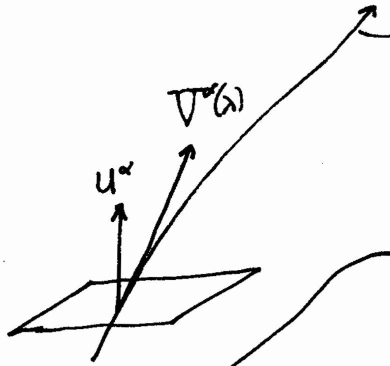
Splitting Derivatives Along a Curve



$\frac{dx^\alpha}{d\lambda} = V^\alpha$ tangent to curve in spacetime

Intrinsic or "absolute" derivative along the curve:

$$\frac{DX^\alpha}{d\lambda} = \frac{dX^\alpha}{d\lambda} + \Gamma^\alpha{}_{\beta\gamma} V^\beta X^\gamma$$

is well defined if X^α is defined only along the curve.

If X^α is a vector field (i.e., defined on spacetime) then

$$\frac{DX^\alpha}{d\lambda} = \nabla_V X^\alpha = X^\alpha{}_{;\beta} V^\beta$$
$$= X^\alpha{}_{,\beta} V^\beta + \Gamma^\alpha{}_{\beta\gamma} V^\beta X^\gamma$$

chain rule:
$\lambda = \lambda(\bar\lambda)$
$\frac{d}{d\lambda} = \frac{d\bar\lambda}{d\lambda}\frac{d}{d\bar\lambda}$
to change parametrization

is the value of the covariant directional derivative of the field evaluated along the curve. The original expression is well defined since if one extends a vector field defined only along a curve in a neighborhood of the curve, it reduces to the second expression but is independent of the extension.

We need derivatives along curves to describe test particle motions and properties, or to describe the geometry of the curves themselves. These derivatives can be "measured" by an observer family, and in the same way that a derivative of a field (defined everywhere) corresponds to a derivative of a field only defined along the curve, we can first project the derivatives on fields.

$$\begin{cases} \pounds_V X^\alpha = X^\alpha{}_{;\beta} V^\beta - V_{;\beta} X^\beta & \xrightarrow{\text{general}} \quad \pounds(u)_V X^\alpha = P(u)^\alpha{}_\beta \pounds_V X^\beta \\ \nabla_V X^\alpha = X^\alpha{}_{;\beta} V^\beta & \xrightarrow{\text{spatial}} \quad \nabla(u)_V X^\alpha = P(u)^\alpha{}_\beta \nabla_{P(u)V} X^\alpha \\ & \xrightarrow{\text{temporal}} \quad \nabla_{(fw)}(u) X^\alpha = P(u)^\alpha{}_\beta \nabla_u X^\beta \\ & \xrightarrow{\text{only temporal}} \quad \nabla_{(lie)}(u) X^\alpha = \pounds(u)_u X^\alpha \end{cases}$$

direction of differentiation

relation between Lie and covariant derivative:

$$\nabla_V X^\alpha = \pounds_V X^\alpha + V^\alpha{}_{;\beta} X^\beta$$

now set $V^\alpha = u^\alpha$ and project:

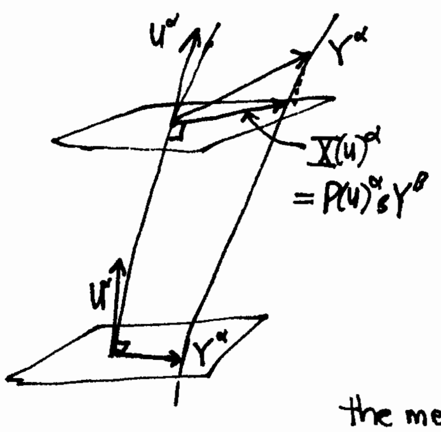
$$\nabla_{(fw)}(u) X^\alpha = \nabla_{lie}(u) X^\alpha + P(u)^\alpha{}_\gamma \underbrace{u^\gamma{}_{;\beta}}_{-a(u)^\gamma u_\beta + [\nabla_{(fw)}(u)\, u]^\gamma{}_{;\beta}} X^\beta$$

$$\nabla_{(fw)}(u) X^\alpha = \nabla_{(lie)}(u) X^\alpha + a(u)^\alpha(-u_\beta X^\beta) + [\Theta(u)^\alpha{}_\beta - \omega(u)^\alpha{}_\beta] X^\beta$$

if set $X^\alpha = u^\alpha$:
$$\nabla_{(fw)}(u) u^\alpha = \underbrace{\nabla_{(lie)}(u) u^\alpha}_{P(u)^\alpha{}_\delta \underbrace{\pounds_u u^\delta}_{[u,u]^\gamma = 0}} + a(u)^\beta = a(u)^\beta$$

if X^α spatial :
$$\nabla_{(fw)}(u) X^\alpha = \nabla_{(lie)}(u) X^\alpha + \underbrace{[\Theta(u)^\alpha{}_\beta - \omega(u)^\alpha{}_\beta] X^\beta}_{}$$

the kinematical quantities (rotation, expansion, and shear) describe the difference between projected Lie and parallel transport along u^α.



Suppose Y^α is a vector field dragged along by u^α, i.e., $\pounds_u Y^\alpha = 0$ (the field is invariant under the group of transformations generated by u^α)

If Y^α is initially spatial, it wont remain so under dragging along since

$$\pounds_u g_{\alpha\beta} = u_{\alpha;\beta} + u_{\beta;\alpha} = -2a(u)_{(\alpha} u_{\beta)} + \Theta(u)_{\alpha\beta}$$
$$(\to \pounds(u)_u g_{\alpha\beta} = \Theta(u)_{\alpha\beta})$$

the metric is not invariant in general and inner products are not preserved. [The first term tilts the local rest space of u^α in the time direction, while the second deforms the spatial inner product relationships.] We can project Y^α to get a spatial vector:

$$\nabla_{(lie)}(u) X^\alpha = \pounds(u)_u X^\alpha = P(u)^\alpha{}_\beta \pounds_u [P(u)^\beta{}_\gamma Y^\gamma] = P(u)^\alpha{}_\beta [\underbrace{(\pounds_u P(u)^\beta{}_\gamma)}_{=0} \cdot Y^\gamma + P(u)^\beta{}_\gamma \underbrace{\pounds_u Y^\gamma}_{=0}] = 0$$

$$\text{but } \pounds(u)_u P(u)^\alpha{}_\beta = \text{"}P(u)\text{"}[\pounds_u][\delta^\alpha{}_\beta + u^\alpha u_\beta] = \text{"}P(u)\text{"}[\underbrace{(\pounds_u u^\alpha)}_{0} u_\beta + u^\alpha \underbrace{\pounds_u u_\beta}_{=0}] = 0$$

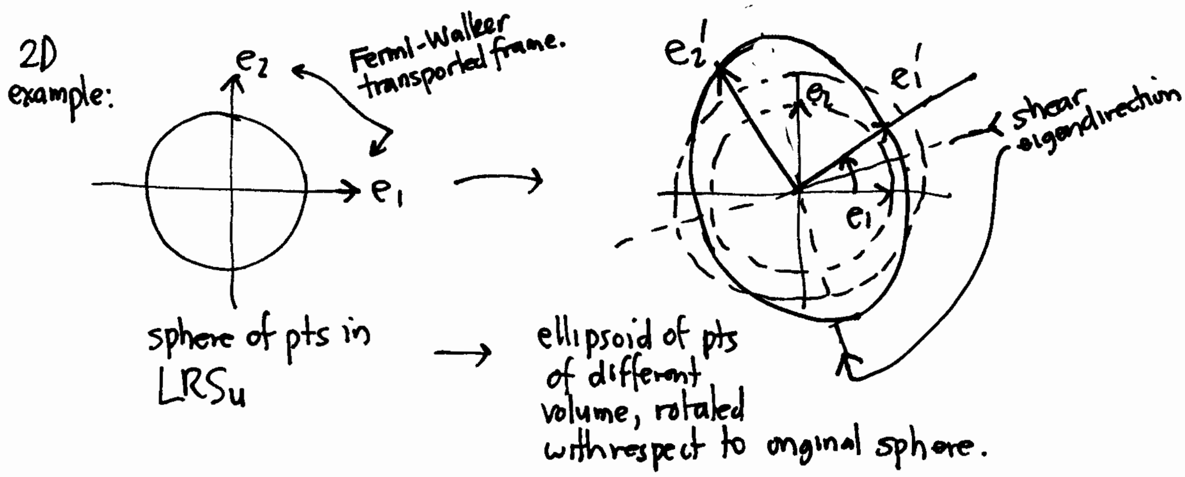
and it undergoes "projected" or "spatial" Lie transport along u^α.

Thus :
$$\underbrace{\nabla_{(fw)}(u) X^\alpha}_{} = [\tfrac{1}{3}\Theta(u)\delta^\alpha{}_\beta + \sigma(u)^\alpha{}_\beta - \omega(u)^\alpha{}_\beta] X^\beta, \quad \text{if } X^\alpha = P(u)^\alpha{}_\beta Y^\beta!$$

measures the change in X^α with respect to a FW transported frame along u^α, i.e. $\nabla_{(fw)}(u) e^\alpha{}_{(a)} = 0 \quad a = 1,2,3$.

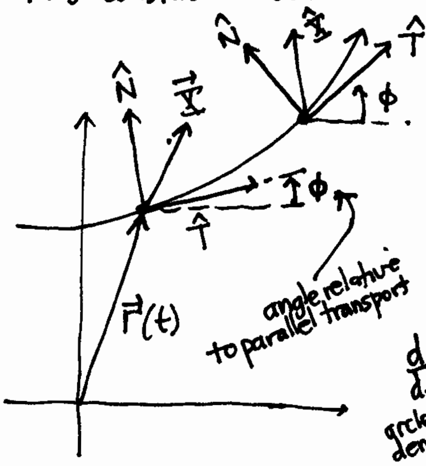
The dragged-along vector field Y^α can be thought of as a "connecting vector field" in the sense that for small enough ϵ one can identify the vector ϵY^α as having its tip attached to the world line of a nearby observer in the family. $eX(u)^\alpha$ can then be thought of as the "relative position vector" of the nearby observer, in the sense that one identifies points in the local rest space of the original observer (in the tangent space) with corresponding points in the spacetime manifold.

The vorticity ("rotation") then describes an overall rotation $\omega(u)^\alpha{}_\beta X^\beta = [\vec{\omega}(u) \times_u \vec{X}]^\alpha$ of the nearby observers in the local rest space with angular ~~velocity~~ velocity $\vec{\omega}(u)^\alpha$ compared to FW transported axes, an overall ~~rotation~~ with expansion rate $\theta(u) = \theta(u)^\alpha{}_\alpha$, and a volume preserving deformation with rate $\sigma(u)^\alpha{}_\beta$.



But we haven't explained the connection between the transport associated with $\nabla_{(fw)}(u)$ and spacetime Fermi-Walker transport along a curve.

First consider a curve in the Euclidean plane:



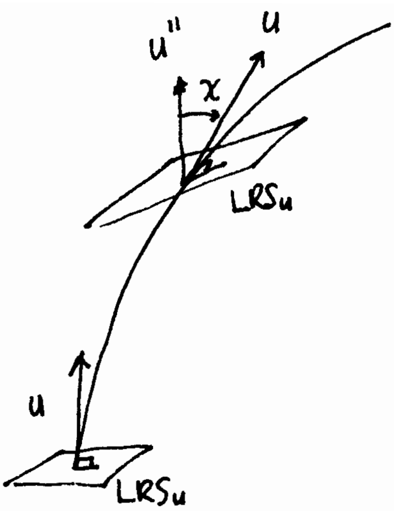
The unit tangent and normal both rotate along the curve in order to remain tangent and orthogonal respectively ($\hat{N}$ also reflects at inflection pts).

$$\left.\begin{aligned} \hat{T}' &= \phi' \hat{z} \times \hat{T} \\ \hat{N}' &= \underbrace{\phi' \hat{z}}_{\vec{\omega}} \times \hat{N} \end{aligned}\right\} \rightarrow \vec{X}' = \vec{\omega} \times \vec{X} = \Omega^i{}_j X^j e_j \quad \text{where } \Omega_{ij} = 2\hat{T}_{[i}\hat{T}'_{j]}$$

$\frac{d}{d\ell}$ arclength derivative along curve

action of representation of Lie algebra of rotation group → acceleration

angular velocity describes rate of change of rotation of tangent space to curve <u>relative to parallel transport</u>



Now consider a world line in spacetime. One would like a transport along the curve which keeps u^α tangent and keeps LRS_u orthogonal to the curve but introduces no additional rotation of the local rest space. Thus if $\{e_a\}$ is an orthonormal basis of LRS_u at one point, it will remain so along the curve, and $\{u, e_a\}$ will be an orthonormal frame as well, adapted to the local space + time split of the observer.

The unit tangent u^α is related to the parallel transport $u^{(\parallel)\alpha}$ of u^α along u^α by a parameter-dependent Lorentz transformation of the tangent space (since they are both timelike unit vectors). The "minimal" such Lorentz transformation along the curve (a continuous function of the curve parameter) that moves $u^{(\parallel)\alpha}$ into u^α is a boost in the "velocity-acceleration" plane in which the curve is instantaneously pseudorotating in the tangent space.

The anti-symmetric tensor $\Omega^{\alpha\beta} = u^{[\alpha} a(u)^{\beta]}$ is the pseudo-rotation angular velocity of this minimal boost in that plane, so we subtract it from the covariant derivative to obtain the FW derivative associated with the new transport we are seeking [we could add <u>any</u> antisymmetric tensor to the covariant derivative and obtain a new derivative whose transport involves <u>any</u> Lorentz transformation along the curve relative to parallel transport]:

$$
\begin{aligned}
{}^{(4)}\nabla_{fw}(u)\, X^\alpha &= \nabla_u X^\alpha - [u\wedge a(u)]^\alpha{}_\beta \qquad \Omega^\alpha{}_\beta \\
&= X^\alpha{}_{;\beta}u^\beta - [u^\alpha a(u)_\beta - a(u)^\alpha u_\beta] X^\beta
\end{aligned}
$$

note $${}^{(4)}\nabla_{fw}(u)\, u^\alpha = \underbrace{u^\alpha{}_{;\beta}u^\beta}_{a(u)^\alpha} - [u^\alpha \underbrace{a(u)_\delta u^\delta}_{0} - a(u)^\alpha \underbrace{u_\delta u^\delta}_{-1}] = a(u)^\alpha - a(u)^\alpha = 0$$

so u^α is transported along the curve by this transport.

$$
\begin{aligned}
{}^{(4)}\nabla_{fw}(u)\, g_{\alpha\beta} &= \underbrace{\nabla_u g_{\alpha\beta}}_{0} - g_{\gamma\beta}[u^\gamma a(u)_\alpha - a(u)^\gamma u_\alpha] - g_{\alpha\gamma}[u^\gamma a(u)_\beta - a(u)^\gamma u_\beta] \\
&= \underbrace{-u_\beta a(u)_\alpha + a(u)_\beta u_\alpha - u_\alpha a(u)_\beta + a(u)_\alpha u_\beta}_{-2\Omega_{(\alpha\beta)} = 0 \text{ by antisymmetry}} = 0
\end{aligned}
$$

so inner products are preserved as expected and therefore spatial tensors remain spatial (orthogonal to u^α) under this transport. In other words we can use it to ~~drag ato~~ transport along the curve an "adapted orthonormal frame", adapted to the observer decomposition of time and space.

The physical interpretation is that the change in spatial fields relative to FW propagated axes is zero when transported by FW transport they change in spacetime only to maintain orthogonality with respect to u^α with no additional spatial rotation — and FW propagated axes correspond to test gyro directions which do not rotate with respect to the local spacetime geometry.

Suppose we project the spacetime FW derivative:

$$P(u)^\alpha{}_\beta {}^{(4)}\nabla_{fw}(u) X^\beta = P(u)^\alpha{}_\beta [\nabla_u X^\beta + a(u)^\beta u_\gamma X^\gamma - \underbrace{u^\beta}_{0} a(u)_\gamma X^\gamma]$$
$$= \nabla_{(fw)}(u) X^\alpha - a(u)^\alpha [-u_\gamma X^\gamma]$$

so unless X^α is spatial, they disagree, but this is okay because along u^α:

$$P(u)^\alpha{}_\beta \, {}^4\underbrace{\nabla_{(fw)}(u) u^\beta}_{0} = \nabla_{(fw)}(u) u^\alpha - a(u)^\alpha [1] = a(u)^\alpha - a(u)^\alpha = 0$$

we get a vanishing spacetime FW derivative, but we need a nontrivial derivative to measure changes in u^α itself, and that's what $\nabla_{(fw)}(u)$ does for us: it coincides with the spacetime FW derivative on spatial fields, but gives the covariant derivative of u along u itself, but not of any coefficients since

$$\nabla_{(fw)}(u)[f u^\alpha] = P(u)^\alpha{}_\beta \nabla_u [f u^\alpha] = P(u)^\alpha{}_\beta [f \underbrace{\nabla_u u^\beta}_{a(u)^\beta} + (\nabla_u f) \underbrace{u^\beta}_{0}]$$
$$= f\, a(u)^\alpha .$$

While we are introducing new derivatives along the curve, we can add the vorticity tensor to the derivative to generate an additional spatial rotation of the local rest space which attempts to follow nearby observers without undergoing their expansion & shear:

$${}^4\nabla_{(cfw)}(u) X^\alpha = {}^4\nabla_{(fw)}(u) X^\alpha + \underbrace{\omega(u)^\alpha{}_\beta}_{\substack{\text{antisymmetric spatial tensor,} \\ \text{angular velocity of a spatial rotation.}}} X^\beta$$

${}^4\nabla_{(cfw)}(u)\, g_{\alpha\beta} = 0$ (again since $\omega(u)_{(\alpha\beta)} = 0$) so inner products are preserved as expected and projecting it:

$$P(u)^\alpha{}_\beta\, {}^{(4)}\nabla_{(cfw)}(u) X^\beta = P(u)^\alpha{}_\beta\, {}^{(4)}\nabla_{(fw)}(u) X^\beta + \omega(u)^\alpha{}_\beta X^\beta$$

$$= \underbrace{\nabla_{(fw)}(u) X^\beta + \omega(u)^\alpha{}_\beta X^\beta}\; - \underbrace{a(u)^\alpha [-u_\gamma X^\gamma]}$$

$$\equiv \underbrace{\nabla_{(cfw)}(u) X^\beta}$$

spatial corotating FW derivative (agrees with spacetime one for spatial fields).

gives nonzero derivative of u^α compared to spacetime "corotating" FW derivative

The associated transport anchors an orthonormal frame as best it can to the nearby observers without being deformed by their shear and expansion.

Thus we have 3 natural spatial (in the sense that derivatives of spatial fields are spatial) temporal derivatives along u (temporal in the sense of the direction of differentiation)

$$\nabla_{(cfw)}(u) X^\alpha = \nabla_{(fw)}(u) X^\alpha + \omega(u)^\alpha{}_\beta X^\beta = \nabla_{(lie)}(u) X^\alpha + \underbrace{\theta(u)^\alpha{}_\beta} X^\beta$$

If $\theta(u)^\alpha{}_\beta = 0$, then $\nabla_{(fw)} = \nabla_{(lie)}(u)$.

If $\omega(u)^\alpha{}_\beta = 0$, then $\nabla_{(cfw)} = \nabla_{(fw)}(u)$.

Note
$$\mathcal{L}_u g_{\alpha\beta} = u_{\alpha;\beta} + u_{\beta;\alpha} = (f\xi_\alpha)_{;\beta} + f\xi_{\beta;\alpha})$$
$$u = f\xi \qquad = f[\xi_{\alpha;\beta} + \xi_{\beta;\alpha}] + \xi_\alpha . f_{;\beta} + \xi_\beta f_{;\alpha}$$

project: $\mathcal{L}(u)_u\, g_{\alpha\beta} = f\, \underbrace{\mathcal{L}(u)_\xi\, g_{\alpha\beta}} + 0 \qquad = P(u) 2 u_{(\beta;\alpha)} = 2\theta(u)_{\alpha\beta}$

if zero (ξ = Killing vector field) then expansion tensor zero

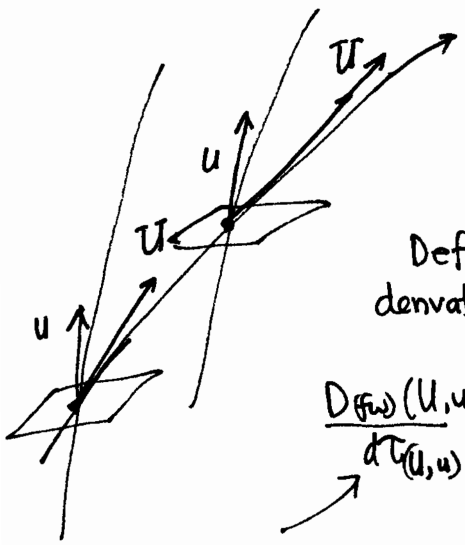
So if u^α is a Killing observer (normalized Killing vector field), the expansion tensor is zero (no deformation) and the Lie and Cfw temporal derivatives agree. When $\theta(u)_{\alpha\beta} \neq 0$, the projected Lie transport is not compatible with orthonormality.

We also have three temporal derivatives for tensors defined only along the worldline of u^α:

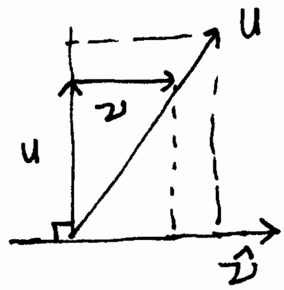
$$\frac{D_{(cfw)}(u,u) X^\alpha}{d\tau_u} = \frac{D_{(fw)}(u,u) X^\alpha}{d\tau_u} + \omega(u)^\alpha{}_\beta X^\beta = \frac{D_{(lie)}(u,u) X^\alpha}{d\tau_u} + \theta(u)^\alpha{}_\beta X^\beta$$

which agree with the others when acting on fields defined on spacetime.

Although we can no split the absolute derivative along an arbitrary curve, let's do it along another worldline:



$$U^\alpha = \gamma(u^\alpha + \nu(U,u)^\alpha) = \frac{dx^\alpha}{d\tau_U}$$



Define the corresponding three derivatives for a spatial field X^α:

$$\frac{D_{(fw)}(U,u)X^\alpha}{d\tau_{(U,u)}} = P(u)^\alpha{}_\beta \frac{DX^\beta}{d\tau_{(U,u)}} = \frac{D_{(cfw)}(U,u)X^\alpha}{d\tau_{(U,u)}} - \omega(u)^\alpha{}_\beta$$

$$= \frac{D_{(lie)}(U,u)X^\alpha}{d\tau_{(U,u)}} + \theta(u)^\alpha{}_\beta - \omega(u)^\alpha{}_\beta$$

$$\frac{d\tau_{(U,u)}}{d\tau_U} = \gamma(U,u) = [1 - \nu(U,u)^\alpha \nu(U,u)_\alpha]^{-1/2}$$

Spatial projection of the force equation $a(U)^\alpha = \nabla_U U^\alpha = f(U)^\alpha$ then leads to 3 different forms of the spatial "equation of motion"

$$\frac{D_{(tem)}(U,u)\, p(U,u)^\alpha}{d\tau_{(U,u)}} = F(U,u)^\alpha + F^{(G)}_{(tem)}(U,u)^\alpha, \quad tem = cfw, fw, lie.$$

and the temporal projection

$$\frac{dE(U,u)}{d\tau_{(U,u)}} = \nu(U,u)_\alpha [F(U,u)^\alpha + F^{(G)}_{(tem)}(U,u)^\alpha]$$

where $\gamma(U,u) = E(U,u)$ is the energy per unit mass

where $$F^{(G)}_{(tem)}(U,u) = \gamma(U,u)\,[\,\vec{g}(u)^\alpha + [?]\,\nu(U,u)^\alpha \times_u \vec{H}(u) - [?]\,\theta(u)^\alpha{}_\beta \nu(U,u)^\beta]$$

$\frac{1}{2}$ or 1

1 or 2

GEM force
Gravito Electro Magnetic

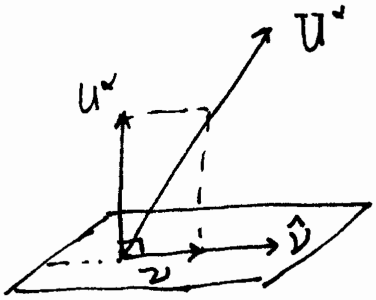
GE force field (acceleration field)

GM vector force field (vorticity field)

SG (spatial geometry) tensor force field (expansion field)

where there was no gravitational force in spacetime, the observers introduce 3 gravitational force fields due to their motion & relative motion in spacetime, but the form of each depends on the choice of spatial "reference frame" with respect to which changes in spacetime fields are measured, ie, what transport along the world line is used.

Relative direction of motion decomposition of LRS$_u$



Given 2 4-velocities we can perform a $4 = 1+3 = 1+(2+1)$ orthogonal decomposition of the tangent space by further decomposing the local rest space of u^α with respect to the direction

$$\hat{\nu}(U,u)^\alpha = \underbrace{\|\nu(U,u)\|}_{[\nu(U,u)^\alpha \nu(U,u)_\alpha]^{1/2}}{}^{-1}\, \nu(U,u)^\alpha \qquad \text{(unit vector)}$$

of relative motion:

$$P(u)^\alpha{}_\beta = \underbrace{\left[P(u)^\alpha{}_\beta - \hat{\nu}(U,u)^\alpha \hat{\nu}(U,u)_\beta\right]}_{P_u^{(\perp)}(U,u)^\alpha{}_\beta} + \underbrace{\left[\hat{\nu}(U,u)^\alpha \hat{\nu}(U,u)_\beta\right]}_{P_u^{(\parallel)}(U,u)^\alpha{}_\beta}$$

perpendicular to $\hat{\nu}$ $\qquad$ parallel to $\hat{\nu}$

projection in LRS$_u$ (as opposed to LRS$_U$)

U^α relative to u^α

If we first define the relative acceleration

$$a_{(\text{tem})}(U,u)^\alpha = \frac{D_{(\text{tem})}(U,u)\,\nu(U,u)^\alpha}{d\tau_{(U,u)}} \qquad \text{(one for each derivative)}$$

then decompose it:

$$a_{(\text{tem})}(U,u)^\alpha = \underbrace{a^{(\perp)}_{(\text{tem})}(U,u)^\alpha}_{} + \underbrace{a^{(\parallel)}_{(\text{tem})}(U,u)^\alpha}_{}$$

transverse relative acceleration or relative "centripetal" acceleration $\qquad$ longitudinal relative acceleration or linear relative acceleration

$$\frac{D_{(\text{tem})}(U,u)\,\hat{\nu}^\alpha}{d\ell_{(U,u)}} = \|\nu\|^{-1} \frac{D_{(\text{tem})}\hat{\nu}^\alpha}{d\tau} = \kappa\, \hat{n}^\alpha$$

magnitude . "normal" unit vector

$$= \frac{d\|\nu(U,u)\|}{d\tau_{(U,u)}}\, \hat{\nu}(U,u)^\alpha$$

for tem = cfw, fw

$$\frac{d\ell_{(U,u)}}{d\tau_{(U,u)}} = \|\nu(U,u)\|$$

But note:

$$\frac{D_{\text{tem}}(\|\nu\|\hat{\nu}^\alpha)}{d\tau_u} = \|\nu\| \underbrace{\frac{D_{\text{tem}}\hat{\nu}^\alpha}{d\tau}}_{\|\nu\|\kappa \hat{n}^\alpha} + \hat{\nu}^\alpha \frac{d\|\nu\|}{d\tau} = \underbrace{\frac{\|\nu\|^2}{\rho}\hat{n}^\alpha}_{a^{(\perp)\alpha} \text{ for tem = cfw, fw}} + \underbrace{\hat{\nu}^\alpha \frac{d\|\nu\|}{d\tau}}_{a^{(\parallel)\alpha} \text{ for}}$$

$\kappa = 1/\rho$

For tem = cfw, fw $\hat{n}_{(\text{tem})}(U,u)^\alpha$ is orthogonal to $\hat{\nu}(U,u)^\alpha$,
but not for tem = lie since the derivative does not respect inner products:

$$\|\hat{\nu}\|^2 = 1 \rightarrow 0 = \frac{D_{(\text{lie})}}{d\tau}(\|\hat{\nu}\|^2) = \frac{D_{(\text{lie})}}{d\tau}(g_{\alpha\beta}\hat{\nu}^\alpha\hat{\nu}^\beta) = 2\hat{\nu}_\beta \underbrace{\frac{D\hat{\nu}^\beta}{d\tau}}_{K\hat{n}^\beta_{(\text{lie})}} + \underbrace{\frac{D_{\text{lie}}g_{\alpha\beta}}{d\tau}}_{2\theta(u)_{\alpha\beta}}\hat{\nu}_\alpha\hat{\nu}_\beta$$

$$\rightarrow \quad K\hat{\nu}_\beta \hat{n}^\beta_{(\text{lie})} = -2\theta(u)_{\alpha\beta}\hat{\nu}^\alpha\hat{\nu}^\beta$$

which is only zero if the relative velocity is along a zero eigenvalue eigendirection of the expansion tensor.

Thus for tem = cfw, fw we have a complete analogy with the decomposition of acceleration in multivariable calculus on $\mathbb{R}^3$ or in elementary physics and:

$$a^{(\perp)}_{(\text{tem})}(U,u)^\alpha = \frac{\|\nu(U,u)\|^2}{\rho_{(\text{tem})}(U,u)}\hat{n}_{(\text{tem})}(U,u)^\alpha$$

relative centripetal acceleration. (tem = cfw, fw.)

Similarly we can decompose the spatial equation of motion
" $\frac{DP}{d\tau} = F + F^{(G)}$ " which has an extra gamma factor since
$p(U,u)^\alpha = \gamma(U,u)\nu(U,u)^\alpha$.

Example: For the static observers in Schwarzschild, circular orbits have purely transverse relative acceleration, while radial orbits have purely longitudinal or linear acceleration. In the former case one can think of the circular orbit condition on the forces as a balancing of the gravitational force (GE $\vec{g}$ force) and a "centrifugal" force associated with the sign reversal of the centripetal acceleration term in the spatial equations of motion. [All 3 derivatives are the same since $\theta(u)^\alpha{}_\beta = \omega(u)^\alpha{}_\beta = 0$.]

Example / exercise. In Kerr these (cfw) and (fw) derivatives are distinct for the static Killing observers. Again one can study circular orbits in the equatorial plane & the radial orbits along the axis of symmetry. What else can you do?