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# Gravitomagnetic Clock Effects in Black Hole Spacetimes

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**Abstract**

Gravitomagnetic clock effects for circularly rotating orbits in black hole spacetimes are studied from a relative observer point of view, clarifying the roles played by special observer families.

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## Preface

One of us (R.J.) knew Ruggiero through our mutual friend Giovanni Platania for several decades and was happy that finally, with a collaborator (D.B.) relocated to Napoli, it appeared that we could begin to interact more directly both socially and professionally. Indeed we all had a wonderful dinner together in the summer of 2000. This potential was cut short prematurely, saddening both of us tremendously. In our many years of crossing paths, Ruggiero was always such a gentleman, with obvious class, and had a good sense of optimism and humor and a special Neopolitan way of looking at the world that we will never forget. He continues to live on in our hearts and memories with special affection.

# 1 Introduction

Recently, various clock effects in stationary axially symmetric spacetimes have renewed interest in better understanding them and possibly eventually measuring some of them. It is precisely the presence of a rotating source of the gravitational field that leads to these effects, first in black hole spacetimes which have the most interest as the exterior fields of physically interesting time independent axially symmetric sources, and secondarily in other members of the larger symmetry class to which they belong, including the Gödel and van Stockum spacetimes, which provide alternative examples of what can happen in theory. Rotating sources thus mean nontrivial gravitomagnetism; the Sagnac effect, synchronization gaps, and other clock comparison effects all fall (or should fall) under this umbrella topic referred to as "gravitomagnetic clock effects." In fact, one can study them all in parallel and their relationships using a general approach set in the context of stationary axisymmetry.

It is important to consider the observers who are associated with each of these effects, since most of the effects are observer-dependent. Gravitoelectromagnetism is basically a long word for a way of describing spacetimes in terms of families of test observers, and so it appropriate to use its related tools in approaching the present questions. Each such family measures spacetime quantities relative to its own local space and time directions, thus performing a "relative observer analysis." The quotient of the spacetime by the family of world lines of the observer family is useful for considering and making sense of the "closed circular loops" that we normally think of when visualizing these effects. In fact for stationary circular world lines, it is really only the geometry of the intrinsically flat timelike cylinder containing them which is relevant, where the observer splittings are easy to picture and the local splitting of each tangent space extends to the whole cylinder by the symmetry, with global complications which are at the heart of the various clock effects.

One must distinguish three distinct gravitomagnetic clock effects for a pair of oppositely rotating circular geodesic test particles (oppositely rotating with respect to an intermediate observer) (see [1] and references therein):

1. the observer-dependent single-clock clock effect: the difference between the periods of the oppositely rotating geodesic test particles as measured by the observer's clock [2],

2. the observer-dependent two-clock clock effect: the difference between the periods of the oppositely rotating geodesic test particles as measured by their own clocks for one revolution with respect to an observer [3, 4, 5], and

3. the observer-independent two-clock clock effect: the difference between the periods of the oppositely rotating geodesic test particles as measured by their own clocks between two crossing events [6, 7, 8, 9].

In the first two cases, for a given observer, one compares the periods of one revolution of these orbits (starting from and returning to the same observer world line) measured either by the observer's own clock (single-clock effect) or by the clocks carried along the two orbits (two-clock effect). In the third case, no observer enters the calculation.

The Sagnac effect is similar but with a pair of oppositely rotating accelerated photons replacing the geodesic test particle pair. The observer-dependent single-clock effect in this new context is the Sagnac effect. The desynchronization effect is similar but with accelerated spacelike curves (orthogonal to the observer family) replacing the null curves of the Sagnac effect. This observer-dependent single-clock effect, divided by 2, is the synchronization gap which measures the failure to synchronize the observer family proper times around a single loop.

Here we draw some connections between the various clock effects, the Sagnac and desynchronization effects, and the usual symmetry adapted coordinates in black hole spacetimes, while extending some previous work.

## 2 Metric splitting, observer-adapted frames, lapse/shift notation

An orthogonally transitive stationary axially symmetric line element is usually written in the Boyer-Lindquist-like coordinates $\{t, r, \theta, \phi\}$ adapted to the time-like Killing observers:

$$ds^2 \quad = \quad ds^2_{(t,\phi)} + ds^2_{(r,\theta)} \ , \tag{1}$$

where

$$\begin{aligned} ds^2_{(t,\phi)} &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \ , \\ ds^2_{(r,\theta)} &= g_{rr}dr^2 + g_{\theta\theta}d\theta^2; \end{aligned} \tag{2}$$

Decomposing the $t, \phi$ 2-metric into perfect square form defines the lapse ($M$), shift ($M_\phi$) and spatial metric quantity ($\gamma_{\phi\phi}$) in the so-called threading point of view of the Killing observers

$$ds^2_{(r,\theta)} = -M^2(dt - M_\phi d\phi)^2 + \gamma_{\phi\phi}d\phi^2 \ . \tag{3}$$

The 4-velocities of circularly rotating timelike orbits/observers (i.e. at fixed $r$ and $\theta$) form a 1-parameter family most simply parametrized by their coordinate “angular velocity” $\zeta$

$$U = \Gamma(\partial_t + \zeta\partial_\phi) \ , \tag{4}$$

where $\Gamma = dt/d\tau_U > 0$ is a normalization factor defined by

$$\Gamma = \{-[g_{tt} + 2\zeta g_{t\phi} + \zeta^2 g_{\phi\phi}]\}^{-1/2} \ . \tag{5}$$

In order for $U$ to be timelike, $\zeta$ must belong to the interval $(\zeta_-, \zeta_+)$ between the roots of the quadratic equation $\Gamma^{-2} = 0$ in $\zeta$ corresponding to the two null directions, namely

$$\zeta_\pm \quad = \quad [-g_{t\phi} \pm ({g_{t\phi}}^2 - g_{\phi\phi}g_{tt})^{1/2}]/g_{\phi\phi} \ . \tag{6}$$

(Note $\zeta_- \leq 0 \leq \zeta_+$.) Thus apart for an arbitrary normalization factor $\Gamma_{(\rm null)}$, the photon 4-momentum vector $P_\pm$ is of the same form as $U$

$$P_\pm = \Gamma_{(\rm null)}(\partial_t + \zeta_\pm \partial_\phi) \ . \tag{7}$$

We assume that the source of the rotating gravitational field is rotating in the positive sense, i.e. in the increasing $\phi$ direction. Corotating and counter-rotating orbits will refer to motion in the positive ($\zeta > 0$) and negative ($\zeta < 0$) $\phi$ direction respectively in this coordinate system. For example, the 4-velocities of the corotating ($+$) and counter-rotating ($-$) circularly rotating timelike geodesics are

$$U_\pm = \Gamma_\pm(\partial_t + \dot\phi_\pm \partial_\phi) \ , \tag{8}$$

where it is assumed that $\zeta_- < \dot\phi_- < 0 < \dot\phi_+ < \zeta^+$. These represent the orbits of freely falling test particles.

It is convenient to introduce new coordinates adapted to a generic circularly rotating stationary observer $U$

$$\tilde t = t, \qquad \tilde\phi = \phi - \zeta t \ , \tag{9}$$

so that the new form of the metric is

$$ds^2_{(\tilde t,\tilde\phi)} = g_{\tilde t\tilde t} d\tilde t^2 + 2 g_{\tilde t\tilde\phi} d\tilde t d\tilde\phi + g_{\tilde\phi\tilde\phi} d\tilde\phi^2 = ds^2_{(t,\phi)} \ , \tag{10}$$

or equivalently

$$ds^2_{(\tilde r,\tilde\theta)} = -\tilde M^2 (dt - \tilde M_{\tilde\phi} d\tilde\phi)^2 + \tilde\gamma_{\tilde\phi\tilde\phi} d\tilde\phi^2 \ . \tag{11}$$

The 4-velocity of the observer $U$ can be represented in exactly the same form as with respect to the original $(t,\phi)$ coordinates

$$U = \tilde\Gamma[\partial_{\tilde t} + \tilde\zeta \partial_{\tilde\phi}] = [-g_{\tilde t\tilde t}]^{-1/2} \partial_{\tilde t} \ , \tag{12}$$

where $\tilde\zeta = \zeta - \zeta = 0$ and $\tilde\Gamma = (-g_{\tilde t\tilde t})^{-1/2}$. Finally, the direction $\bar U$ orthogonal to $U$ has the form Eq. (4) with the replacement

$$\zeta \to \bar\zeta = -\frac{g_{tt} + \zeta g_{t\phi}}{g_{t\phi} + \zeta g_{\phi\phi}}. \tag{13}$$

The quantity $\bar\zeta$ is an inverse angular velocity measuring the local effective rate of change of the synchronization gap with respect to angle.

Table 1 summarizes the representations of the new observer 4-velocity and unit 1-form and their orthogonal counterparts. The new lapse, shift and spatial metric quantities are explicitly

$$\begin{aligned}
\tilde M^2 &= -g_{\tilde t\tilde t} = -(g_{tt} + 2\zeta g_{t\phi} + \zeta^2 g_{\phi\phi}) = \Gamma^{-2} \ , \\
\tilde M_{\tilde\phi} &= -g_{\tilde t\tilde\phi}/g_{\tilde t\tilde t} = \Gamma^2 (g_{t\phi+\zeta g_{\phi\phi}}) = (\bar\zeta - \zeta)^{-1} \ , \\
\tilde\gamma_{\tilde\phi\tilde\phi} &= g_{\tilde\phi\tilde\phi} + \tilde M^2 \tilde M_{\tilde\phi}^2 = \gamma_{\phi\phi} g_{tt}/g_{\tilde t\tilde t} \ .
\end{aligned} \tag{14}$$

The coordinate time $t = \tilde{t}$ can be used to parametrize the world lines of the observer $U$ or any other timelike or null world line. This may be converted to a proper time with respect that observer differentially by

$$d\tau_U = \Gamma^{-1} dt = \tilde{M} d\tilde{t} \ . \tag{15}$$

Integrating this along a world line segment gives the equivalent proper time elapsed along an observer world line between the beginning and ending time hypersurface.

Table 1: New stationary observer quantities in terms of old and new adapted coordinates.

| | $(t, \phi)$ | $(\tilde{t}, \tilde{\phi})$ |
|---|---|---|
| $U$ | $= \Gamma[\partial_t + \zeta \partial_\phi]$ | $= \tilde{M}^{-1} \partial_{\tilde{t}}$ |
| $U^\flat$ | $= \Gamma(g_{tt} + \zeta g_{t\phi})[dt - 1/\bar{\zeta}\, d\phi]$ | $= -\tilde{M}[d\tilde{t} - \tilde{M}_{\tilde{\phi}} d\tilde{\phi}]$ |
| $\bar{U}$ | $= \bar{\Gamma}[\partial_t + \bar{\zeta} \partial_\phi]$ | $= (\tilde{\gamma}_{\tilde{\phi}\tilde{\phi}})^{-1/2}[\partial_{\tilde{\phi}} + \tilde{M}_{\tilde{\phi}} \partial_{\tilde{t}}]$ |
| $\bar{U}^\flat$ | $= -\Gamma^2/\bar{\Gamma}\, (g_{t\phi} + \zeta g_{\phi\phi})\zeta[dt - 1/\zeta\, d\phi]$ | $= (\tilde{\gamma}_{\tilde{\phi}\tilde{\phi}})^{1/2} d\tilde{\phi}$ |

# 3 Clock effects

Consider a pair of oppositely rotating orbits, $U_1$ and $U_2$, as seen by an intermediate observer $U$, with angular velocities $\zeta_1$, $\zeta_2$ and $\zeta$ respectively with respect to the original coordinates $(t, \phi)$ satisfying $\zeta_1 < \zeta < \zeta_2$. We have in mind identifying $U_{1,2}$ either with the oppositely rotating geodesic pair (counter-rotating: $U_1 = U_-$, corotating: $U_2 = U_+$) or with oppositely rotating photons (counter-rotating: $U_1 = P_-$, corotating: $U_2 = P_+$) and new coordinates $(\tilde{t}, \tilde{\phi})$ will be adapted to the observer $U$ as described in the previous section. The unit tangent vector along the orthogonal spacelike counterpart of timelike rotating curves $U_{1,2}$ (i.e. corresponding to the orthogonal direction in the $t - \phi$ space, oriented in the sense of increasing $\phi$) will be denoted by $\bar{U}_{1,2}$.

We now consider each of the various clock effects in turn, analyzed from the relative-observer point of view.

## 3.1 Timelike observer-dependent single-clock clock effect

The timelike single-clock clock effect relative to the observer $U$ measures the observer proper time elapsed between the arrivals of the two geodesics (one co-rotating and the other counter-rotating) which depart from the same event on that observer worldline (i.e. the geodesics each make a single spacetime loop beginning and ending at the observer worldline). Denoting by $C^+_{U_\pm}$ the line with unit tangent vector $U_\pm$ oriented according the direction of increasing time, this single-clock proper time difference is just the difference of the observer periods of the two loops

$$\begin{aligned}\Delta\tau_{(1-c)}(U,U_-,U_+) &= \int_{C^+_{U_+}} d\tau_U - \int_{C^+_{U_-}} d\tau_U \\ &= \frac{1}{\Gamma}[\int_{C^+_{U_+}} d\tilde t - \int_{C^+_{U_-}} d\tilde t] \\ &= \frac{2\pi}{\Gamma}[\frac{1}{\dot\phi_+ - \zeta} + \frac{1}{\dot\phi_- - \zeta}] \\ &= -\frac{4\pi}{\Gamma}\frac{\zeta - \zeta_{(\rm gmp)}}{(\zeta - \dot\phi_-)(\zeta - \dot\phi_+)}\ ,\end{aligned} \tag{16}$$

where $\zeta_{(\rm gmp)} = (\dot\phi_+ + \dot\phi_-)/2$ is the angular velocity of the 'geodesic meeting point observers' [6, 7], whose world lines pass through the successive alternating meeting points of each pair of corotating and counter-rotating geodesics, corresponding to undergoing complete revolutions with respect to those observers.

The computation of these integrals is conveniently done by changing the integration variable from $\tilde t$ to $\tilde\phi$ and noting the following facts a) along the world lines $C^+_{U_\pm}$ the 1-form $\bar U_\pm \propto d\tilde t - \frac{1}{\zeta - \dot\phi_\pm} d\tilde\phi$ vanishes identically; b) along $C^+_{U_-}$ (counter-rotating), $\tilde\phi$ goes from 0 to $-2\pi$ while along $C^+_{U_+}$ (co-rotating), $\tilde\phi$ goes from 0 to $2\pi$.

As a function of the observer angular velocity $\zeta$, the single clock effect $\Delta\tau_{(1-c)}(U,U_-,U_+)$ is positive for $\zeta > \zeta_{(\rm gmp)}$ (which means the corotating geodesic returns later than the counter-rotating one), zero corresponding to $U = U_{(\rm gmp)}$ and negative for $\zeta < \zeta_{(\rm gmp)}$, and it is a monotonically increasing function of $\zeta$, from $\zeta = \dot\phi_-$ (where a vertical asymptote is located) to $\zeta = \dot\phi_-$ (where another vertical asymptote is located).

## 3.2 Lightlike observer-dependent single-clock clock effect or Sagnac effect

The lightlike single-clock clock effect relative to the observer $U$ [10, 11, 12, 13, 14, 15] is a measure of the observer proper time elapsed between the arrivals of the two photons (one co-rotating and the other counter-rotating) at a particular observer, having both departed from a single event on that observer's world line (i.e. the photons each make a single spacetime loop around the observer $U$). Denoting by $C^+_{P_\pm}$ the photon world line with null tangent vector $P_\pm$ oriented according the direction of increasing time this single-clock proper time difference is just the Sagnac effect

$$\Delta\tau_{(1-c)}(U,P_-,P_+) = \int_{C^+_{P_+}} d\tau_U - \int_{C^+_{P_-}} d\tau_U$$

$$
\begin{aligned}
&= \frac{1}{\Gamma}[\int_{C^+_{P_+}} d\tilde{t} - \int_{C^+_{P_-}} d\tilde{t}] \\
&= \frac{2\pi}{\Gamma}[\frac{1}{\zeta_+ - \zeta} + \frac{1}{\zeta_- - \zeta}] \\
&= -\frac{4\pi}{\Gamma}\frac{\zeta - \zeta_{(\rm nmp)}}{(\zeta-\zeta_-)(\zeta-\zeta_+)} \ , \qquad (17)
\end{aligned}
$$

where $\zeta_{(\rm nmp)} = (\zeta_+ + \zeta_-)/2$ is the angular velocity of the 'null meeting point observers', whose world lines contains the alternating successive meeting points of the two corotating and counter-rotating photons after making complete revolutions with respect to those observers. These observers are also called 'slicing observers' because their with 4-velocity $n$ is orthogonal to the $t = \textit{constant}$ hypersurfaces which form a slicing of the spacetime itself $\zeta_{(\rm sl)} = \zeta_{(\rm nmp)}$.

The Sagnac effect $\Delta\tau_{(1-\rm c)}(U, P_-, P_+)$ vanishes when $U = U_{(\rm nmp)} = n$ and its behavior as a function of $\zeta$ is similar to that of $\Delta\tau_{(1-\rm c)}(U, U_-, U_+)$ but the two vertical asymptotes correspond now to the photon angular velocities.

One can study the relations that might exist between the timelike single-clock clock effect and its lightlike analogue. Recently an observer family has been found [1] for which

$$\Delta\tau_{(1-\rm c)}(U, U_-, U_+) = \Delta\tau_{(1-\rm c)}(U, P_-, P_+) \ . \qquad (18)$$

In other words the Sagnac effect and 1-clock clock effect agree. These observers turn out to be the so-called 'extremely accelerated observers' [16, 17, 18]. Their 4-velocity is related to those of the co- and counter-rotating geodesics by a renormalized average

$$U_{(\rm ext)} = \frac{U_+ + U_-}{||U_+ + U_-||} \ , \qquad (19)$$

and their angular velocity is therefore:

$$\zeta_{(\rm ext)} = \frac{\Gamma_+\dot{\phi}_+ + \Gamma_-\dot{\phi}_-}{\Gamma_+ + \Gamma_-} \ . \qquad (20)$$

The most important additional properties of these observers are briefly summarized here:

1. They see the two geodesics (co-rotating and counter-rotating) with equal magnitude but opposite sign relative velocities:

$$\nu(U_+, U_{(\rm ext)}) = -\nu(U_-, U_{(\rm ext)}); \qquad (21)$$

2. Their 4-acceleration is extremal among the whole family of circular orbits parametrized by the angular velocity:

$$\partial_\zeta a(U)|_{\zeta=\zeta_{(\rm ext)}} = 0; \qquad (22)$$

3. They are "intrinsically non-rotating". In fact the first and second torsion of the spacetime Frenet-Serret frame associated with them are both vanishing. Consequently the spacetime Frenet-Serret angular velocity is also vanishing and the (intrinsic) spatial frame is Fermi-Walker dragged along their own world lines. Thus if the extremely accelerated observers carry a test gyroscope along with them, there is no gyroscope precession when referred to their intrinsic Frenet-Serret axes.

### 3.3 Observer-dependent two-clock clock effect

The two-clock clock effect with respect to a generic observer $U$ is defined as the difference between the proper periods of two geodesic loops (one co-rotating and the other counter-rotating) with respect to the observer $U$:

$$
\begin{aligned}
\Delta\tau_{(2-\mathrm{c})}(U, U_-, U_+) &= \int_{C^+_{U_+}} d\tau_{U_+} - \int_{C^+_{U_-}} d\tau_{U_-} \\
&= \frac{1}{\Gamma_+}\int_{C^+_{U_+}} d\tilde{t} - \frac{1}{\Gamma_-}\int_{C^+_{U_-}} d\tilde{t} \\
&= \frac{2\pi}{\Gamma_+(\dot\phi_+ - \zeta)} + \frac{2\pi}{\Gamma_-(\dot\phi_- - \zeta)} \\
&= 2\pi\frac{\Gamma_+ + \Gamma_-}{\Gamma_+\Gamma_-}\frac{\zeta - \zeta_{(\mathrm{ext})}}{(\zeta - \dot\phi_-)(\dot\phi_+ - \zeta)}\ ,
\end{aligned} \tag{23}
$$

This shows another important property which must be added to those enumerated in the previous subsection for the extremely accelerated observers:

$$
\Delta\tau_{(2-\mathrm{c})}(U_{(\mathrm{ext})}, U_-, U_+) = 0\ . \tag{24}
$$

Their observer-dependent 2-clock clock effect is zero.

### 3.4 Observer-independent two-clock clock effect

The observer-independent two-clock clock effect coincides with the observer-dependent two-clock clock effect with respect to the geodesic meeting point observers $U_{(\mathrm{gmp})}$, whose worldline contains the alternating successive meeting points of the two oppositely rotating geodesics:

$$
\begin{aligned}
\Delta\tau_{(2-\mathrm{c})}(U_{(\mathrm{gmp})}, U_-, U_+) &= 2\pi\frac{\Gamma_+ + \Gamma_-}{\Gamma_+\Gamma_-}\frac{\zeta_{(\mathrm{gmp})} - \zeta_{(\mathrm{ext})}}{(\zeta_{(\mathrm{gmp})} - \dot\phi_-)(\dot\phi_+ - \zeta_{(\mathrm{gmp})})} \\
&= 4\pi\frac{\Gamma_- - \Gamma_+}{\Gamma_+\Gamma_-}\frac{1}{\dot\phi_+ - \dot\phi_-}\ .
\end{aligned} \tag{25}
$$

This effect is observer-independent since it only involves the meeting points of the two geodesics after a full revolution (during which they meet twice after the initial departure event).

### 3.5 Synchronization gap

The amount of observer proper time $\tau_U$ elapsing before the arrival of the orthogonal spatial orbit with unit tangent vector $\bar{U}$ is what has been called the synchronization gap [19]

$$\Delta\tau_{(\mathrm{SG})}(U,\bar{U}) = \int_{C^+_{\bar{U}}} d\tau_U = \frac{2\pi}{\Gamma}\frac{1}{(\bar{\zeta}-\zeta)} \ . \tag{26}$$

It is nonzero for observer families with nonzero vorticity, which prevents the synchronization of the proper times on the whole family of observer world lines by an orthogonal slicing. Orthogonal synchronization can always be carried out by curves orthogonal to the observer world lines, but if they complete a loop around the symmetry axis, the initial and final times are desynchronized, hence a 'desynchronization effect' occurs. The synchronization gap is a measure of this effect. It vanishes for the slicing observers ($\bar{\zeta}_{(\mathrm{sl})} \to \infty$) and is just half the Sagnac effect:

$$\Delta\tau_{(\mathrm{SG})}(U,\bar{U}) = \frac{1}{2}\Delta\tau_{(1-\mathrm{c})}(U,P_-,P_+) \ . \tag{27}$$

# 4 Kerr black holes

We conclude by specializing our results to the equatorial plane of the Kerr spacetime. The $t$-$\phi$ 2-metric in Boyer-Lindquist coordinates is

$$ds^2_{(t,\phi)} = (-1+2\mathcal{M}/r)dt^2 - 4a\mathcal{M}/r dt d\phi + (r^2+a^2+2a^2\mathcal{M}/r)d\phi^2 \ , \tag{28}$$

and the geodesic angular velocities are

$$\dot{\phi}_\pm^{-1} = a \pm (r^3/\mathcal{M})^{1/2} \ . \tag{29}$$

In the limit $r\to\infty$ (which corresponds to a realistic situation for experiments in the solar system) the coordinate gamma factors reduce to

$$\Gamma_\pm \simeq 1 + \frac{3\mathcal{M}}{2r} + \frac{27\mathcal{M}^2}{8r^2} \mp 3\frac{a}{r}(\frac{\mathcal{M}}{r})^{3/2} \ , \tag{30}$$

so that

$$\begin{aligned}
\zeta_{(\mathrm{nmp})} &\simeq -2\frac{a\mathcal{M}}{r^3} \ , \\
\zeta_{(\mathrm{gmp})} &\simeq -\frac{a\mathcal{M}}{r^3} \ , \\
\zeta_{(\mathrm{ext})} &\simeq -\frac{a\mathcal{M}}{r^3}(1+3\frac{\mathcal{M}}{r}) \ .
\end{aligned} \tag{31}$$

Denoting by $m$ the 4-velocity of the static observers (with $\zeta=0$) one finds

$$\begin{aligned}
\Delta\tau_{(1-c)}(m,U_-,U_+) &\simeq 4\pi a\ ,\\
\Delta\tau_{(1-c)}(U_{(\rm ext)},U_-,U_+) &\simeq -\frac{12\pi a\mathcal{M}}{r}(1+\frac{2\mathcal{M}}{r})\ ,\\
\Delta\tau_{(2-c)}(m,U_-,U_+) &\simeq 4\pi a(1+\frac{3\mathcal{M}}{2r})\ ,\\
\Delta\tau_{(2-c)}(U_{(\rm gmp)},U_-,U_+) &\simeq \frac{12\pi a\mathcal{M}}{r}(1+\frac{3\mathcal{M}}{2r})\ .
\end{aligned} \tag{32}$$

## 5 The Carter observers

In black hole spacetimes, the 4-velocity $u_{(\rm car)}$ of the Carter observers [20] in the equatorial plane is tied to those of the usual threading observers ($m$) and the circular geodesics ($U_\pm$) by the following pair of parallel relationships at the global and local levels

$$\begin{aligned}
\int_{C_{\bar u_{(\rm car)}}} dt &= \frac{1}{2}[\int_{C_{U_+}} dt - \int_{C_{U_-}} dt]\ ,\\
\frac{1}{\bar\zeta_{(\rm car)}} &= \frac{1}{2}[\frac{1}{\dot\phi_+}+\frac{1}{\dot\phi_-}]\ ,
\end{aligned} \tag{33}$$

where $\zeta_{(\rm car)}=a/(r^2+a^2)$, $\bar\zeta_{(\rm car)}=1/a$. The right hand side of the first equation is half the coordinate single-clock clock effect for the threading observers (compare with Eq. (16)), while its left hand side is the corresponding coordinate synchronization gap for the Carter observers, where the loop is defined with respect to the threading observers (compare with Eq. (17)).

This is in direct analogy with the same pair of corresponding equations for the photon orbits and the threading observer Sagnac effect

$$\begin{aligned}
\int_{C_{\bar m}} dt &= \frac{1}{2}[\int_{C_{P_+}} dt - \int_{C_{P_-}} dt]\ ,\\
\frac{1}{\bar\zeta_{(\rm thd)}} &= \frac{1}{2}[\frac{1}{\zeta_+}+\frac{1}{\zeta_-}]\ .
\end{aligned} \tag{34}$$

These reciprocal averaging relationships are complementary to the direct averaging relationships $\zeta_{(\rm gmp)}=(\dot\phi_+ + \dot\phi_-)/2$, $\zeta_{(\rm sl)}=\zeta_{(\rm nmp)}=(\zeta_+ + \zeta_-)/2$. All four such relationships may be restated in terms of averaging relationships among the coordinate-time renormalized 4-velocities and corresponding 1-forms and their orthogonal counterparts (i.e. the coefficient of $\partial_t$ or $dt$ is 1). In contrast, the extremely accelerated observers are defined by an averaging relationship for the unit 4-velocities.

## 6 Conclusions

The various gravitomagnetic clock effects in stationary axially symmetric spacetimes have been discussed from a relative observer point of view, including the closely related Sagnac and desynchronization effects. The analysis has shown the special roles played by the 'geodesic meeting point observers' and the 'extremely accelerated observers', which have been previously introduced in the literature in a different context (test particle motion, relativistic definition of inertial forces). The discussion is also relevant in view of possible solar system experiments to test the validity of the theory of general relativity.

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## References

[1] Bini D Jantzen R T and Mashhoon B 2001 to appear in *Class. Quantum Grav.*

[2] Bini D Carini P and Jantzen R T 1997 *Int. J. Mod. Phys.* **D6** 143

[3] Cohen J M and Mashhoon B 1993 *Phys. Lett. A* **181** 353

[4] Bonnor W B and Steadman B R 1999 *Class. Quantum Grav.* **16** 1853

[5] Semerák O 1999 *Class. Quantum Grav.* **16** 3769

[6] Mitskevich N V and Pulido Garcia I 1970 *Sov. Phys. Dokl.* **15** 591 (see reference [7])

[7] Ferrari V and Mashhoon B 1984 *Phys. Rev.* **D30** 295

[8] Tartaglia A 2000 *Class. Quantum Grav* **17** 783

[9] Tartaglia A 2000 *Class. Quantum Grav.* **17** 2381

[10] Post E J 1967 *Rev. Mod. Phys.* **39** 475

[11] Ashtekar A and Magnon A 1975 *J. Math. Phys.* **16** 341

[12] Henriksen R H and Nelson L A 1985 *Can. J. Phys.* **63** 1393

[13] Anderson R Bilger H R and Stedman G E 1994 *Am. J. Phys.* **62** 975

[14] Stedman G E 1985 *Contemp. Phys.* **26** 311

[15] Stedman G E 1997 *Rep. Prog. Phys.* **60** 615

[16] de Felice F 1994 *Class. Quantum Grav.* **11** 1283; 1995 *Class. Quantum Grav.* **12** 1119

[17] Semerák O 1996 *Gen. Rel. Grav.* **28** 1151

[18] Page D 1998 *Class. Quantum Grav.* **15** 1669

[19] Bazański S L 1997 in *Mathematics of Gravitation Part II, Gravitational Wave Detection* Banach Center Publications **41** Polish Academy of Sciences Warsaw 273

[20] Carter B 1968 *Commun. Math. Phys.* **10** 280