$$

Thus only g_3 is affected by the source, the variables orthogonal to g_3 continuing to satisfy the vacuum equations. To discuss the evolution of the metric one must make assumptions about the homogeneity groups of M_3 and M_7 . If the spatial curvature on M_3 is such that the equation for g_3 is linear, i.e., the Taub time gauge potential is linear in g_3 , then adding the source only changes the coefficient, i.e., “varies the parameters” in the evolution equations, which have the same solutions but a different dependence on the parameters.

In the simplest case M_3 and M_7 are both flat and the vacuum dynamics is just “free motion.” With the Freund-Rubin source, the variables orthogonal to g_3 remain free and the single degree of freedom g_3 decouples from them leading to the 1-dimensional exponential scattering problem (in $\ln g_3$); the solution is the Bianchi type II Taub solution involving one nontrivial curvature term, apart from a variation of parameters. The variables g_3 and g_7 are not orthogonal, however. One must introduce

$$\begin{aligned} \mathbf{g} &= e^{2(\beta^3 \mathbf{e}_3 + \beta^{3\perp} \mathbf{e}_{3\perp})} \begin{pmatrix} \tilde{\mathbf{g}}_3 & 0 \\ 0 & \tilde{\mathbf{g}}_7 \end{pmatrix}, & \mathbf{e}_3 &= \text{diag}(1, 1, 1, -\frac{1}{2}, \dots, -\frac{1}{2}) \\ & & \mathbf{e}_{3\perp} &= \text{diag}(0, 0, 0, 1, \dots, 1), \\ g_3 &= e^{3\beta^3}, & g_7 &= e^{7(-\frac{1}{2}\beta^3 + \beta^{3\perp})}. \end{aligned} \quad (4.14)$$

The β^3 degree of freedom decouples from $\beta^{3\perp}$ which together with the remaining anisotropy variables remains free. The picture in the β^3 - $\beta^{3\perp}$ plane is exactly the Bianchi type II Taub solution with a variation of parameters. In this plane, $\ln g_7$ has slope $\frac{1}{2}$; one sees that free motion in $\beta^{3\perp}$ (parallel to the vertical potential contours) leads to monotonic expansion or contraction in g_7 . The initial asymptotic free state scatters off the static potential into a final free state with a velocity reflected from the potential contour direction, leading to an expansion from zero to a maximum and then recollapse in g_3 . Although this model²⁰ is not realistic, it does exhibit phases of preferential expansion or contraction of the ordinary and extra dimensions. (The solutions of the Ricci evolution equations for \mathbf{g}_3 in Taub time gauge are in fact the same as the $d = 4$ spatially flat case with a positive cosmological constant, but the proper time is a different function of the Taub time in the two cases due to the additional dimensions.)

If instead one gives M_3 isotropic spatial curvature by letting its metric be the diagonal Bianchi type V metric, some special solutions may be obtained from the $d = 4$ vacuum solution (found by Joseph⁴⁸) by a variation of parameters. This is easily seen by considering this solution in Taub time gauge $N^2 \sim g_3$ instead of the usual time gauge $N^2 = g_{33}$ in which that solution is usually presented (discussed for spatially homogeneous models by Siklos⁴⁹). The free motion in the latter time gauge is represented by hyperbolic tangents instead of exponentials as occur in Taub time gauge, while the single curvature driven variable is described by hyperbolic sines in both gauges for this case. In Taub time gauge one sees that the Joseph solution is again the Taub solution with one curvature term (Bianchi type II) in disguised form.

For a diagonal Bianchi type V metric (with $C_{\bar{b}\bar{c}}^{\bar{a}} = a\delta_{\bar{b}\bar{c}}^{\bar{a}}$) one has

$$\mathbf{g}_3 = e^{2\beta^0} e^{2(\beta^+ \mathbf{e}_+ + \beta^- \mathbf{e}_-)} , \quad \mathbf{e}_+ = \text{diag}(1, 1, -2) , \quad \mathbf{e}_- = \sqrt{3} \text{diag}(1, -1, 0) , \quad (4.15)$$

but the supermomentum constraint suppresses the β^+ degree of freedom (one may set $\beta^+ = 0$), resulting in a Hamiltonian system for the remaining two variables with $g_{33} = g_3^{1/3}$. In Taub time gauge the curvature potential is

$$NU^{curv} = (2 \cdot 10 \cdot 9) g_7 g_3 g^{33} a^2 , = (2 \cdot 10 \cdot 9) g_7 g_3^{2/3} a^2 . \quad (4.16)$$

If one imposes that g_7 be proportional to $g_3^{1/3}$, this potential becomes proportional to g_3 and simply adds on to the Freund-Rubin potential. The only difference from the vacuum case is that the coefficient of g_3 in this potential is different; the same general solution is obtained with a variation of parameters, i.e., the Taub solution occurs again. Apparently from the work of Demaret et al²⁰ and Lorentz-Petzold²² this additional condition is compatible with the dynamics. If one could consistently impose the constraint that this potential depend on $\beta^{3\perp}$ only, then one would obtain a system equivalent to the semisimple Taub solution which describes the evolution in the presence of two exponential potentials depending on orthogonal variables. In general the two potentials are not orthogonal and the evolution equations do not decouple in Taub time gauge;

perhaps a direct analysis of the decoupling possibilities with power law lapses would lead to the solution.

The Joseph solution is itself a special case of the Ellis-MacCallum diagonal type VI_h solution,⁵⁰ obtained by Lie algebra contraction of that solution. The same variation of parameters extends to this case and the whole family of $d = 4$ Taub solutions, as well as the type VI₀ analog of the locally rotationally symmetric Taub metrics and the Kantowski-Sachs geometry, whose vacuum solution is related by analytic continuation to the Bianchi type III (= VI₋₁) value of the Ellis-MacCallum solution. These are discussed by Lorentz.²² All of these are disguised versions of the Taub family of solutions which represents the dynamics of one or two nontrivial gravitational degrees of freedom, independent of the dimension. This variation of parameter idea is not special to $d = 11$ supergravity in the higher dimensional context, but as already mentioned above, has implications for other theories as well.

Consider the general case of a decomposable spatial metric from the point of view of the Ricci evolution equations

$$\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 & 0 \\ 0 & \mathbf{g}_2 \end{pmatrix}, \quad \mathbf{R} - \mathbf{E} = \begin{pmatrix} \mathbf{R}_1 - \mathbf{E}_1 & 0 \\ 0 & \mathbf{R}_2 - \mathbf{E}_2 \end{pmatrix}. \quad (4.17)$$

Here \mathbf{R}_1 and \mathbf{R}_2 are the Ricci tensor mixed component matrices of the individual homogeneous metrics on the factor manifolds M_1 and M_2 ; the source energy-momentum must also be in block diagonal form for consistency with the Einstein equations. The Ricci evolution equations are

$$0 = x^{-1}(x\mathbf{g}^{-1}\dot{\mathbf{g}})' + 2N^2(\mathbf{R} - \mathbf{E}), \quad x \equiv N^{-1}g^{1/2}. \quad (4.18)$$

and in Taub time gauge $x \sim const$

$$0 = (\mathbf{g}^{-1}\dot{\mathbf{g}})' + (2 \cdot 10 \cdot 9)^2 \cdot 2g_1g_2(\mathbf{R} - \mathbf{E}). \quad (4.19)$$

In the diagonal case $\mathbf{g} = \mathbf{g}_D$, the time derivative term is simply $(\ln \mathbf{g}_D)'$.

The only coupling between \mathbf{g}_1 and \mathbf{g}_2 in this time gauge occurs through the factor g_1g_2 in the driving term. When $\mathbf{R}_2 - \mathbf{E}_2 = 0$, for example, \mathbf{g}_2 has free dynamics and one can rescale \mathbf{g}_1 by a power of g_2 to decouple the remaining evolution equations from g_2

$$\begin{aligned} \mathbf{g}_1 &= g_2^\zeta \bar{\mathbf{g}}_1, & \zeta &= (D_1 - 1)^{-1}, & D_1 &= \dim M_1, \\ 0 &= (\bar{\mathbf{g}}_1^{-1}\dot{\bar{\mathbf{g}}}_1)' + (2 \cdot 10 \cdot 9)^2 \cdot 2\bar{g}_1(\bar{\mathbf{R}}_1 - \bar{\mathbf{E}}_1). \end{aligned} \quad (4.20)$$

This redefinition of variables leads to the Taub time Ricci evolution equations for the spacetime $R \times M_1$ with spatial metric matrix $\bar{\mathbf{g}}_1$. The conserved energy associated with those equations, however, must be nonzero due to the contribution of the free motion in the extra dimensions to the higher dimensional energy. This was noted by Barrow and Stein-Schabes for a $d = 5$ Kaluza-Klein generalization of the Bianchi type IX models, for which generalized Taub solutions were mentioned.²³

When the spatial metric is not decomposable, but still has the block diagonal form (4.17), a number of decoupling possibilities occur depending on the form of the driving term. Suppose that \mathbf{g}_2 is the 1-dimensional matrix (g_{DD}) so $g_2 = g_{DD}$. Then the gauge $N = g_{DD}^{1/2}$ leads to $x = g_1^{1/2}$; if $N^2(\mathbf{R} - \mathbf{E})$ is independent of g_{DD} , then \mathbf{g}_1 immediately decouples from g_{DD} , while $\ln g_{DD}$ satisfies a linear equation with a \mathbf{g}_1 dependent source. When the trace of $N^2(\mathbf{R}_1 - \mathbf{E}_1)$ depends only on x , taking the trace of the first block of the Ricci evolution equations leads to a decoupled equation for x , the kinetic term reducing to $2x^{-1}\ddot{x}$. If $N^2 \text{Tr}(\mathbf{R}_1 - \mathbf{E}_1)$ is a constant, say $-\chi^2$, this equation is in fact $\ddot{x} - \chi^2 x = 0$, leading to exponential or hyperbolic solutions when $\chi^2 > 0$, trigonometric solutions when $\chi^2 < 0$ and linear solutions when $\chi = 0$. (This is again the 1-dimensional scattering problem, this time in the variable $\ln x$.) This situation occurs, for example, if the spatial section is the semidirect product of a $(D - 1)$ -dimensional abelian subgroup and a 1-dimensional group of automorphisms of that subgroup, the 1-dimensional group being associated with the D th direction. The block diagonal (“symmetric case”) $D = 3$ nonsemisimple models are of this type, and this feature allows a uniform treatment in the time gauge $N = g_{33}^{1/2}$ of most of the known vacuum solutions of this class, which are related by variation of parameters to scalar field (and hence higher dimensional Kaluza-Klein), stiff perfect fluid (including tilt along the distinguished direction) and spatially self-similar generalizations.⁴² Some solutions of the evolution equations, not allowed by the original super-Hamiltonian constraint, may satisfy that constraint after a variation of parameters, leading to solutions with no analog in the original case. Such solutions occur when a certain parameter changes sign.

For x to decouple, it is sufficient that $N^2 \text{Tr}(\mathbf{R}_1 - \mathbf{E}_1)$ depend only on x . This occurs if this term arises from a curvature term associated with an isotropic coset space factor manifold to which the D th direction belongs. Friedmann-Robertson-Walker or Kantowski-Sachs factors or even semisimple Taub factors lead to this behavior. In the $D = 3$ semisimple Taub case, the group manifold S_3 , as an S_1 fiber bundle over S_2 ,⁵¹ is locally a product, hence choosing $N = A^{1/2}$ (i.e., $(m, n) = (1, 0)$), where A is associated with the isotropic S_2 base of the fiber bundle, leads to a decoupled equation for $x = (AB)^{1/2}$. One may also rescale the lapse by a power of x without losing this decoupling (i.e., $m = n + 1$). For example, in the Taub case, the lapse $N = x^{-2}A = B^{-1}$ leads to the $(0, -1)$ decoupling of x , yielding quadratic solutions.

These ideas were used by Lorentz-Petzold (without being fully understood) in his discussion of some supergravity solutions²² and Brans-Dicke solutions.⁵² The latter are equivalent to a $d = 5$ Kaluza-Klein model in the vacuum case, and his choice of decoupling variable “ g ” involving the scalar field is exactly the variable x in the higher dimensional formulation. Adding perfect fluids with certain equations of state leads to a variation of parameters in the evolution equations without breaking this decoupling. All exact solutions can be understood in terms of the variation of parameters idea.

Another situation arises when the lapse depends only on g , say $N = g^{(1-\zeta^2)/2}$ so $x = g^{\zeta^2/2}$. The trace of the Ricci evolution equations then gives a decoupled

equation for x as long as $N^2 \text{Tr}(\mathbf{R} - \mathbf{E})$ is a constant, say $-\chi^2$; the equation is then $\ddot{x} - \zeta^2 \chi^2 x = 0$, again having the same solutions as discussed above depending on the signs of ζ^2 and χ^2 . For example, in the $d = 4$ case, a Bianchi type I perfect fluid with equation of state $p = (\gamma - 1)\rho$ has $\mathbf{E} \sim g^{-\gamma/2}$ and hence $N \sim g^{\gamma/4}$ makes $N^2 \mathbf{E}$ constant; a cosmological constant alone has \mathbf{E} constant, so in this latter case x has the same solutions as above but in proper time gauge. On the other hand for the Bianchi type I fluid case, one can also choose $N = g^{(\gamma-1)/2}$ so that $x \sim g^{1-\gamma/2}$ and $N^2 \mathbf{E} \sim x^{-1}$, which leads to quadratic solutions for x ; this was an intermediate step in the work of Jacobs.⁵³

All these exact solutions are characterized by the fact that the source energy-momentum (including a possible cosmological constant) may be represented in terms of the metric and constants of the motion, leading to an entirely geometric system. This occurs only when the source has either discrete or continuous additional symmetry.³⁷ In this class of models at most two “nontrivial” gravitational modes can be excited, as in the semisimple Taub case, the existence of exact solutions depending at least on a partial decoupling of the modes.

5 Concluding Remarks on Exact Solutions

At this point the overview of existing special exact solutions has shown us that they all depend crucially on decoupling ideas that involve the DeWitt geometry, the symmetry of the potentials or driving forces, the choice of time variable and the choice of evolution equations. The variation of parameters idea and its relation to the constraints turns out to be a very important one, sometimes involving the parameters of the solutions of the equations and sometimes the parameters of the equations themselves. Although the super-Hamiltonian constraint has been emphasized because of the models discussed here, the supermomentum constraint plays a similar role when models are considered which excite the supermomentum.⁴² All of these ideas seem never to be systematically applied. Most special solutions are found almost by accident and never later understood in a broader context. Studying the appropriate sections of the *Exact Solutions* book⁵⁴ with these ideas in mind, one begins to see some order in the seemingly unrelated list of exact solutions in ordinary spacetime. In higher dimensions the situation is similar.

But are particular exact cosmological solutions really significant? What do we gain by finding new very special solutions of some particular set of field equations? This is a timely question, given that the exact solutions industry often seems to be only weakly coupled to reality. One might imagine that special solutions might be interesting if they exhibit some “stable” property of the field equations in some sense or if the solutions exhibit some “typical” behavior during certain phases of the evolution of the universe, at least within the restricted symmetry class considered if not in a larger context.

For the spatially homogeneous case, ordinary differential equations permit a sophisticated analysis of the qualitative behavior of their solutions.^{55–58} In an appropriate formulation of the field equations, certain special exact solutions

play the role of “singular points” and “separatrixes” between singular points on an appropriate phase space. In the context of the parameter space of all symmetry types of a given dimension, exact solutions of a lesser symmetry type often characterize phases of evolution of a higher symmetry type.³⁸ Singular points are associated with “exact power law” solutions^{59,60} of the field equations, which in turn are geometrically characterized by self-similar evolution, i.e., the existence of additional spacetime symmetry in the form of a homothetic symmetry which shifts the homogeneous hypersurfaces along the time direction. This reduces the field equations to algebraic equations in certain parameters which determine the metric completely. The simplest of these is the Kasner solution. The Taub solutions and their generalizations act as separatrixes in this picture. These kinds of solutions are important in understanding typical behavior at very early times toward the big bang and at very late times coming out of the big bang.

Perhaps some of the energy devoted to finding particular exact solutions (a tree level activity) should be shifted to understanding some of the broader questions that arise in this area of research. The average level of “new exact solution” papers seems to have remained close to the level of two decades ago when this game first drew widespread participation. As we have seen, those exact solutions which exist are very special and have very nice mathematical properties which relate them to one another due to the richness of the Einstein equations. If these simple ideas are not commonly understood, how can one hope to have reasonable progress on more complicated questions?

6 Chaos?

One deeper mathematical question which seems to fascinate many people is the extent to which the “oscillatory approach to the initial singularity” found in the $d = 4$ spatially homogeneous vacuum semisimple case and in the (nonstiff) perfect fluid case with anisotropic spatial curvature might be a “general property” of cosmological solutions of the classical Einstein equations.^{28,29} This has aroused considerable controversy, more recently acquiring the trendier name of gravitational chaos²⁹ and spreading into higher dimensional purely gravitational models.^{23–27,30,31}

In the $d = 4$ semisimple case, the vacuum supermomentum constraints restrict the spatial metric to be diagonalizable, the three spatial gauge degrees of freedom being intimately connected with the remaining three “offdiagonal modes” which are not excited in vacuum. (Diagonalization depends on the choice of the Lie algebra basis; the Killing metric must be diagonal in a basis in which the spatial metric is diagonal. Such a frame is then also an eigenbasis of the extrinsic curvature, namely a “Kasner frame”^{28,46}.) In Taub time gauge the spatial curvature potentials at fixed α have “essentially closed” potential contours on the flat subspace of the two β variables, and for increasing values of the potential, these contour lines resemble more and more closely straight line contours joined together at vertices which are either closed or have open channels which run out to infinity with a width which goes to zero. The “straight”

contours are exponential and move outward with α -speed $\frac{1}{2}$ as $\alpha \rightarrow 0$, while the system point moves with unit α -speed (a null geodesic) when the potential is negligible compared to the kinetic energy. Thus the system point continually overtakes and scatters off the receding exponential potentials as long as the relative velocity between the system point in its free state and some exponential potential is always negative (approach). In fact the geometry is such that the relative velocity between the system point and the exponential potentials which it is chasing is nonpositive in every direction but the vertex directions, where it vanishes. Motion exactly along a vertex direction only occurs for the Taub solutions which describe motion down the center of an open channel extending to infinity. Thus in general one has the Mixmaster behavior⁴¹ in which the system point rattles around indefinitely within the expanding potential as $\alpha \rightarrow 0$. On the other hand the nonsemisimple vacuum models have an open set of directions in which the system point may escape, i.e., the absence of at least one of the “straight” potential terms leads to an open set of directions in which the potential asymptotically goes to zero, and the system point eventually ends up in an asymptotic free state.

In the group manifold case, which is more general than the coset space case, each “straight” potential contour line is associated with the structure constant component C^a_{bc} with $a \neq b \neq c$, which enters as a quadratic factor in the potential.¹ For the $d = 4$ semisimple case, all such terms are necessarily present, leading to the essentially closed potential contours. In higher dimensions several crucial differences occur. One still needs semisimplicity for the existence of essentially closed potential contours; these potentials are best described in a frame in which the Killing metric is diagonal as in the $d = 4$ case. However, the Jacobi identity no longer allows all structure constants C^a_{bc} with distinct indices to be nonzero, removing certain potential terms.

Furthermore, only some of the offdiagonal modes are restricted by the supermomentum constraints, so the vacuum case is no longer diagonalizable and certain offdiagonal modes are nontrivial in the sense that the curvature potential necessarily depends on them and they cannot be suppressed without additional symmetry imposition, much like that which reduces the Mixmaster models to the essentially different Taub models. This leads to the excitation of nonzero curvature modes of the (rescaled) DeWitt metric. For those models which are diagonalizable and therefore no longer general, the absence of all possible structure constants with distinct indices apparently leads to a vertex geometry such that an open set of vertex directions exists along which the “free” system point can chase the potential with positive relative velocity (recession); the system point then eventually ends up in an asymptotic free state.^{26,27}

Without imposing the Jacobi identity, Demaret et al³⁰ have shown that this happens only for $d > 10$, relevant to the general inhomogeneous case which still remains somewhat controversial. For the homogeneous case, the Jacobi identities seem to reduce the critical dimension to $d > 4$ in diagonalizable models. However, in nondiagonal models, the eigenvectors of the extrinsic curvature no longer coincide with the time independent invariant frame vectors and the transformation to the eigenvector frame induces more nonzero structure functions.

Since it is the eigenvector frame (Kasner frame) structure functions which are relevant to the existence of an asymptotic free state, the question still remains open for those dimensions $d \in [5, 10]$ for which semisimple groups exist.

Of course chaos even in $d = 4$ dimensions is sabotaged by the presence of a scalar field since it contributes a term to the super-Hamiltonian which reduces the free motion speed of the system point (timelike motion), therefore allowing an open set of vertex directions (or eventually all directions) which permit an asymptotic free state.^{9,37} Moreover, since the free motion phases are translations in $\alpha \sim \ln g$, one very quickly approaches the initial singularity $g = 0$, slamming into the Planck scale where the classical Einstein equations break down and the indefinite approach to the classical singularity is no longer particularly relevant. The classical chaos does lead to quantum consequences, however, at least at the naive level which allows calculation.⁶¹

7 Conclusion

Whether one is interested in mathematical questions or more physical questions, a broader point of view is extremely valuable. This article has focused on certain mathematical questions as a means of illustrating this point. Staring at equations or their solutions in a 2-dimensional representation is not particularly illuminating. The rich geometrical structure of the Einstein equations allows one instead to exploit “hidden” relationships and visualize many concepts which remain somewhat obscure in their brute form representation. Even when visualization may be difficult or not particularly relevant, a perspective which views a given problem “from above” is certainly more effective in not only resolving that problem, but in understanding the result and placing it into context. The latter is, after all, more important than the former.

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