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The kinematical role of automorphisms in the orthonormal frame approach to Bianchi cosmology

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The automorphism group and frame commutator relations in the orthonormal frame approach to Bianchi cosmology are used to construct an explicit coordinate representation of the orthonormal frame itself (and hence of the spacetime metric) which depends algebraically on the connection coefficients. This is not possible in general inhomogeneous models where differential equations must instead be solved. The shift vector field required for this procedure is intimately related to the true Smarr–York minimal strain and minimal distortion shifts. © *1999 American Institute of Physics.* [S0022-2488(99)01701-6]

I. INTRODUCTION

When studying spatially homogeneous (SH) Bianchi cosmology, two complementary approaches have been taken, one using orthonormal frames in which the metric components are fixed and the dynamics resides in the commutation functions (see references in Ref. 1) and the other using computational frames[2] in which the commutation functions are fixed and the dynamics resides in the metric components (see references in Ref. 3). The relationship between these two has been clouded by the fact that one usually uses a synchronous frame in the computational frame approach to Bianchi cosmology. By instead choosing a computational frame based on a suitable shift vector field intimately related to the true Smarr–York minimal strain and minimal distortion shifts,[4,5] one can construct an explicit coordinate representation of all the orthonormal frame vectors (and therefore a coordinate representation of the spacetime metric) using an algebraic procedure involving the commutator functions and commutator relations of the orthonormal frame approach.

For general inhomogeneous models, such an algebraic procedure is not possible and one is instead forced to solve differential equations resulting from the commutator relations in order to obtain a coordinate representation of the orthonormal frame. The closest one might come to the present SH construction in more general inhomogeneous cases would be to use a computational frame with a minimal strain or minimal distortion shift vector field. It is remarkable that the usual orthonormal frame approach to Bianchi cosmology is so closely connected to these general ideas about fixing the coordinate gauge freedom in evolving a spacetime from initial data.

Bianchi cosmology has long served as a testing ground for exploring features of general relativity both in generalizing aspects of these highly symmetric models to the broader context of more general inhomogeneous spacetimes and in specializing results from the general theory to explore them in SH models which facilitate computations. While one may not need the metric explicitly to answer questions about Bianchi models alone, an explicit representation of the metric is essential for answering many interesting questions about general inhomogeneous spacetimes.

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TABLE I. Canonical structure constants for each Bianchi symmetry type. Note the special case Bianchi type III$=$VI$_{-1}$ with $\hat{a}=1$.

	Class A						Class B			
Type	IX	VIII	VII$_0$	VI$_0$	II	I	VII$_h$	VI$_h$	IV	V
$\hat{n}_1$	1	1	1	1	1	0	1	1	1	0
$\hat{n}_2$	1	1	1	-1	0	0	1	-1	0	0
$\hat{n}_3$	1	-1	0	0	0	0	0	0	0	0
$\hat{a}\geqslant 0$	0	0	0	0	0	0	$\hat{a}$	$\hat{a}$	1	1

Thus relating results of Bianchi cosmology to a wider setting requires the construction of the metric.

The outline of this article is as follows. In Sec. II the symmetry group properties of this class of spacetimes are summarized. In Sec. III the essentials of the orthonormal frame approach are reviewed. In Sec. IV the general framework is described for constructing the metric in a computational frame starting from the commutation functions of an orthonormal frame and then a discussion of why it works is given. In Sec. V the metric is explicitly constructed for each Bianchi type. The last section ends with concluding remarks.

II. SYMMETRY PROPERTIES

In the present discussion we consider only those symmetry types for which the full spacetime symmetry group admits a simply transitive 3-dimensional subgroup G acting on the spatially homogeneous (SH) hypersurfaces, i.e., a Bianchi group action. Such spacetimes admit a class of spatial frames $\{\hat{\mathbf{e}}_i\}$ $(i=1,2,3)$ tangent to each hypersurface which are not only invariant under the action of the group but which have structure or commutator functions $\hat{C}^k_{ij}$ which are constants throughout the spacetime. These invariant vector fields thus themselves generate a transformation group which turns out to be isomorphic to the original Bianchi group. The constant structure functions are defined by

$$[\hat{\mathbf{e}}_i,\hat{\mathbf{e}}_j]=\hat{C}^k_{ij}\hat{\mathbf{e}}_k, \tag{2.1}$$

and may be represented in the form (see e.g., Ref. 6)

$$\hat{C}^k_{ij}=2\hat{a}_{[i}\delta^k_{j]}+\epsilon_{ijl}\hat{n}^{lk}, \tag{2.2}$$

where the Jacobi identities $\hat{C}^l{}_{m[i}\hat{C}^m{}_{jk]}=0$ require

$$0=\hat{n}^{ij}\hat{a}_j. \tag{2.3}$$

A useful parameter h may be defined by

$$\hat{a}_i\hat{a}_j=\frac{1}{2}\,h\epsilon_{ikl}\epsilon_{jmn}\hat{n}^{km}\hat{n}^{ln}. \tag{2.4}$$

The Bianchi symmetry types may be divided into 2 symmetry classes, A and B, depending on whether $\hat{a}_i$ is zero or not.

It is often convenient to choose a gauge in which the structure constants $\hat{n}_{ij}$ are diagonal, and the covector $\hat{a}_i$ is aligned with one of the basis directions, here chosen to be the third one,

$$\hat{n}_{ij}=\mathrm{diag}(\hat{n}_1,\hat{n}_2,\hat{n}_3),\quad \hat{a}_i=(0,0,\hat{a}). \tag{2.5}$$

This ''diagonal-alignment'' gauge will be assumed here. Canonical choices of the structure constants for each Bianchi type are given in this form in Table I.[6] The Bianchi symmetry group action

TABLE II. Dimensions of the adjoint and automorphism groups for each Bianchi symmetry type.

Type	VIII, IX	IV, VI, VII	III	V	II	I
dim(Ad)	3	3	2	3	2	0
dim(Aut)	3	4	4	6	6	9

is assumed to be a global action by a simply connected group. (The discussion of local symmetry actions which are not global is considerably more complicated.)

The automorphism matrix group of the Lie algebra with structure constants $\hat{C}^i{}_{jk}$ is the subgroup of linear transformations of its basis $\{\hat{\mathbf{e}}_i\}$ which leaves those constants invariant,

$$B^k{}_l \hat{C}^l{}_{mn} B^{-1m}{}_i B^{-1n}{}_j = \hat{C}^k{}_{ij} . \tag{2.6}$$

The Lie algebra of this matrix group consists of the matrix derivations of the original Lie algebra,

$$F^k{}_l \hat{C}^l{}_{ij} = \hat{C}^k{}_{lj} F^l{}_i + \hat{C}^k{}_{il} F^l{}_j . \tag{2.7}$$

The matrix adjoint group is the subgroup of inner automorphisms generated by the matrix Lie algebra whose basis consists of a linearly independent subset of the adjoint matrices $\hat{k}_i$ defined by $[\hat{k}_i]^j{}_k = \hat{C}^j{}_{ik}$ representing the inner derivations (Lie bracketing by elements of the original Lie algebra),

$$\pounds_{\hat{\mathbf{e}}_i} \hat{\mathbf{e}}_j = [\hat{k}_i]^l{}_j \hat{\mathbf{e}}_l . \tag{2.8}$$

These matrices satisfy the derivation property (2.7) due to the Jacobi identities. Automorphisms which are not inner are called outer automorphisms. The dimensions of the adjoint and automorphism groups are given in Table II. Their differences represent the number of independent outer automorphism generators which exist in any basis of the full matrix automorphism Lie algebra which includes a basis of the matrix adjoint Lie algebra. The automorphism structure summarized in Table II is important for the algebraic procedure for constructing the metric from the commutator functions.

III. THE ORTHONORMAL FRAME APPROACH

Let $\{\mathbf{e}_a\}$ be a SH orthonormal frame $(a=0,1,2,3)$ with dual frame $\{\boldsymbol{\omega}^a\}$ (satisfying $\langle \boldsymbol{\omega}^a, \mathbf{e}_b \rangle = \delta^a{}_b$) so that the metric takes the form

$$\mathbf{g} = \eta_{ab} \boldsymbol{\omega}^a \boldsymbol{\omega}^b , \tag{3.1}$$

where $(\eta_{ab}) = \mathrm{diag}(-1,1,1,1)$. Choose $\mathbf{e}_0 = \mathbf{n} = n^a \mathbf{e}_a$ to be the unit normal vector field of the SH hypersurfaces. The remaining frame vector fields are then tangent to the SH hypersurfaces and so are related to any set of invariant spatial frame vectors $\hat{\mathbf{e}}_i$ by a linear transformation which is constant on any given such hypersurface.

The full set of commutator functions $\gamma^a{}_{bc}$ are defined by

$$[\mathbf{e}_a , \mathbf{e}_b] = \gamma^c{}_{ab} \mathbf{e}_c , \quad \mathbf{d}\boldsymbol{\omega}^a = -\tfrac{1}{2} \gamma^a{}_{bc} \boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c . \tag{3.2}$$

As a consequence of the symmetry and hypersurface-forming condition, the normal $\mathbf{n}$ has zero acceleration and rotation. Thus, making a $3+1$ decomposition of these functions leads to[1]

$$\gamma^\alpha{}_{0\beta} = -\theta^\alpha_\beta + \epsilon^\alpha{}_{\beta\gamma} \Omega^\gamma , \qquad \gamma^\gamma{}_{\alpha\beta} = 2 a_{[\alpha} \delta^\gamma{}_{\beta]} + \epsilon_{\alpha\beta\delta} n^{\delta\gamma} , \tag{3.3}$$

where $\epsilon^{\alpha\beta\gamma}$ is the permutation symbol satisfying $\epsilon^{123}=1$ and $\alpha=1,2,3$. The quantity Ω^{α} can be interpreted as the local angular velocity of a spatial frame $\{\mathbf{e}_\alpha\}$ with respect to a second spatial frame $\{\overline{\mathbf{e}}_\alpha\}$ which is Fermi-propagated along $\mathbf{e}_0=\mathbf{n}$. The quantity θ_{ab} is the expansion tensor, which is often represented in terms of the trace-free shear tensor σ_{ab}, the expansion scalar Θ, and the spatial metric $h_{ab}=g_{ab}+n_a n_b$ as $\theta_{ab}=\sigma_{ab}+\frac{1}{3}\Theta h_{ab}$. The purely spatial components $\gamma^{\alpha}{}_{\beta\gamma}$ have been decomposed in the same way as $\hat{C}^{k}{}_{ij}$ and are assumed to have the same diagonal-alignment gauge form as the canonical structure constants for each Bianchi type, with the correspondingly defined structure functions (n_1,n_2,n_3,a) having the same signs (when nonzero) as the corresponding structure constants $(\hat{n}_1,\hat{n}_2,\hat{n}_3,\hat{a})$. The symbol n_α will be used only to designate these structure functions in the remainder of the paper, and not the covariant spatial components of the normal vector field, which are no longer needed.

The $3+1$ decomposition of the Jacobi identities,

$$[[\mathbf{e}_a,\mathbf{e}_b],\mathbf{e}_c]+[[\mathbf{e}_b,\mathbf{e}_c],\mathbf{e}_a]+[[\mathbf{e}_c,\mathbf{e}_a],\mathbf{e}_b]=0\Leftrightarrow \mathbf{e}_{[a}(\gamma^{d}{}_{bc]})+\gamma^{e}{}_{[ab}\gamma^{d}{}_{c]e}=0, \tag{3.4}$$

leads to

$$\mathbf{e}_0(a^{\alpha})=-a_\beta\theta^{\alpha\beta}-\epsilon^{\alpha\beta\gamma}a_\beta\Omega_\gamma, \tag{3.5}$$

$$\mathbf{e}_0(n^{\alpha\beta})=-\Theta n^{\alpha\beta}+2\,\theta^{(\alpha}{}_{\gamma}n^{\beta)\gamma}-2\,\epsilon^{\gamma\delta(\alpha}n^{\beta)}{}_{\gamma}\Omega_\delta, \tag{3.6}$$

$$0=a_\beta n^{\alpha\beta}. \tag{3.7}$$

The diagonal-alignment conditions imposed on the structure functions cause certain components of the first two derivative equations to have an identically zero left hand side, leading to certain relationships among components of Ω^{α} and $\theta_{\alpha\beta}$ when the right hand side is not also identically zero. Two relationships follow from the first two components of Eq. (3.5) when $a\neq 0$ and three from the off-diagonal components of (3.6), all of which must be identically zero. These restrictions are

$$a(\theta_{13}-\Omega_2)=0=a(\theta_{23}+\Omega_1), \tag{3.8}$$

and

$$(n_1+n_2)\theta_{12}+(n_1-n_2)\Omega_3=0, \tag{3.9}$$

and its two cyclic permutations.

IV. CONSTRUCTION OF THE METRIC: GENERAL FRAMEWORK

Once the commutation functions have been obtained as explicit functions of some time function t (for example, by solving the constraint and evolution equations for some metric theory with a particular choice of the lapse function N), it is possible to construct the spacetime metric explicitly in terms of local coordinates $\{t,x^i\}$ $(i=1,2,3)$ adapted to the SH hypersurfaces. This is accomplished without solving any additional differential equations, using the particular structure that is associated with the SH symmetry.

Here the orthonormal spatial frame $\{\mathbf{e}_\alpha\}$ will be expressed first in terms of an invariant spatial frame $\{\hat{\mathbf{e}}_i\}$ with canonical structure constant values for its structure functions and then in terms of local coordinates. The full orthonormal frame is related to the computational frame $\{\boldsymbol{\partial}_t,\hat{\mathbf{e}}_i\}$ by

$$\mathbf{e}_0=N(t)^{-1}(\boldsymbol{\partial}_t-\hat{N}^i(t,\mathbf{x})\hat{\mathbf{e}}_i),\quad \mathbf{e}_\alpha=\hat{e}(t)_\alpha{}^i\hat{\mathbf{e}}_i, \tag{4.1}$$

where N is the lapse function and $\vec{\mathbf{N}}=\hat{N}^i\hat{\mathbf{e}}_i$ is the shift vector field. The corresponding dual 1-forms are given by

$$\boldsymbol{\omega}^0 = N(t)\mathbf{d}t, \quad \boldsymbol{\omega}^\alpha = \hat{e}^\alpha{}_i[\hat{\boldsymbol{\omega}}^i + \hat{N}(t,\mathbf{x})^i\mathbf{d}t], \tag{4.2}$$

where $\hat{D} = (\hat{e}^\alpha{}_i)$ is the inverse matrix of $(\hat{e}_\alpha{}^j)$ (i.e., $\hat{e}^\alpha{}_i\hat{e}_\alpha{}^j = \delta^j{}_i$).

This leads to the following spacetime metric:

$$\mathbf{g} = -N(t)^2\ \mathbf{d}t^2 + \hat{g}_{ij}(t)[\hat{\boldsymbol{\omega}}^i + \hat{N}^i(t,\mathbf{x})\mathbf{d}t][\hat{\boldsymbol{\omega}}^j + \hat{N}^j(t,\mathbf{x})\mathbf{d}t], \tag{4.3}$$

where the SH components of the spatial metric tensor in the spatial frame $\{\hat{\mathbf{e}}_i\}$ are given by

$$\hat{g}_{ij} = \delta_{\alpha\beta}\hat{e}^\alpha{}_i\hat{e}^\beta{}_j, \tag{4.4}$$

or simply $(\hat{g}_{ij}) = \hat{D}^T\hat{D}$ in matrix notation.

The computational frame $\{\boldsymbol{\partial}_t, \hat{\mathbf{e}}_i\}$ is characterized by the Lie dragging condition $\pounds_{\boldsymbol{\partial}_t}\hat{\mathbf{e}}_i = 0$ which implies the time-independent local coordinate expression $\hat{\mathbf{e}}_i = \hat{e}_i{}^j(x^k)\boldsymbol{\partial}_j$ for the invariant spatial frame, whose constant commutator functions $\hat{C}^k{}_{ij}$ are assumed to have their canonical values given in Table I. Explicit coordinate expressions for $\hat{e}_i{}^j(x^k)$ follow from the representation of the left invariant vector fields in canonical coordinates of the second kind in Refs. 7 and 8. These spatial coordinates will be assumed throughout this article. Similarly, explicit coordinate expressions for a basis of the homogeneity Killing vector fields follow from the representation of the right invariant vector fields in these coordinates.[7,8]

The lapse function is SH, but the associated shift vector field is not necessarily SH. In fact exploiting the action of the outer automorphisms requires an inhomogeneous shift. However, the shift Lie derivative of the spatial frame $\hat{\mathbf{e}}_i$ and its dual must be SH,

$$\pounds_{\vec{\mathbf{N}}}\hat{\mathbf{e}}_i = -\hat{A}(t)^j{}_i\hat{\mathbf{e}}_j, \quad \pounds_{\vec{\mathbf{N}}}\hat{\boldsymbol{\omega}}^j = \hat{A}(t)^j{}_i\hat{\boldsymbol{\omega}}^i, \tag{4.5}$$

in order that the Lie derivative of the induced spatial metric be SH for any component matrix $(\hat{g}_{ij})$,

$$\pounds_{\vec{\mathbf{N}}}\hat{g}_{ij} = 2\hat{g}_{k(i}\hat{A}^k{}_{j)}. \tag{4.6}$$

This in turn guarantees that the extrinsic curvature (sign-reversed expansion tensor) be SH under the same condition,

$$\hat{K}_{ij} = \frac{1}{2}N^{-1}[-\dot{\hat{g}}_{ij} + \pounds_{\vec{\mathbf{N}}}\hat{g}_{ij}] = -\hat{e}_i{}^\alpha e_j{}^\beta\theta_{\alpha\beta}. \tag{4.7}$$

Thus, for a fixed value of t, this restricts the shift to the finite-dimensional space of derivations of the Lie algebra of invariant spatial vector fields.[7] This derivation Lie algebra (containing the homogeneity Killing vector fields which correspond to the trivial zero derivation) generates an action of the automorphism-translation group of the Bianchi homogeneity group G on each SH hypersurface which induces the action of the matrix automorphism group on the invariant spatial vector fields under Lie dragging. Given a basis for the matrix derivation Lie algebra $\{\kappa_P\}$ (so that P is an index taking values from 1 to the dimension of the automorphism group given in Table II), one can construct a corresponding basis $\{\boldsymbol{\xi}_P\}$ for the Lie algebra of derivation vector fields modulo Killing vector fields by the relation

$$\pounds_{\boldsymbol{\xi}_P}\hat{\mathbf{e}}_i = [\kappa_P]^j{}_i\hat{\mathbf{e}}_j. \tag{4.8}$$

One can then express the shift and its corresponding derivation matrix as time-dependent linear combinations of these respective bases with SH coefficients,

$$-\hat{A}(t) = M^P(t)\kappa_P \rightarrow \vec{\mathbf{N}} = M^P(t)\boldsymbol{\xi}_P. \tag{4.9}$$

This relationship may then be used to determine the shift vector field from the matrix $\hat{A}$. The basis vector fields $\boldsymbol{\xi}_P$ can be taken from a subset of invariant spatial frame vector fields $\hat{\mathbf{e}}_i$ which generate the inner automorphisms, plus some independent outer automorphism generators. Coordinate expressions for the additional independent outer automorphism vector field generators for each symmetry type, most of which may be found in Appendix 3 of Ref. 9, will be given below, while expressions for the invariant spatial vector fields themselves have already been discussed. All of these follow from known results for 3-dimensional Lie groups.

In the diagonal-alignment gauge the action of the matrix automorphism group on the spatial metric induced by the action of Lie dragging by the automorphism-translations,

$$\hat{g}_{ij} \rightarrow B^m{}_i B^n{}_j \hat{g}_{mn}, \tag{4.10}$$

has orbits which may be parametrized by a submanifold of the space of diagonal metric matrices (not unique when diagonal automorphism matrices exist). It is exactly this fact which allows one to assume a shift for which the spatial metric component matrix is confined to such a diagonal submanifold, i.e., making the matrix $\hat{D}$ diagonal. This shift generates a time-dependent matrix automorphism which transforms the orthogonal (zero shift) gauge metric matrix into the diagonal matrix $(\hat{g}_{ij})$.

One may evaluate the relationship between the structure functions of the original orthonormal frame and the computational frame by inserting the expressions (4.1) into Eqs. (3.2) and (3.3) leading to

$$\gamma^\alpha{}_{0\beta} = N^{-1}[-\dot{\hat{e}}^\alpha{}_i \hat{e}_\beta{}^i + \hat{e}^\alpha{}_i \hat{A}^i{}_j \hat{e}_\beta{}^i] = -\theta^\alpha_\beta + \epsilon^\alpha{}_{\beta\gamma}\Omega^\gamma, \tag{4.11}$$

$$\gamma^\alpha{}_{\beta\gamma} = \hat{e}^\alpha{}_i \hat{C}^i{}_{jk} \hat{e}_\beta{}^j \hat{e}_\gamma{}^k. \tag{4.12}$$

The first of these in matrix notation takes the form

$$(\gamma^\alpha{}_{0\beta}) = N^{-1}[-\dot{\hat{D}}\hat{D}^{-1} + \hat{D}\hat{A}\hat{D}^{-1}] = (-\theta^\alpha{}_\beta + \epsilon^\alpha{}_{\beta gg}\Omega^\gamma). \tag{4.13}$$

The shift may be chosen so that the matrix $\hat{D}$ is diagonal and positive-definite,

$$\hat{D} = (\hat{e}^\alpha{}_i) = \mathrm{diag}(e^{\beta^1}, e^{\beta^2}, e^{\beta^3}), \tag{4.14}$$

as will always be assumed here, with the number of independent components equal to three minus the number of independent diagonal automorphisms. This matrix represents the time-dependent rescaling of the orthogonal spatial frame $\{\hat{\mathbf{e}}_i\}$ which normalizes it to the orthonormal spatial frame $\{\mathbf{e}_\alpha\}$ and transforms the nonzero structure constants of the first frame into the corresponding nonzero time-dependent structure functions of the second frame by Eq. (4.12),

$$n_1 = e^{\beta^1 - \beta^2 - \beta^3}\hat{n}_1, \quad n_2 = e^{\beta^2 - \beta^3 - \beta^1}\hat{n}_2, \quad n_3 = e^{\beta^3 - \beta^1 - \beta^2}\hat{n}_3, \tag{4.15}$$

$$a = e^{-\beta^3}\hat{a}. \tag{4.16}$$

When $\hat{D}$ is diagonal, the (index-lowered) symmetric part of Eq. (4.13) is equivalent to the orthonormal components of the mixed form of Eq. (4.7), evaluating the extrinsic curvature or sign-reversed expansion tensor,

$$N^{-1}[-\dot{\hat{D}}\hat{D}^{-1} + \tfrac{1}{2}(\hat{D}\hat{A}\hat{D}^{-1}) + \tfrac{1}{2}(\hat{D}\hat{A}\hat{D}^{-1})^T] = \hat{D}(\hat{K}^i{}_j)\hat{D}^{-1}, \tag{4.17}$$

while the antisymmetric part of Eq. (4.13) relates the shift Lie derivative term involving the matrix $\hat{A}$ to the local angular velocity Ω^α of the spatial orthonormal frame [or to the off-diagonal components of the expansion tensor, due to Jacobi identities of the form (3.8) and (3.9)]. The particular way in which this diagonal matrix $\hat{D}$ is fixed when some freedom remains allows one to specialize the shift vector field either to a true minimal strain or minimal distortion shift. These shifts minimize the contribution of the diagonal time derivative term,

$$-\dot{\hat{D}}\hat{D}^{-1} = -(\ln \hat{D})^{\cdot} = \mathrm{diag}(\dot{\beta}^1, \dot{\beta}^2, \dot{\beta}^3), \tag{4.18}$$

to the formula (4.17) for the extrinsic curvature tensor in two different ways.

A true minimal strain shift is one for which the diagonal time-derivative term $-\dot{\hat{D}}\hat{D}^{-1}$ in Eq. (4.17) is orthogonal to the remaining two $\hat{A}$ terms in that expression under the trace inner product for second rank tensors, while the true minimal distortion shift is the one for which the trace-free part of $-\dot{\hat{D}}\hat{D}^{-1}$ is instead orthogonal to the trace-free part of the remaining two $\hat{A}$ terms in that expression.[5] The assumption that $\hat{D}$ is diagonal is consistent with the generic off-diagonal part of the second term in Eq. (4.13) for all Bianchi types in the diagonal-alignment gauge, leaving only the diagonal orthogonality conditions to be analyzed. By representing the logarithm of $\hat{D}$ as an arbitrary linear combination of a set of diagonal matrices which are each orthogonal to the matrix generators of the matrix automorphism group, one obtains a true minimal strain shift vector field. By representing it instead so that the trace-free parts are orthogonal, one obtains a true minimal distortion shift vector field. Since only the true minimal strain and distortion shifts are relevant to the SH case, the modifier ‘‘true’’ will be implicitly understood below.

A. The construction procedure

The procedure for constructing a coordinate representation of the metric consists of the following steps.

(1) Represent the diagonal matrix $\hat{D}$ in terms of a minimal set of variables which parametrize the quotient space of the diagonal metric matrices under the action of the diagonal automorphisms. When 1 or 2 independent diagonal automorphism generators exist, there is no natural choice for these variables, and a parameter ζ describes the most useful variations, allowing one to specialize to a minimal strain or minimal distortion shift if desired. This is done by choosing to parametrize $\ln \hat{D}$ or its trace-free part, respectively, so that it is always orthogonal to the diagonal matrix automorphism generators.

(2) Express the minimal diagonal variables in terms of the spatial commutation functions using Eqs. (4.15) and (4.16).

(3) Construct the diagonal (κ_D) and off-diagonal (κ_O) matrix automorphism generators, so that the matrix $-\hat{A}$ can be expressed as a linear combination of them, $-\hat{A} = M^D\kappa_D + M^O\kappa_O$.

(4) Use the diagonal components of Eq. (4.13) to express the time derivatives of the minimal $\hat{D}$ variables as functions of the diagonal components of $\theta_{\alpha\beta}$ and then use these results in the solution of the same equation for the automorphism coefficients (M^D, M^O) to express the latter entirely in terms of the commutation functions. Note that some of these may be expressed in several equivalent ways due to the Jacobi identities (3.8) and (3.9).

(5) Give coordinate expressions for the basis $\{\boldsymbol{\xi}_P\}$ of the automorphism vector fields (modulo Killing vector fields). $\{\hat{\mathbf{e}}_i\}$ provide a basis of the inner automorphism generators corresponding to the adjoint matrices $\{\hat{k}_i\}$ so one only needs coordinate expressions for the remaining independent outer automorphism vector fields which may be easily found from their matrices using the condition (4.8).

(6) Re-express the automorphism matrices as a linear combination of a linearly independent subset of the adjoint matrices $\hat{k}_i$ and the remaining outer automorphisms, so that one can then re-express the shift vector field $\vec{\mathbf{N}} = M^D\boldsymbol{\xi}_D + M^O\boldsymbol{\xi}_O$ as the same linear combination of the

corresponding invariant vectors $\hat{\mathbf{e}}_i$ and the remaining inhomogeneous outer automorphism generators.

The final result then gives a coordinate representation of the orthonormal frame vectors and of the metric whose time dependence is completely determined by the commutation functions through the diagonal matrix $\hat{D}=(\hat{e}^{\alpha}{}_i)$ and the shift coefficients M^D. The spatial homogeneity determines the remaining spatial coordinate dependence.

B. Why it works

Before carrying out these steps explicitly for the various symmetry types, it is worth explaining why the symmetry allows this procedure to work. As discussed in Ref. 7, the ''minigauge group'' of symmetry compatible diffeomorphisms for Bianchi cosmology is the subgroup of the spacetime diffeomorphism group which maps into itself both the space of SH spatial vector fields and also leaves invariant the normal vector field $\mathbf{n}$ to the SH hypersurfaces. Its corresponding Lie algebra consists of vector fields of the form $\mathbf{X}=X^{\perp}\mathbf{n}+\vec{\mathbf{X}}$, where $X^{\perp}$ is SH and the spatial vector field $\vec{\mathbf{X}}$ belongs to the ''automorphism-translation'' Lie algebra on each SH hypersurface.

Since the Bianchi symmetry group G acts simply transitively on its orbits, each orbit is diffeomorphic to G with its action on the orbit corresponding to its action on itself by left translation, and one may map the semidirect product group of automorphisms and (left or right) translations $\mathrm{Aut}(G)\otimes_s L(G)=\mathrm{Aut}(G)\otimes_s R(G)$ onto each orbit, the generators of which define this ''automorphism-translation'' Lie algebra. It is characterized by the condition that the Lie derivatives of the SH spatial vector fields by its elements are themselves SH. The SH spatial vector fields correspond to the left invariant vector fields on G, while the spacetime Killing vector fields generating the action of G correspond to the right invariant vector fields. A minigauge group diffeomorphism of the spacetime is then a 1-parameter (i.e., time-dependent) family of such automorphism-translations acting on the family of SH orbits.

The subgroup of diffeomorphisms generated by the SH spatial vector fields, when acting on the space of SH spatial vector fields by Lie dragging, induces the action of the linear inner automorphism or adjoint group on that space, while the full symmetry compatible diffeomorphism subgroup of automorphism-translations induces the action of the whole linear automorphism group. When expressed in terms of a given invariant spatial frame, these groups are represented by their corresponding matrix groups, which are entirely determined by the values of the structure constants for that frame.

Associated with every computational frame $\{\boldsymbol{\partial}_t,\hat{\mathbf{e}}_i\}$ is an equivalence class of comoving coordinate systems $\{t,x^i\}$ which establish an identification of the spacetime manifold with the product manifold $R\times G$. The usual synchronous gauge frame has the time lines aligned with the unit normal vector field $\mathbf{n}$, and any other symmetry compatible computational frame with the same structure constants is related to it by the action of a time-dependent automorphism matrix induced by the action of the related shift vector field in Lie dragging the original invariant spatial frame. Conversely time-dependent changes of an invariant spatial frame by a time-dependent automorphism matrix are equivalent to the choice of a new time direction for the computational frame. Thus one can reconstruct the associated shift for a new computational frame from a knowledge of the time-dependent automorphism which relates the spatial frames, modulo spacetime Killing vector fields which commute with the spatial frame vectors and induce no change in them nor in the spatial metric or extrinsic curvature, but only change the direction of the time lines of the associated comoving coordinate system since $\boldsymbol{\partial}_t=N\mathbf{n}+\vec{\mathbf{N}}$.

Given a choice of SH spatial frame $\{\hat{\mathbf{e}}_i\}$, the action of the symmetry compatible diffeomorphism group of automorphism-translations on this frame by Lie dragging induces the action (4.10) of the matrix automorphism group on the space of SH inner product matrices. For the diagonal-alignment gauge choice of the structure constants of such a frame, the orbits of this action can be parametrized by a submanifold of the diagonal inner product matrices (corresponding to orthogonal spatial frames). Thus, starting from a frame with arbitrary inner products in synchronous gauge, one can always produce from it an orthogonal frame with minimal freedom in those diagonal inner product components by using the automorphism matrix freedom to choose an appropriate new time direction via the corresponding generating shift vector field. Conversely, given any orthogonal invariant spatial frame with constant structure functions, one can always pick a vector field to complete it to a spacetime computational frame. This is equivalent to picking the shift which generates the time-dependent automorphism matrix which transforms the synchronous gauge inner product matrix to the one of the orthogonal computational frame. Solving the key equation (4.13) for a symmetry compatible shift vector field under the assumption that $\hat{D}$ is a diagonal matrix parametrizing the orbits of the matrix automorphism group determines this desired shift.

V. CONSTRUCTION OF THE METRIC: SPECIFIC BIANCHI TYPES

The metric is simultaneously constructed for each subset of Bianchi types listed in Table II according to common adjoint and automorphism dimensions, except for Bianchi type III$=$VI$_{-1}$, which is included with the type VI and VII discussion, and type IV which is treated separately. This is an artifact of the choices of automorphism matrix parametrizations made for the convenience of calculation.

A. Bianchi types VIII and IX

In this case there are no diagonal or outer automorphisms, $\hat{a}=0=a$, and $n_1 n_2 n_3 \neq 0 \neq \hat{n}_1 \hat{n}_2 \hat{n}_3$, and the equations (4.15) uniquely determine the β^α, yielding

$$e^{-2\beta^\alpha}=(n_\beta n_\gamma)/(\hat{n}_\beta \hat{n}_\gamma), \tag{5.1}$$

where (α,β,γ) is a cyclic permutation of (1,2,3). Given the canonical choice of $(\hat{n}_\alpha)$ for each symmetry type, the β^α variables are then expressed in terms of the structure functions n_α.

The relationship (4.13) may then be used to solve for the matrix automorphism generator $\hat{A}^i{}_j$. For these symmetry types, the matrix automorphism generators are off-diagonal, and consist entirely of inner automorphism generators belonging to the adjoint Lie algebra of the original Lie algebra with structure constants $\hat{C}^i{}_{jk}$, a basis for which consists of the three off-diagonal adjoint matrices,

$$[\kappa_i]^j{}_k=[\hat{k}_i]^j{}_k=\hat{C}^j{}_{ik}=\epsilon_{ikl}\hat{n}^{jl}, \tag{5.2}$$

so that

$$-\hat{A}^j{}_k=M^i[\kappa_i]^j{}_k=M^i[\hat{k}_i]^j{}_k\,. \tag{5.3}$$

Since there are no diagonal automorphisms, there is no need to examine the diagonal components of the key equation (4.13). Its off-diagonal components immediately determine the off-diagonal matrix $\hat{A}$ in terms of Ω^α and the off-diagonal components of $\theta^\alpha{}_\beta$, the latter of which are related by the off-diagonal components of the Jacobi identities (3.9). The results are

$$N^{-1}M^3=(-\theta_{12}+\Omega_3)/(n_1 e^{\beta^3})=(\theta_{12}+\Omega_3)/(n_2 e^{\beta^3}), \tag{5.4}$$

and its two cyclic permutations.

The shift vector field is then SH and given by

$$\vec{\mathbf{N}}=M^i\boldsymbol{\xi}_i=M^i\hat{\mathbf{e}}_i\,, \tag{5.5}$$

which is both a minimal strain and a minimal distortion shift.

B. Bianchi types VI and VII

For this category of symmetry types, both class A and B including type III=VI$_{-1}$, the matrix automorphism group has one independent diagonal automorphism generator so that one relationship may be imposed on the β^α, which may be parametrized as follows:

$$\begin{aligned}\ln \hat{D} = \text{diag}(\beta^1,\beta^2,\beta^3) &= \beta^- \ \text{diag}(\sqrt{3},-\sqrt{3},0)+\beta^\times \ \text{diag}(\zeta,\zeta,1)\\ &= \text{diag}(\zeta\beta^\times+\sqrt{3}\beta^-,\zeta\beta^\times-\sqrt{3}\beta^-,\beta^\times). \end{aligned}\tag{5.6}$$

The arbitrary parameter ζ, which can be chosen to have any convenient value, reflects a freedom in the shift vector field. The β variables are determined in terms of the structure functions by the two independent components of the equation (4.15),

$$e^{-2\beta^\times}=(n_1n_2)/(\hat{n}_1\hat{n}_2), \quad e^{4\sqrt{3}\beta^-}=(n_1\hat{n}_2)/(\hat{n}_1 n_2). \tag{5.7}$$

Note that in the class B case $a^2=hn^1n^2$ and $\hat{a}^2=h\hat{n}_1\hat{n}_2$ leading to an equivalent expression $e^{-2\beta^\times}=(a/\hat{a})^2$.

A basis of the matrix automorphism Lie algebra consists of the following three off-diagonal matrices whose nonzero entries are given by

$$[\kappa_i]^j{}_k=\epsilon_{ikl}\hat{n}^{jl}, \tag{5.8}$$

for each of the three cyclic permutations of (i,j,k) and the fourth diagonal automorphism generator,

$$\kappa_4=\text{diag}(1,1,0). \tag{5.9}$$

One can then express the matrix $\hat{A}$ as a linear combination of these matrices $\hat{A}=M^i\kappa_i+M^4\kappa_4$. (One could have chosen instead the basis $\{k_1,k_2,k_3-a\kappa_4,\kappa_4\}$ more closely related to the adjoint matrices but then k_1 and k_2 are linearly dependent for type III.)

The key equation (4.13) then becomes

$$N^{-1}[-\dot{\beta}^- \ \text{diag}(\sqrt{3},-\sqrt{3},0)-\dot{\beta}^\times \ \text{diag}(\zeta,\zeta,1)+\hat{D}\hat{A}\hat{D}^{-1}]=(-\theta^\alpha{}_\beta+\epsilon^\alpha{}_{\beta\gamma}\Omega^\gamma), \tag{5.10}$$

where the choice $\zeta=0$ corresponds to a minimal strain shift and the choice $\zeta=1$ corresponds to a minimal distortion shift.

The third diagonal component of this equation yields

$$N^{-1}\dot{\beta}^\times=\theta_{33}. \tag{5.11}$$

Using this in the sum of the first two diagonal components of the same equation leads to

$$N^{-1}M^4=-\tfrac{1}{2}(\theta_{11}+\theta_{22})+\zeta\theta_{33}. \tag{5.12}$$

The off-diagonal components of Eq. (5.10) may be used to determine the coefficients of the three off-diagonal automorphism generators in the same way as in the previous case of types VIII and IX,

$$N^{-1}M^1=(-\theta_{23}+\Omega_1)/(n_2e^{\beta^1}), \quad N^{-1}M^2=(\theta_{13}+\Omega_2)/(n_1e^{\beta^2}), \tag{5.13}$$

$$N^{-1}M^3=(-\theta_{12}+\Omega_3)/(n_1e^{\beta^3})=(\theta_{12}+\Omega_3)/(n_2e^{\beta^3}).$$

The shift vector field itself is then determined once a vector field generator corresponding to the diagonal automorphism is identified. Let $\boldsymbol{\xi}_4$ be the unique time-independent (inhomogeneous) spatial vector field which satisfies

$$\pounds_{\boldsymbol{\xi}_4}\hat{\mathbf{e}}_i=[\mathrm{diag}(1,1,0)]^j{}_i\hat{\mathbf{e}}_j\,. \tag{5.14}$$

It has the expression $\boldsymbol{\xi}_4=-x^1\partial_1-x^2\partial_2$ in coordinates that correspond to canonical coordinates of the second kind.[9]

Except for Bianchi type III (VI_h with $h=-1$) the first three automorphism matrices may be expressed in terms of the adjoint matrices $\hat{k}_i$ and the matrix κ_4 in the following way:

$$\kappa_1=(1+h)^{-1}[\hat{k}_1+(\hat{a}/\hat{n}_1)\hat{k}_2],\quad \kappa_2=(1+h)^{-1}[\hat{k}_2-(\hat{a}/\hat{n}_2)\hat{k}_1],\quad \kappa_3=\hat{k}_3-\hat{a}\kappa_4, \tag{5.15}$$

allowing one to expand $\hat{A}$ in terms of these latter four matrices instead. The desired shift vector field is then the same linear combination of the corresponding vector fields $\hat{\mathbf{e}}_i,\boldsymbol{\xi}_4$,

$$\vec{\mathbf{N}}=(1+h)^{-1}[M^1-(\hat{a}/\hat{n}_2)M^2]\hat{\mathbf{e}}_1+(1+h)^{-1}[M^2+(\hat{a}/\hat{n}_1)M^1]\hat{\mathbf{e}}_2+M^3\hat{\mathbf{e}}_3+(M^4-\hat{a}M^3)\boldsymbol{\xi}_4. \tag{5.16}$$

In the type III case, there is one less independent adjoint matrix and one more independent outer automorphism matrix whose corresponding vector field generator must be evaluated. Introduce the combination

$$\kappa_5=\kappa_1+\kappa_2=\begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix},\quad \boldsymbol{\xi}_5=-x^3(\partial_1+\partial_2)=-x^3(\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2). \tag{5.17}$$

Then the shift vector field can be chosen to be instead

$$\vec{\mathbf{N}}=\tfrac{1}{2}(M^1+M^2)\hat{\mathbf{e}}_1+\tfrac{1}{2}(M^1-M^2)\boldsymbol{\xi}_5+M^3\hat{\mathbf{e}}_3+(M^4-\hat{a}M^3)\boldsymbol{\xi}_4. \tag{5.18}$$

C. Bianchi type IV

Bianchi type IV is very similar to the previous case, but with slightly different algebra. One can use the same parametrization of $\hat{D}$ in terms of β^- and $\beta^\times$ but their relation to the structure functions following from Eqs. (4.15) and (4.16) is now

$$e^{-\beta^\times}=a/\hat{a},\quad e^{2\sqrt{3}\beta^-}=(n_1\hat{a})/(\hat{n}_1a). \tag{5.19}$$

A basis of the matrix automorphism Lie algebra consists of the following three off-diagonal matrices,

$$\kappa_1=\hat{k}_1,\quad \kappa_2=\hat{k}_2,\quad \kappa_3=\hat{k}_3-\hat{a}\ \mathrm{diag}(1,1,0), \tag{5.20}$$

and the fourth diagonal automorphism generator,

$$\kappa_4 = \mathrm{diag}(1,1,0). \tag{5.21}$$

One can then express the matrix $\hat{A}$ as a linear combination of these matrices $\hat{A} = M^i \kappa_i + M^4 \kappa_4$.

The key equation (5.10) and its immediate consequence (5.11) remain the same, leading to the same expression (5.12) for M^4 and to the same values of ζ for the minimal strain and minimal distortion shifts. The off-diagonal components of this equation lead to the following expressions for the remaining automorphism coefficients:

$$N^{-1}M^1 = -[(\theta_{13} + \Omega_2) - (n_1/a)(-\theta_{23} + \Omega_1)]/(ae^{\beta^1}),$$

$$N^{-1}M^2 = (\theta_{23} - \Omega_1)/(ae^{\beta^2}), \quad N^{-1}M^3 = (\theta_{12} - \Omega_3)/(n_1 e^{\beta^3}). \tag{5.22}$$

Let $\boldsymbol{\xi}_4$ be the same as in the previous case. The first three automorphism matrices may be expressed in terms of the adjoint matrices $\hat{k}_i$ and κ_4,

$$\kappa_1 = -[\hat{k}_2 + (\hat{n}_1/\hat{a})\hat{k}_1]/\hat{a}, \quad \kappa_2 = (-\hat{n}_1/\hat{a})\hat{k}_1, \kappa_3 = k_3 - \hat{a}\kappa_4, \tag{5.23}$$

allowing one to expand $-\hat{A}$ in terms of these latter four matrices instead. Then the desired shift vector field is

$$\vec{\mathbf{N}} = -[(\hat{n}_1 M^1/\hat{a}^2) + (\hat{n}_1 M^2/\hat{a})]\hat{\mathbf{e}}_1 - (M^1/\hat{a})\hat{\mathbf{e}}_2 + M^3\hat{\mathbf{e}}_3 + (M^4 - \hat{a}M^3)\boldsymbol{\xi}_4, \tag{5.24}$$

with the same remarks about the minimal strain and distortion conditions holding as in the previous case.

D. Bianchi type V

The parametrization of $\hat{D}$ for types VI, VII, IV with $\beta^- = 0$ is appropriate here, with the one independent β variable now given by

$$e^{-\beta^\times} = a/\hat{a}, \tag{5.25}$$

as in the type IV case. In addition to the three adjoint matrices $\kappa_i = \hat{k}_i$, one must introduce three outer automorphism matrices:

$$\kappa_4 = \mathrm{diag}(1,-1,0), \quad \kappa_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \kappa_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.26}$$

so that one has $-\hat{A} = M^i \hat{k}_i + M^4\kappa_4 + M^5\kappa_5 + M^6\kappa_6$. Note that $\hat{k}_3 = \hat{a}$ diag(1,1,0) is diagonal and $\hat{a} = 1$ by assumption. Since the diagonal matrix κ_4 is trace-free and orthogonal to $\ln \hat{D}$, and $\hat{k}_3$ is the same diagonal automorphism matrix generator as in the previous two cases, the same respective values of ζ yield the minimal strain and minimal distortion shifts.

As before the third diagonal component of the key equation (4.13) yields the same result $N^{-1}\dot{\beta}^\times = \theta_{33}$, while its diagonal components together yield

$$N^{-1}M^3=[-\tfrac{1}{2}(\theta_{11}+\theta_{22})+\zeta\theta_{33}],\quad N^{-1}M^4=\tfrac{1}{2}(-\theta_{11}+\theta_{22}). \tag{5.27}$$

The off-diagonal components then give the remaining automorphism coefficients:

$$\begin{aligned} N^{-1}M^1&=(\theta_{13}+\Omega_2)/(ae^{\beta^1}),\quad N^{-1}M^2=(\theta_{23}-\Omega_1)/(ae^{\beta^2}),\\ N^{-1}M^5&=(-\theta_{12}+\Omega_3)e^{\beta^2-\beta^1},\quad N^{-1}M^6=-(\theta_{21}+\Omega_3)e^{\beta^1-\beta^2}. \end{aligned} \tag{5.28}$$

Let

$$\boldsymbol{\xi}_4=-x^1\partial_1+x^2\partial_2,\quad \boldsymbol{\xi}_5=-x^2\partial_1,\quad \boldsymbol{\xi}_6=-x^1\partial_2, \tag{5.29}$$

be the explicit expressions for the outer automorphism vector fields corresponding to $\kappa_4,\kappa_5,\kappa_6$.[9] Then the desired shift vector field is

$$\vec{\mathbf{N}}=M^i\hat{\mathbf{e}}_i+M^4\boldsymbol{\xi}_4+M^5\boldsymbol{\xi}_5+M^6\boldsymbol{\xi}_6, \tag{5.30}$$

with the same minimal distortion and strain features as in the previous Class B models.

E. Bianchi type II

For this symmetry type, the matrix automorphism group has two additional diagonal automorphism generators leading to two relationships among the β^α, which may be parametrized as follows:

$$\ln\hat{D}=\mathrm{diag}(\beta^1,\beta^2,\beta^3)=\beta^\dagger\ \mathrm{diag}((4\zeta-1)/3,\,(2\zeta+1)/3,\,(2\zeta+1)/3). \tag{5.31}$$

The arbitrary parameter ζ, which can be chosen to have any convenient value, reflects a freedom in the shift vector field. The variable $\beta^\dagger$ is determined in terms of one nonvanishing structure function by the one independent component of the equation (4.15),

$$e^{-\beta^\dagger}=n_1/\hat{n}_1. \tag{5.32}$$

A basis of the matrix automorphism Lie algebra consists of the following off-diagonal matrices:

$$\kappa_2=\hat{k}_2,\quad \kappa_3=\hat{k}_3,\qquad \kappa_1=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad \kappa_4=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}, \tag{5.33}$$

plus the additional two diagonal matrices,

$$\kappa_5=\mathrm{diag}(2,1,1),\quad \kappa_6=\mathrm{diag}(0,1,-1). \tag{5.34}$$

One can then express the matrix $\hat{A}$ as a linear combination of these matrices $-\hat{A}=M^1\kappa_1+M^2\hat{k}_2+M^3\hat{k}_3+M^4\kappa_4+M^5\kappa_5+M^6\kappa_6$.

The key equation (4.13) then becomes

$$N^{-1}[-\dot{\beta}^{\dagger}\ \mathrm{diag}((4\zeta-1)/3,\ (2\zeta+1)/3,\ (2\zeta+1)/3)+\hat{D}\hat{A}\hat{D}^{-1}]=(-\theta^{\alpha}{}_{\beta}+\epsilon^{\alpha}{}_{\beta\gamma}\Omega^{\gamma}), \tag{5.35}$$

where the choice $\zeta=0$ corresponds to a minimal strain shift and the choice $\zeta=1$ corresponds to a minimal distortion shift as in the previous cases. The first diagonal component of this equation minus the sum of the last two diagonal components yields

$$N^{-1}\dot{\beta}^{\dagger}=-\theta_{11}+\theta_{22}+\theta_{33}. \tag{5.36}$$

Using this in the average of the last two diagonal components of the same equation leads to

$$N^{-1}M^{5}=-\zeta\theta_{11}+(\zeta-1)(\theta_{22}+\theta_{33}). \tag{5.37}$$

Taking half the difference of the last two diagonal components of that equation then yields the remaining diagonal automorphism coefficient,

$$N^{-1}M^{6}=-\theta_{22}+\theta_{33}. \tag{5.38}$$

The off-diagonal components yield the remaining coefficients,

$$\begin{gathered} N^{-1}M^{1}=(-\theta_{23}+\Omega_{1})e^{\beta^{3}-\beta^{2}}, \quad N^{-1}M^{2}=-(\theta_{13}+\Omega_{2})/(n_{1}e^{\beta^{2}}), \\ N^{-1}M^{3}=(\theta_{12}-\Omega_{3})/(n_{1}e^{\beta^{3}}), \quad N^{-1}M^{4}=-(\theta_{23}+\Omega_{1})e^{\beta^{2}-\beta^{3}}. \end{gathered} \tag{5.39}$$

The generating vector fields ξ_1,ξ_4,ξ_5,ξ_6, corresponding to outer automorphism matrices expressed in canonical coordinates of the second kind are[9]

$$\begin{gathered} \boldsymbol{\xi}_{1}=-x^{3}\partial_{2}+\tfrac{1}{2}(x^{3})^{2}\partial_{1}, \quad \boldsymbol{\xi}_{5}=-2x^{1}\partial_{1}-x^{2}\partial_{2}-x^{3}\partial_{3}, \\ \boldsymbol{\xi}_{4}=-x^{2}\partial_{3}+\tfrac{1}{2}(x^{2})^{2}\partial_{1}, \quad \boldsymbol{\xi}_{6}=-x^{2}\partial_{2}+x^{3}\partial_{3}. \end{gathered} \tag{5.40}$$

Finally the desired shift vector field is

$$\vec{\mathbf{N}}=M^{1}\xi_{1}+M^{2}\hat{\mathbf{e}}_{2}+M^{3}\hat{\mathbf{e}}_{3}+M^{4}\xi_{4}+M^{5}\xi_{5}+M^{6}\xi_{6}. \tag{5.41}$$

F. Bianchi type I

For this symmetry type, there exist no nontrivial inner matrix automorphisms and the matrix automorphisms are just the entire general linear group, so the problem is somewhat simpler. Let

$$[E^{i}{}_{j}]^{k}{}_{l}=\delta^{i}{}_{l}\delta^{k}{}_{j} \tag{5.42}$$

be the component definition of the natural basis of 3×3 matrices. In Cartesian coordinates adapted to the translational symmetry, a basis of the corresponding vector field Lie algebra is then[9]

$$\boldsymbol{\xi}^{i}{}_{j}=-[E^{i}{}_{j}]^{k}{}_{l}x^{l}\partial_{k}. \tag{5.43}$$

One may set $\hat{D}$ equal to the identity, so that (4.13) reduces to

$$N^{-1}\hat{A}^{i}{}_{j}=\delta^{i}{}_{\beta}\delta^{\alpha}{}_{j}(-\theta^{\alpha}{}_{\beta}+\epsilon^{\alpha}{}_{\beta\gamma}\Omega^{\gamma}), \tag{5.44}$$

and the corresponding (inhomogeneous) shift vector field is

$$\vec{\mathbf{N}} = -\hat{A}^i{}_j \boldsymbol{\xi}^j{}_i, \tag{5.45}$$

where the entries of the matrix $\hat{A}$ themselves play the role of the automorphism coefficients M^D, M^O of the previous cases.

This is clearly both a minimal strain shift and a minimal distortion shift. However, the minimal distortion shift for all the other Bianchi types still has a time-dependent isotropic part of $\hat{D}$ present. To restore this possibility, one can relax the condition on $\hat{D}$ to allow it to have a purely isotropic time-dependent part ($\beta^1=\beta^2=\beta^3$) while retaining the minimal distortion condition on the shift, though not the minimal strain condition. Then the parametrization,

$$\ln \hat{D} = \beta^0 \,\mathrm{diag}(1,1,1), \tag{5.46}$$

leads to an arbitrary contribution to the pure trace part of $\hat{A}$,

$$\mathrm{Tr}\, \hat{A} = -\Theta + 3N^{-1}\dot{\beta}^0. \tag{5.47}$$

VI. CONCLUDING REMARKS

The essential ingredient of the above procedure for constructing a coordinate representation of the orthonormal frame is the hypersurface transitivity of the symmetry group. It can therefore be carried over to the temporally homogeneous case and to some extent to their corresponding hypersurface self-similar cases. While the latter are not hypersurface-homogeneous, they are conformally hypersurface-homogeneous with a particular conformal factor.

The construction procedure for SH models is valid for any choice of lapse function. However, it suggests a preferred class of lapse functions, namely those which depend only on the commutator functions through an algebraic relationship. A lapse of this type is used to produce a dimensionless time variable in the dynamical systems formulation[1] of the orthonormal frame equations.

Geometrical objects on spacetime can be assigned dimensions under constant spacetime conformal transformations. In a computational frame the lapse has dimension 1 and the shift dimension 0 provided that a dimensionless time variable is used, while the spatial metric has dimension 2 and hence $\hat{D}$ dimension 1, except for Bianchi type I where $\hat{D}$ is the identity and the spatial coordinates instead have dimension 1. By re-introducing an isotropic degree of freedom in $\hat{D}$ to carry the dimension as described above for this case, the spatial coordinates become dimensionless. In an orthonormal frame the commutation functions have dimension -1.

In the dynamical systems analysis of the orthonormal frame equations, dimensionless variables are obtained by dividing the commutation functions by a function of them which has the same dimension. Usually one chooses the expansion scalar Θ, which leads to the so-called expansion normalized variables.[1] To produce a dimensionless time variable, the lapse must be a function constructed from the commutator functions with dimension 1. In the expansion normalized approach, the associated lapse is $N \propto \Theta^{-1}$.

The minimal distortion shift choice of the present approach is naturally adapted to any such dimensionless formulation since it allows the factorization of the metric into an overall conformal factor carrying the dimension and the remaining part which is therefore automatically expressed in terms of dimensionless variables. This can be seen from the fact that by choosing the minimal distortion shift for all Bianchi types provides a purely isotropic β variable parametrizing part of $\hat{D}$ that carries the dimension, making the remaining variables, if any, dimensionless. Allowing the extra isotropic degree of freedom for Bianchi type I, one can re-express the metric for all Bianchi types in the form in which the square of the lapse is an overall conformal factor times a new unphysical metric. In the minimal distortion gauge, if the lapse is chosen to be a function of the commutation functions with dimension 1, then the new unphysical metric is then expressed entirely in terms of dimensionless combinations of the commutator functions.

In the special case of a Bianchi spacetime with an additional homothetic Killing vector not tangent to the SH hypersurfaces, all dimensionless quantities take constant values[10,11] under these conditions, and the spacetime metric in the minimal distortion gauge is explicitly stationary except for an overall conformal factor which is exponential in the dimensionless time variable. Thus one automatically obtains the standard form for a transitively self-similar spacetime metric adapted to the homothetic Killing vector field (compare Ref. 9).

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