Reinterpretation of the Mass Formula for Black Holes

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Abstract

Since these elementary considerations about the black hole mass formula were not given in the original articles, nor seem to be found in any reviews since those early days of the renaissance of general relativity, it seems useful to present them now.

The black hole mass formula

A charged rotating black hole is completely determined by the values of the black hole mass m, charge e and specific angular momentum $a = L/m$ parameters [1]. The corresponding spacetime is described by the Kerr-Newmann metric [2, 3] usually expressed in Boyer-Lindquist spherical-like coordinates (t, r, θ, ϕ) [4]. We use geometrical units in which $c = 1$. When the inequality

$$m^2 - e^2 - a^2 \geq 0 \tag{1}$$

is satisfied, the black hole has an outer horizon at the Boyer-Lindquist radial coordinate value

$$r_+ = m + (m^2 - a^2 - e^2)^{1/2} \tag{2}$$

whose area is

$$A = 4\pi\rho_{(\mathrm{h})}^2 = 16\pi m_{(\mathrm{i})}^2 \tag{3}$$

and (coordinate) angular velocity is

$$\Omega_{(\mathrm{h})} = \frac{a}{r_+^2 + a^2} = \frac{a}{4m_{(\mathrm{i})}^2}\ , \tag{4}$$

where

$$\rho_{(\mathrm{h})} = (r_+^2 + a^2)^{1/2} = 2m_{(\mathrm{i})} \tag{5}$$

is the quasi-spheroidal [4, 7] cylindrical coordinate $\rho = (r^2 + a^2)^{1/2} \sin\theta$ of the outer horizon evaluated at the equatorial plane. An extreme black hole satisfies the equality in (1), corresponding to a surface in the parameter space where the condition $r_+ = m$ then holds.

The quantity $m_{(\mathrm{i})} = \frac{1}{2}\rho_{(\mathrm{h})} = \frac{1}{2}(r_+^2 + a^2)^{1/2}$ is the irreducible mass of the black hole. It parametrizes the mass m through the Christodoulou-Ruffini mass formula [5, 6], which can be put into various suggestive forms

$$\begin{aligned} m^2 &= \left(m_{(\mathrm{i})} + \frac{e^2}{2\rho_{(\mathrm{h})}}\right)^2 + \left(\frac{L}{\rho_{(\mathrm{h})}}\right)^2 && (6) \\ &= m_{(\mathrm{r})}^2 + p_{(\mathrm{h})}^2 && (7) \\ &= \left(\gamma m_{(\mathrm{r})}\right)^2\ , && (8) \end{aligned}$$

where

$$m_{(r)} = [m_{(i)} + e^2/(2\rho_{(h)})] = [m_{(i)} + e^2/(4m_{(i)})] \tag{9}$$

is an effective rest mass, $p_{(h)} = L/\rho_{(h)}$ is an effective momentum with corresponding speed

$$v_{(h)} = p_{(h)}/m = \frac{L}{m\rho_{(h)}} = \frac{a}{\rho_{(h)}} = \frac{a}{2m_{(i)}} = \rho_{(h)}\Omega_{(h)} \ , \tag{10}$$

and associated gamma factor

$$\gamma = [1 - v_{(h)}^2]^{-1/2} = [1 - a^2/(4m_{(i)}^2)]^{-1/2} \ . \tag{11}$$

Note that $v_{(h)}^2$ is a curious product of the angular velocity and angular momentum (the conjugate momentum variable) divided by the mass of the black hole

$$v_{(h)}^2 = a\Omega_{(h)} = \frac{L}{m}\Omega_{(h)} \ . \tag{12}$$

Rewriting a formula in two ways using either $m_{(i)}$ or $\rho_{(h)}$ in geometrical units corresponds to a mass variable or length variable in CGS units respectively, which is helpful in the interpretation of the obvious analogy between these various formulas for black hole quantities and certain formulas from mechanics and electromagnetism.

For example, in the small angular momentum limit, one has

$$m = m_{(i)} + \frac{e^2}{2\rho_{(h)}} + K \ , \tag{13}$$

where

$$K = \frac{1}{2}m_{(r)}v_{(h)}^2 = \frac{L^2}{2I_{(h)}} = \frac{1}{2}I_{(h)}\Omega_{(h)}^2 \tag{14}$$

and

$$I_{(h)} = m_{(r)}\rho_{(h)}^2 \tag{15}$$

seems to play the role of a moment of inertia in this analogy, in this limit.

Solving the mass formula differential equation

Considerations involving the Penrose process [8] led to the following exact differential relationship between the black hole parameters for holes connected by reversible transformations as discussed by Christodoulou and Ruffini [6]

$$dm = \frac{(L/m)dL + r_+ede}{r_+^2 + L^2/m^2} \ . \tag{16}$$

Changing variables from L to a, this has the equivalent form

$$dm = \frac{ma}{r_+^2}da + \frac{e}{r_+}de \ . \tag{17}$$

Re-arranging terms yields

$$\frac{m - r_+}{r_+}ada = r_+dm - ede - ada = (r_+ - m)dr_+ \ , \tag{18}$$

using the definition of r_+ and then with further re-arranging

$$0 = (r_+ - m)(r_+dr_+ + ada) = r_+dm - ede - ada = (r_+ - m)d(r_+^2 + a^2) = (r_+ - m)d(4m_{(i)}^2) \ . \tag{19}$$

Thus the solution either is the extreme black hole condition $r_+ = m$ or the irreducible mass $m_{(i)}$ is constant, leading in the second case to the black hole mass formula when this condition is solved for m. These two conditions together imply that the surface of constant irreducible mass is tangent to the extreme case surface at its curve of intersection within the parameter space.

Inequalities and the extreme case

The differential equation (16) which produces the mass formula as its solution is only valid on the part of the parameter space corresponding to extreme or under extreme black holes where r_+ is real and a horizon exists. The mass formula represents a family of surfaces in this part of the (m, e, L) parameter space parametrized by the irreducible mass $m_{(\mathrm{i})}$. One may solve the mass formula for this parameter to see the implicit form of these surfaces. Multiplying through by the factor $m_{(\mathrm{i})}^2$ leads to a quadratic equation in $m_{(\mathrm{i})}^2$. Only the plus root solution is relevant

$$m_{(\mathrm{i})}^2 = [m^2 - e^2/2 + m(m^2 - e^2 - a^2)^{1/2}]/2 \ . \tag{20}$$

This is equivalent to the original formula (5) for $m_{(\mathrm{i})}$, using (5) and the defining equation for the horizon coordinate

$$r_+^2 - 2mr_+ + e^2 + a^2 = 0 \ . \tag{21}$$

Re-expressing the inequality (1) in terms of the irreducible mass leads to

$$m^2 - e^2 - L^2/m^2 = [m_{(\mathrm{i})}^2 - (e^2/4 + L^2)/(4m_{(\mathrm{i})}^2)]^2 \geq 0 \tag{22}$$

but the expression inside the brackets must itself be nonnegative in order that the formula for $m_{(\mathrm{i})}$ be consistent (the radical must equal $m_{(\mathrm{i})}^2 - [m^2 - e^2/2]/2$ so the latter expression must be nonnegative, leading to this result), so one obtains the inequality

$$\frac{e^4}{16m_{(\mathrm{i})}^4} + \frac{L^2}{4m_{(\mathrm{i})}^4} \leq 1 \ . \tag{23}$$

This is equivalent to the simpler relation for the parameters $(m_{(\mathrm{i})}, e, a)$

$$\frac{e^2}{4m_{(\mathrm{i})}^2} + \frac{a^2}{2m_{(\mathrm{i})}^2} \leq 1 \ , \tag{24}$$

which is an ellipse in the e-a plane, suggesting the natural parametrization (determining m from equations (6) and (11))

$$e = 2m_{(\mathrm{i})} \sin\zeta \ , \ a = \sqrt{2}m_{(\mathrm{i})} \cos\zeta \ , \ m = \sqrt{2}m_{(\mathrm{i})}\sqrt{1 + \sin^2\zeta} \ , \ \zeta \in [0, 2\pi) \ . \tag{25}$$

An alternative is

$$a = 2m_{(\mathrm{i})} \sin\psi \ , \ m = 2m_{(\mathrm{i})} \cos\psi \ , \ e = 2m_{(\mathrm{i})}\sqrt{\cos 2\psi} \ , \ \psi \in [0, 2\pi) \ . \tag{26}$$

The trigonometric parametrization adapted to the corresponding fourth power condition used by Christodoulou and Ruffini [6] is instead

$$L = 2m_{(\mathrm{i})}^2 \cos\chi \ , \ e = \pm 2m_{(\mathrm{i})}\sqrt{\sin\chi} \ , \ m = \sqrt{2}m_{(\mathrm{i})}\sqrt{1 + \sin\chi} \ , \ \chi \in [0, \pi] \ . \tag{27}$$

Another form for this inequality is

$$|v_{(\mathrm{h})}| \leq \frac{1}{\sqrt{2}}\left(1 - \frac{e^2}{4m_{(\mathrm{i})}^2}\right)^{1/2} \ . \tag{28}$$

Subsequent interpretations

No discussion of the black hole mass formula would not be complete without a few words on its variations that have appeared since its original formulation associated with black hole thermodynamics, initiated by Bekenstein in his PhD thesis at Princeton [9, 10], whose re-interpretation led to the four laws of black hole mechanics [11]. Smarr [12] gives an alternative parametrization of the 3-parameter black hole family in a similar vein.

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