Adapted slicings of space-times possessing simply transitive similarity groups

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(Received 7 November 1985; accepted for publication 3 January 1986)

The relationship between the hypersurface-homogeneous slicing of an exact power law metric space-time and slicings adapted to spatial self-similarities is discussed in a group theoretical setting.

Exact power law metrics, recently introduced by Wainwright[1] and generalized by Jantzen and Rosquist[2] are four-dimensional space-time metrics possessing a transitive group of homothetic motions[3] or "similarity transformations." These metrics arise naturally in the qualitative analysis of the general relativistic dynamics of spatially homogeneous and spatially self-similar space-times as singular points of a Hamiltonian system of ordinary differential equations for conformally invariant variables.[4] Starting with a spatially homogeneous or spatially self-similar space-time, the existence of a homothetic Killing vector field not tangent to the orbits of the three-dimensional symmetry group leads to a simply transitive similarity group of the space-time. The most familiar class of such space-times is the Kasner solution,[5] an exact solution of the vacuum Einstein equations that plays an important role as an asymptotic solution during certain phases of the evolution of more general spatially homogeneous or spatially self-similar space-times.

The simply transitive case may be treated in exactly the same way as the space-time-homogeneous metrics studied by Ozsváth[6] and Farnsworth and Kerr.[7] The space-time manifold may be identified with the four-dimensional manifold of the symmetry group and the group action with the natural left action of the group on itself by left translation. The isometry group of the space-time is necessarily a three-dimensional subgroup[8] acting simply transitively on its orbits, a family of three-dimensional hypersurfaces that are the right cosets of the isometry subgroup. The space-time is therefore hypersurface homogeneous[9] and admits a preferred slicing, namely by the family of orbits of the isometry subgroup. The object of this paper is to discuss other possible adapted slicings of such space-times, namely by the orbits (right cosets) of nontrivial three-dimensional similarity subgroups, when they exist. These subgroups are related to the original isometry subgroup by a certain family of Lie group deformations. The relationship between these various subgroups provides a group theoretical procedure for finding the hypersurface-homogeneous slicing of the space-time associated with a spatially self-similar exact power law metric.

Let H_4 be the similarity group with Lie algebra h_4 of left-invariant vector fields. Let $\tilde{h}_4$ be the Lie algebra of right-invariant vector fields, the generators of the left action of H_4 on itself by left translation, and let $\{e_\alpha\}$ and $\{\tilde{e}_\alpha\}$ be bases of h_4 and $\tilde{h}_4$, respectively, which agree at the tangent space at the identity of the group. Let the dual spaces h_4^* and $\tilde{h}_4^*$ be identified with the spaces of, respectively, left-invariant and right-invariant one-forms on H_4, and let $\{\omega^\alpha\}$ and $\{\tilde{\omega}^\alpha\}$ be the respective dual bases defined by $\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta = \tilde{\omega}^\alpha(\tilde{e}_\beta)$, where Greek letters run from 1 to 4. Both $\{e_\alpha\}$ and $\{\tilde{e}_\alpha\}$ are global frames on H_4. One has

$$\begin{aligned} &[e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta} e_\gamma, \quad [\tilde{e}_\alpha, \tilde{e}_\beta] = -C^\gamma{}_{\alpha\beta}\tilde{e}_\gamma, \\ &[e_\alpha, \tilde{e}_\beta] = 0, \\ &d\omega^\alpha = -\tfrac{1}{2} C^\alpha{}_{\beta\gamma}\omega^\beta \wedge \omega^\gamma, \\ &d\tilde{\omega}^\alpha = \tfrac{1}{2} C^\alpha{}_{\beta\gamma}\tilde{\omega}^\beta \wedge \tilde{\omega}^\gamma . \end{aligned} \tag{1}$$

The similarity condition on the Lorentz metric g

$$\pounds_{\tilde{e}_\alpha} g = 2 f_\alpha g , \tag{2}$$

where f_α are constants, not all of which vanish if H_4 is a nontrivial similarity group as assumed, defines a nonzero right-invariant one-form $f = f_\alpha \tilde{\omega}^\alpha$. From the identity $\pounds_{[X,Y]} = [\pounds_X, \pounds_Y]$, one immediately derives the conditions

$$f_\alpha C^\alpha{}_{\beta\gamma} = 0 , \tag{3}$$

which has the following consequences for the one-form f:

$$\begin{aligned} &df = \tfrac{1}{2} f_\alpha C^\alpha{}_{\beta\gamma}\tilde{\omega}^\beta \wedge \tilde{\omega}^\gamma = 0 , \\ &\pounds_{\tilde{e}_\alpha} f = f_\beta C^\beta{}_{\alpha\gamma}\tilde{\omega}^\gamma = 0 . \end{aligned} \tag{4}$$

The first relation shows f to be a closed one-form and, in the case in which H_4 is simply connected, exact as well. The second relation shows f to be invariant under the coadjoint representation of $\tilde{h}_4$ on $\tilde{h}_4^*$ and since $\tilde{h}_4$ generates the left translations, f is left invariant and hence bi-invariant. Since e_α and $\tilde{e}_\alpha$ coincide at the identity of the group, f has the same components in either basis

$$\begin{aligned} f &= f_\alpha \tilde{\omega}^\alpha = f_\alpha \omega^\alpha \\ &\in h_4^* \cap \tilde{h}_4^* = \{\gamma_\alpha \omega^\alpha \mid \gamma_\alpha C^\alpha{}_{\beta\gamma} = 0,\ (\gamma_\alpha) \in \mathbf{R}^4\} . \end{aligned} \tag{5}$$

One can always choose the basis so that $f_\alpha = \delta^4{}_\alpha$, in which case $C^4{}_{\beta\gamma} = 0$ and the only nonzero structure constant tensor components are of the form $C^a{}_{bc}$ and $C^a{}_{4b}$, where Latin indices run from 1 to 3, leading to the relations

$$\pounds_{\tilde{e}_a} g = 0, \quad \pounds_{\tilde{e}_4} g = 2g, \tag{6}$$

and

$$[\tilde{e}_a, \tilde{e}_b] = -C^c{}_{ab}\tilde{e}_c, \tag{7a}$$

$$[\tilde{e}_4, \tilde{e}_a] = -C^b{}_{4a}\tilde{e}_b. \tag{7b}$$

Thus the $\{\tilde{e}_a\}$ span an isometry Lie subalgebra to which the vector fields $[\tilde{e}_4, \tilde{e}_a]$ belong, while bracketing by $\tilde{e}_4$, already an inner derivation of the Lie algebra $\tilde{g}_4$, becomes a derivation of the Lie subalgebra $\tilde{h}_3 = \text{span}\{\tilde{e}_a\}$; let G_3 be the subgroup of H_4 that corresponds to this Lie subalgebra. One can add any linear combination of the vector fields $\{\tilde{e}_a\}$ to $\tilde{e}_4$ without changing (6), while (7b) changes by the addition of an arbitrary inner derivation of the Lie subalgebra $\tilde{g}_3$,

$$[\tilde{e}_4 + y^c\tilde{e}_c, \tilde{e}_a] = -(C^b{}_{4a} + y^c C^b{}_{ca})\tilde{e}_b. \tag{8}$$

The essential part of the matrix $(C^b{}_{4a})$ is therefore equivalent to an outer derivation of $\tilde{g}_3$. Recall that the space of outer derivations of a Lie algebra is the vector space quotient of the Lie algebra of derivations of that algebra by the Lie subalgebra of inner derivations of that algebra (adjoint transformations).

If we assume H_4 is simply connected, then the closed bi-invariant one-form $f = \tilde{\omega}^4$ is exact and therefore $\tilde{\omega}^4 = d\zeta$ (which implies $\tilde{e}_4\zeta = e_4\zeta = 1$ by duality). The integral submanifolds of $\tilde{\omega}^4$, namely the hypersurfaces $\zeta = \zeta_{(0)}$, are exactly the orbits of the action of the isometry subgroup G_3, namely the right cosets of G_3 in H_4. Note that $\tilde{\omega}^4$ is also invariant under the basis transformations $\tilde{e}_4 \to \tilde{e}_4 + y^a\tilde{e}_a$, which leave (6) invariant.

Except for a few special cases discussed by Eardley,[8] a space-time metric g with a similarity group is always conformally related to a metric that is invariant under the similarity group

$$g = e^{2\psi}g_{(0)}, \quad \pounds_{\tilde{e}_a} g_{(0)} = 0. \tag{9}$$

From (6) one has $\tilde{e}_4\psi = 1$, $\tilde{e}_a\psi = 0$, so one may assume $\psi = \zeta$. The metric $g_{(0)}$ is space-time homogeneous and is a left-invariant metric on H_4, which therefore may be expressed in the left-invariant frame $\{e_\alpha\}$ with constant components

$$g_{(0)} = g_{(0)\alpha\beta}\omega^\alpha \otimes \omega^\beta, \quad d(g_{(0)\alpha\beta}) = 0. \tag{10}$$

Suppose one considers the change of basis of $\tilde{h}_4$,

$$\tilde{\xi}_a = \tilde{e}_a + b_a\tilde{e}_4, \quad \tilde{\sigma}^a = \tilde{\omega}^a, \tag{11a}$$

$$\tilde{\xi}_4 = \tilde{e}_4, \quad \tilde{\sigma}^4 = \tilde{\omega}^4 - b_a\tilde{\omega}^a, \tag{11b}$$

where the b_a are constants satisfying

$$b_c C^c{}_{ab} = 0, \tag{12a}$$

$$b_c C^c{}_{4[a}b_{b]} = 0. \tag{12b}$$

A simple computation shows that

$$\begin{aligned} &[\tilde{\xi}_a, \tilde{\xi}_b] = -(C^c{}_{ab} - C^c{}_{4[a}b_{b]})\tilde{\xi}_c, \\ &[\tilde{\xi}_4, \tilde{\xi}_a] = -C^b{}_{4a}\tilde{\xi}_b, \\ &d\tilde{\sigma}^4 = -b_a C^a{}_{4b}\tilde{\sigma}^4 \wedge \tilde{\sigma}^b, \quad \tilde{\sigma}^4 \wedge d\tilde{\sigma}^4 = 0, \\ &\pounds_{\tilde{\xi}_a} g = 2b_a g. \end{aligned} \tag{13}$$

Thus the $\{\tilde{\xi}_a\}$ generate a three-dimensional similarity subgroup H_3 whose three-dimensional orbits (right cosets) are the integral submanifolds of the one-form $\tilde{\sigma}_4$. This new slicing of the space-time is adapted to a "hypersurface self-similarity" of the metric, the natural generalization of Eardley's term "spatial self-similarity[8]" to the case where the causal nature of the hypersurface is arbitrary.

The space

$$g_3^* \cap \tilde{g}_3^* = \{\gamma_a\tilde{\omega}^a | \gamma_c C^c{}_{ab} = 0, \ (\gamma_a) \in \mathbb{R}^3\} \tag{14}$$

(when its elements are restricted to the subgroup G_3) is the space of closed or equivalently bi-invariant one-forms on the subgroup G_3, or equivalently, the subspace invariant under the co-adjoint representation of G_3 on the dual space to $\tilde{g}_3$ or g_3. Since $\tilde{e}_4$ acts as a derivation of the Lie subalgebra $\tilde{g}_3$, with matrix $(-C^a{}_{4b})$, it maps the space $g_3^* \cap \tilde{g}_3^*$ into itself, easily verified by an application of the Jacobi identity. The condition (12b) is equivalent to the requirement that (b_a) be an eigenvector of the matrix $(C^a{}_{4b})$:

$$b_b C^b{}_{4a} = \theta b_a. \tag{15}$$

This guarantees that $b_a\tilde{\omega}^a$ remain bi-invariant under the Lie algebra deformation (11a) of $\tilde{g}_3$ into $\tilde{h}_3$. When the eigenvalue θ is zero, then $\tilde{\sigma}^4$ is also bi-invariant on H_4, and, assuming simple connectivity, also exact.

The classification of four-dimensional similarity groups H_4 for a fixed isometry subgroup G_3 is equivalent to a description of the quotient space of the space of outer derivations of $\tilde{g}_3$ by the natural action of the automorphism group of $\tilde{g}_3$. In more explicit terms this classification is just a description of the equivalence classes of derivation matrices $(C^a{}_{4b})$ under the combined action of the group of matrices that leave invariant the structure constant tensor components $C^a{}_{bc}$ and the addition of adjoint matrices associated with these structure constant tensor matrices

$$\begin{aligned} &C^a{}_{4b} \to A^a{}_c A^{-1d}{}_b(C^c{}_{4d} + y^e C^c{}_{ed}), \\ &A^a{}_d C^d{}_{fg} A^{-1f}{}_b A^{-1g}{}_c = C^a{}_{bc}. \end{aligned} \tag{16}$$

The isometry subgroups G_3 may be classified according to the Bianchi–Behr classification.[10] Only nonsemisimple groups G_3 (of Bianchi types I–VII) admit nontrivial spaces $g_3^* \cap \tilde{g}_3^*$; similarly only these types of groups may act as nontrivial self-similarity groups H_3. Suppose one has a spatially self-similar exact power law metric space-time with spatial self-similarity group H_3 and simply transitive similarity group H_4. Knowing the Lie algebra structure of H_4 one can work backward from (13) to find the Killing vector fields and hence the slicing by homogeneous hypersurfaces. The inverse of the transformation (11) then describes the relationship between the original spatial self-similarity group and the isometry group, which are connected by a family of self-similarity subgroups. Any one of these subgroups of H_4 may be used to slice the space-time; clearly the isometry group is the preferred member of this family of subgroups.

To make this discussion more concrete, it is worth examining an explicit example. Consider a spatially self-similar exact power law metric expressed in coordinates adapted to the orbits of a Bianchi type VI_h subgroup H_3 of the full isometry group $H_4 \sim \mathbb{R}^4$ on which $\{x^1, x^2, x^3, \lambda\}$ are taken to be global coordinates.[11] Using the logarithmic time variable $\lambda = \ln t$, where t is the usual cosmic time function, and the symbol $\mathbf{e}^a{}_b$ for the 3×3 matrix whose only nonzero entry is a 1 in the bth row and ath column, this metric has the form

$$
\begin{aligned}
&{}^4g = -\Omega^4 \otimes \Omega^4 + \delta_{ab}\Omega^a \otimes \Omega^b ,\\
&\Omega^4 = e^{bx^3}\, dt = e^{\lambda + bx^3}\, d\lambda ,\\
&(\Omega^a) = e^{(\lambda + bx^3)\mathbf{1} + \boldsymbol{\beta}_{(0)}}(\sigma^a) , \qquad (17)\\
&(\sigma^a) = [\exp\{[(s-1)\lambda - ax^3]\mathbf{I}^{(3)} + (q\lambda - x^3)\mathbf{k}_3^0\}\\
&\qquad + M\mathbf{e}^3{}_2 + N\mathbf{e}^3{}_1] \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbf{I}^{(3)} = \mathrm{diag}(1,1,0), \quad \boldsymbol{\beta}_{(0)} = \mathrm{diag}(\beta^{1}{}_{(0)},\beta^{2}{}_{(0)},\beta^{3}{}_{(0)}) ,\\
&\mathbf{k}_3^0 = -q_0(\mathbf{e}^1{}_2 + \mathbf{e}^2{}_1) . \qquad (18)
\end{aligned}
$$

The quantities $\{\beta^{a}{}_{(0)},s,q,M,N;a,q_0;b\}$ are constants and $h = -a^2 q_0^{-2}$ is the group parameter specifying the Bianchi type of H_3, while $\{\sigma^a, d\lambda\}$ are a basis of left-invariant one-forms on H_4.

The left action of the subgroup H_3 on H_4 is generated by the right-invariant vector fields

$$
\begin{aligned}
\{\xi_1,\xi_2,\xi_3\} = \{&\partial_1,\partial_2,\partial_3 + a(x^1\,\partial_1 + x^2\,\partial_2)\\
&- q_0(x^2\,\partial_1 + x^1\,\partial_2)\} ,
\end{aligned}
$$

which satisfy the final relation of (13) with $b_a = b\delta^3{}_a$. Note that $\tilde\xi_1$ and $\tilde\xi_2$ are Killing vector fields and, when $b \neq 0$, $\tilde\xi_3$ is a homothetic Killing vector field. The remaining linearly independent right-invariant vector field is associated with the transformation

$$
\begin{aligned}
&(x^1,x^2,x^3,\lambda)\\
&\quad \to (x^1 e^{(aq-(s-1))\zeta}, x^2 e^{(aq-(s-1))\zeta}, x^3 + q\zeta, \lambda + \zeta) . \qquad (19)
\end{aligned}
$$

under which the metric scales by the constant factor $e^{2(1+bq)\zeta}$. When $1 + bq = 0$, this is an isometry subgroup generated by the vector field

$$
\tilde\xi_4 = \partial_\lambda + q\,\partial_3 + [aq - (s-1)](x^1\,\partial_1 + x^2\,\partial_2) , \qquad (20)
$$

and $\{\tilde\xi_1,\tilde\xi_2,\tilde\xi_4\}$ span the Killing Lie algebra (a Lie algebra of Bianchi type V, see below) and are tangent to the slicing of the space-time by the orbits of the isometry group. When $1 + bq \neq 0$, correcting for the scale factor so that the generator $\tilde\xi_4$ of this one-parameter similarity subgroup satisfies (6), one has instead

$$
\begin{aligned}
\tilde\xi_4 = (1+bq)^{-1}\{&\partial_\lambda + q\,\partial_3\\
&+ [aq - (s-1)](x^1\,\partial_1 + x^2\,\partial_2)\} , \qquad (21)
\end{aligned}
$$

and consequently

$$
\tilde\sigma^4 = (1+bq)d\lambda = \sigma^4 . \qquad (22)
$$

The Lie brackets of the right invariant basis vector fields are

$$
\begin{aligned}
&\left[\tilde\xi_4, \begin{pmatrix} \tilde\xi_1 \\ \tilde\xi_2 \\ \tilde\xi_3 \end{pmatrix}\right] = -(1+bq)^{-1}[aq-(s-1)]\mathbf{I}^{(3)} \begin{pmatrix} \tilde\xi_1 \\ \tilde\xi_2 \\ \tilde\xi_3 \end{pmatrix},\\
&[\tilde\xi_2,\tilde\xi_3] = -q_0\tilde\xi_1 - a\tilde\xi_2 ,\\
&[\tilde\xi_3,\tilde\xi_1] = a\tilde\xi_1 - q_0\tilde\xi_2 ,\\
&[\tilde\xi_1,\tilde\xi_2] = 0 .
\end{aligned} \qquad (23)
$$

These completely determine the Lie group structure of the simply connected group H_4, once the identity of the group is specified as the point that is the origin of coordinates. The one-forms $\{\sigma^a\}$ defined by (17) and (22) are the left-invariant one-forms dual to the left-invariant basis $\{\xi_\alpha\}$ associated with $\{\tilde\xi_\alpha\}$. In terms of them the metric can assume the form (9), (10) with $\psi = \lambda + bx^3$, $(g_{(0)ab}) = e^{2\beta_{(0)}}$ and $g_{(0)4\alpha} = -\delta_{4\alpha}$. Expressed in language more familiar in general relativistic discussions, the one-forms $\{\sigma^\alpha\}$ are invariant under dragging along by the vector field $\tilde\xi_4$, which generates the slicing of the space-time from the initial hypersurface $H_3 \subset H_4$ by dragging along.

The derivation matrix $(C^a{}_{4b}) = (1+bq)^{-1} \times [aq - (s-1)]\mathbf{I}^{(3)}$ is invariant under the matrix automorphism group of the Lie algebra $\tilde h_3$ of H_3. For all values of h except $h = -1$, which corresponds to the special Bianchi type III, one has

$$
h^*_3 \cap \tilde h^*_3 = \{\gamma_a\omega^a = B\,dx^3 | (\gamma_a) = (0,0,B),\ B \in \mathbb{R}\} \qquad (24)
$$

since $\omega^3 = \tilde\omega^3 = dx^3$. Here $\gamma_a C^a{}_{4b}$ is identically zero for these one-forms, which are therefore also bi-invariant as one-forms on H_4.

The transformation inverse to (11) with b_a replaced by γ_a leads to another exact bi-invariant one-form on H_4:

$$
\begin{aligned}
&\tilde\omega^4 = \tilde\sigma^4 + B\tilde\sigma^3 = d(\lambda + Bx^3) = d\bar\lambda ,\\
&\bar\lambda = \lambda + Bx^3 .
\end{aligned} \qquad (25)
$$

Expressing the metric in terms of $\bar\lambda$ leads to a form adapted to the action of the new homothetic subgroup $\bar H_3$ of H_4:

$$
\begin{aligned}
&\Omega^4 = e^{\bar\lambda + \bar b x^3}(d\bar\lambda - B\,dx^3) ,\\
&(\Omega^a) = e^{(\bar\lambda + \bar b x^3)\mathbf{1} + \beta_{(0)}}(\exp\{[(s-1)\bar\lambda - \bar a x^3]\mathbf{I}^{(3)}\\
&\qquad + (\bar q\bar\lambda - x^3)\bar{\mathbf{k}}_3^0\} + M\mathbf{e}^3{}_2 + N\mathbf{e}^3{}_1) \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix},\\
&\bar b = b - B, \quad \bar a = a - B(s-1) ,\\
&\bar q_0 = (1+qB)q_0, \quad \bar q = q(1+qB)^{-1} ,\\
&\bar h = -\bar a^2\bar q_0^2 .
\end{aligned} \qquad (26)
$$

By choosing $\bar b = 0$, one obtains the slicing by the isometry subgroup, which may be a group of Bianchi type $\mathrm{VI}_{\bar h}$ ($\bar h \neq 0, -\infty$), VI_0 ($\bar q_0 \neq 0$, $\bar a = 0$), V ($\bar q_0 = 0, \bar a \neq 0$), or I ($\bar q_0 = 0 = \bar a$).

For Bianchi type $\mathrm{VI}_{-1} = \mathrm{III}$, setting $a = 1 = q_0$, one has instead

$$
h^*_3 \cap \tilde h^*_3 = \{\gamma_a\omega^a | (\gamma_a) = (\mathscr{B},\mathscr{B},B) \in \mathbb{R}^3\} \qquad (27)
$$

and the automorphism group may be used to reduce (γ_a) to the form (1,1,0). In this case a second linearly independent eigenvector of $(C^b{}_{4b})$ exists, namely (1,1,0) with eigenvalue $\theta = (1+bq)^{-1}(q - s + 1)$, corresponding to the one-form $\sigma^1 + \sigma^2 = e^{-(q-s+1)\lambda}\, d(x^1 + x^2)$, which is bi-invariant when restricted to the subgroup H_3, where $\lambda = 0$. One therefore has the option of considering the transformation inverse to (11) with b_a replaced by $(\mathscr{B},\mathscr{B},0)$, leading to an example in which $\tilde\omega^4$ is not bi-invariant when $\theta \neq 0$.

ACKNOWLEDGMENTS

R. J. gratefully acknowledges the hospitality of the Institute of Theoretical Physics at Stockholm and the Vatican Observatory at Castel Gandolfo.

K. R. was supported by the Swedish Natural Research Council.

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