Some Notes on Splitting Techniques in General Relativity

by bob jantzen [June 1994, June 1996]

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.This built upon the notes of ICM1 and introduced the ideas of splitting spacetime into space plus time.

- ss1994.pdf: 17 pages, 525K

These were continued in June 1996 to precede the Spacetime Splitting 1996 notes, discussing the splitting of derivatives and definition of gravitational forces:

- ss199496.pdf: 9 pages, 300K

Some Notes on Splitting Techniques in General Relativity

Bob Jantzen

June, 1994

Outline

1) orthogonal coordinates on flat or constant curvature manifolds with metric
   [leads to FRW spatial geometry and simplest splitting examples]

2) gaussian normal coordinates on an arbitrary manifold with metric
   [hypersurface metric, intrinsic and extrinsic curvature, relation between them in flat case]

See excerpts from 1987 notes: pp. 1–45

JUNE 1994:

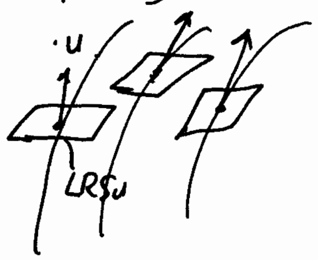
Important pages 2–4, 7–14, 25–29.

The rest is extra for those interested in going deeper into groups of motions.

3) a general "nonlinear reference frame" on a spacetime: nonorthogonal decompositions and orthogonalization at the tangent space level

Observer families and Measurements

One can perform the orthogonal decomposition of tensor fields on spacetime corresponding to a family of observers either abstractly without reference to any frame or coordinate system using the algebra of projection operators or concretely using a frame adapted to the orthogonal splitting or associated with an adapted coordinate system. If the frame is orthonormal, with $e_0 = u$, then the decomposition in terms of 0 and nonzero values of the index is exactly what one does in special relativity globally. No nontrivial projection is needed. For a coordinate system adapted in some way to the observers but not to the orthogonal splitting, a nontrivial projection is required.



Some general remarks about Special Relativity (S.R.) are given.
Some remarks about natural families of observers in G.R. are given.
Some remarks about N.R. (Non Relativistic) problems with noninertial coordinates follow.

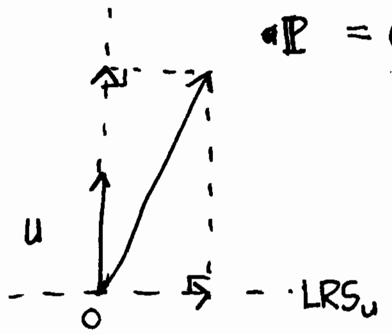
Then a simple 2-D example of Euclidean & Lorentz projections on a linear space, to see the differences between them due to the signature.

Then some calculations with the Rindler family of accelerated observers, first done using components in the original inertial coordinate frame, showing the correspondence with an adapted orthonormal frame, and then in Rindler coordinates adapted to these observers. (Both in 2-D).

Then some examples of splitting the covariant derivative of the observer congruence 4-velocity, and of the Lorentz force equation for a single trajectory not coinciding with an observer worldline. This leads to noninertial spatial gravitational forces

Finally one looks at a general coordinate system adapted in one of two ways to the family of observers. In the general case of rotating observers, this leads to two different points of view which correspond to the observer families which are locally nonrotating or those which are nonrotating with respect to infinity (spatial infinity) in the black hole case.

observer
Measurement is a projection process (orthogonal projection)

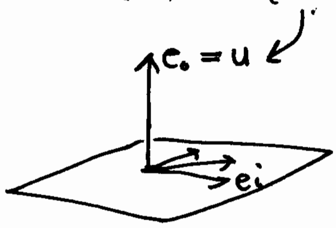


$$\mathbb{P} = \underbrace{(-u\cdot\mathbb{P})}_{\substack{E \\ \text{energy} \\ \text{(scalar)}}}\, u + \underbrace{\underbrace{P(u)}_{\substack{\text{orth} \\ \text{proj} \\ \perp \text{ to } u}}\,\mathbb{P}}_{\vec{P}}$$

$\vec{P}$ three-momentum (vector)

often accomplished by ON frame in SR. (e_0 defines 4-velocity of an observer)

$\{e_\alpha\} = \{e_0, e_i\}$



~~basis~~ ~~point~~ of view

$$\mathbb{P} = \underbrace{\mathbb{P}^0 e_0}_{\substack{\text{temporal} \\ \text{proj}}} + \underbrace{\mathbb{P}^i e_i}_{\substack{\text{spatial} \\ \text{proj}}}$$

component point of view

$(\mathbb{P}^\alpha) \leftrightarrow (\mathbb{P}^0 ; \mathbb{P}^i)$

zero index or not ~~divides~~ groups components.

another example (symmetric tensor)

$$T = \underbrace{T^{00}\, e_0\otimes e_0}_{\text{temporal}} + \underbrace{T^{0i}(e_0\otimes e_i + e_i\otimes e_0)}_{\text{mixed}} + \underbrace{T^{ij}\, e_i\otimes e_j}_{\text{spatial}}$$

$$(T^{\alpha\beta}) \leftrightarrow (\underbrace{T^{00}}_{2\text{ zero}} ; \underbrace{T^{0i}}_{1\text{ zero}} ; \underbrace{T^{ij}}_{0\text{ zero}})$$

↓

T^{00} — spatial scalar

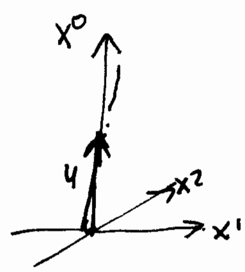
$T^{0i} e_i$, — spatial vector

$T^{ij} e_i\otimes e_j$ — spatial tensor

IN S.R. we are usually using ON coords, so coord components are directly ON components. Can work just with components.

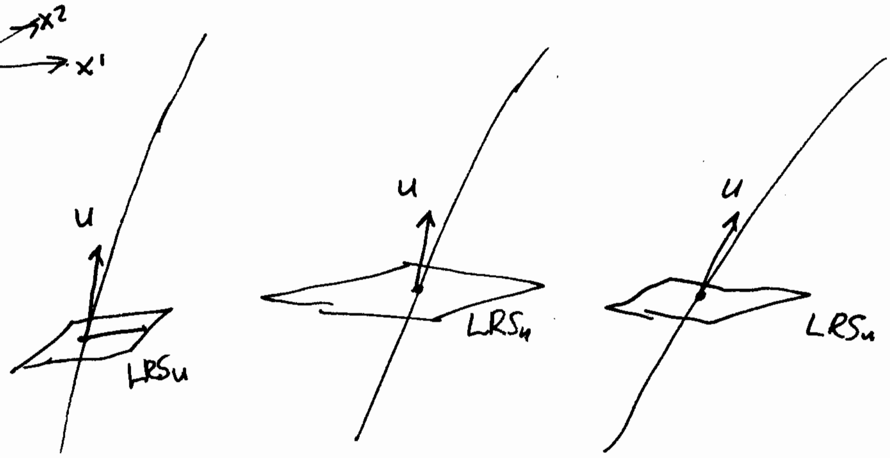
In G.R. better to be more careful since frame itself (or coords) contain nontrivial information. Also no single GLOBAL OBSERVER as in SR.

SR:



. single observer at single event implicitly defines a corresponding observer at every event with ~~the~~ ~~same~~ ~~relative~~ zero relative velocity

GR



In G.R. must introduce a family of observers covering spacetime, equivalent to a family of observer worldlines.

Although one can do this <u>arbitrarily</u>, it is usually better to adapt the choice to the geometry of the spacetime if possible. (preferred observer families).

Any spacetime admitting <u>orthogonal coords</u> has such preferred observers.

$$ds^2 = -dt^2 + R(t)^2 \gamma_{ij}\, dx^i dx^j \qquad \text{FRW}$$

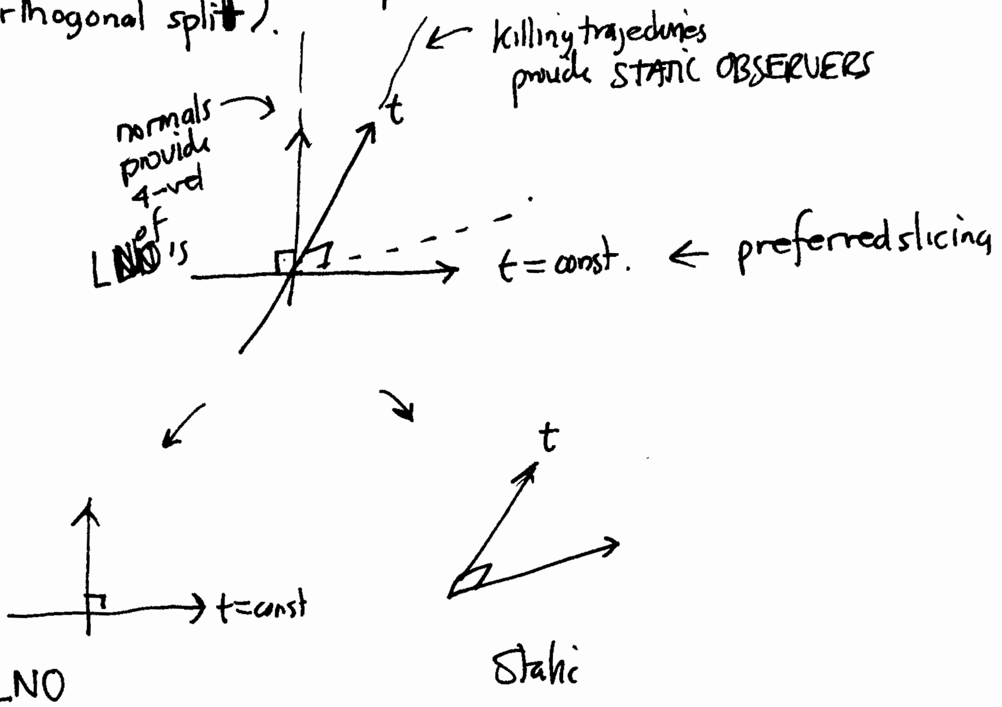
$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \qquad \text{Schw.}$$

But rotating black holes do not. They <u>do</u> have preferred families of observers though.

1) STATIC observers : nonrotating with respect to infinity
2) Locally nonrotating observers : nonrotating with respect to the local spacetime geometry.

These are in relative motion.

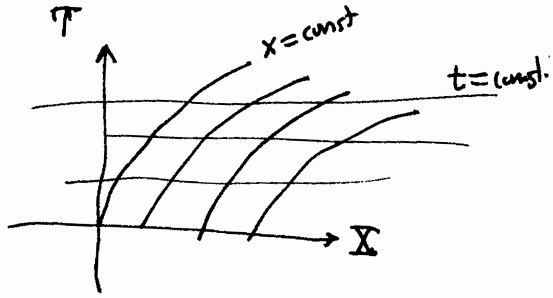
Here ORTHOGONAL PROJECTION is necessary since the natural coordinates do not have an orthogonal decomposition related to either set of observers (nonorthogonal split).



Here one needs to use the Lorentz orthogonal projection procedure to understand what each observer family measures.

The motion of the observers themselves influences what they see, like accelerated coordinates in NR mechanics

N.R.* ~~uniform~~ constant linear acceleration (T, X) inertial coords



$$t = T$$
$$x = X + \tfrac{1}{2} g T^2$$
$\Big\} \to (t, x)$ noninertial coordinates

$x(t)$ path in new coordinates:

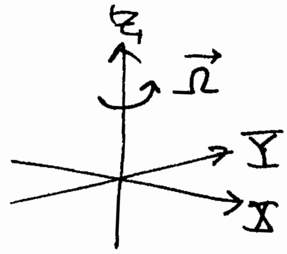
$$\dot{x} = \dot{X} + gT$$
$$\ddot{x} = \underbrace{\ddot{X}}_{F} + g \quad = F + g$$

F: applied force; g: extra constant force like const Grav-field. [due to accelerated coordinate system]

* non relativistic

N.R. rotating coords

T:



$$t = t$$
$$x^i = R^i{}_j X^j \to \ddot{\vec{x}} = \underbrace{-\Omega \times (\Omega \times \vec{x})}_{\text{"}\vec{g}\text{"}} + \dot{\vec{x}} \times \underbrace{(2\vec{\Omega})}_{\text{"}\vec{H}\text{"}}$$

For point fixed in rotating frame

$$\cancel{\ddot{\vec{x}} = \vec{\Omega} \times \dot{\vec{x}} + \dot{\vec{x}} \times (-\vec{\Omega} \times (\vec{\Omega} \times \vec{x}))}$$

[If moving in rotating frame there is the extra term due to the time derivative of the rotating frame components]

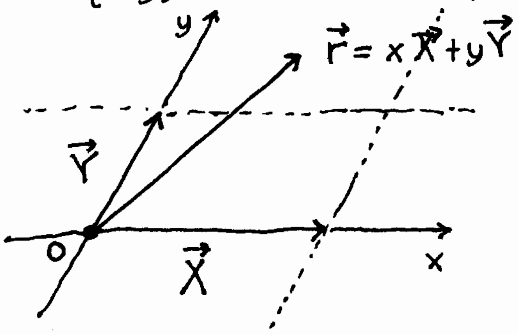
$$\left[\ddot{\vec{x}} = \vec{E} + \dot{\vec{x}} \times \vec{B} \longrightarrow \text{like Lorentz force law} \right]$$

IN GR. need to change time as well for correct local time direction.

The motion of the observers will introduce "noninertial frame" effects into the description of equations of motion of particles under the influence of forces.

Before dealing with those, we look at the process of orthogonal projection in the Lorentz case first looking at the Euclidean case in 2 dimensions.

EXAMPLE: How to orthogonalize nonorthogonal Cartesian coordinates $\{x, y\}$ in the Euclidean plane associated with 2 vectors $\{\vec{X}, \vec{Y}\}$.



$$\|\vec{r}\|^2 = Ax^2 + Bxy + Cy^2 = \text{square of length of } \vec{r}$$

Assume $B^2 - 4AC < 0$, $A > 0$, $C > 0$ for positive-definiteness.

If $B > 0$, then $\vec{X}$ and $\vec{Y}$ form an acute angle as in the diagram.

[Note $A = \vec{X}\cdot\vec{X}$, $C = \vec{Y}\cdot\vec{Y}$, $B = 2\vec{X}\cdot\vec{Y}$]

There are two choices for completing the square:

$$\|\vec{r}\|^2 = A\left(x^2 + \frac{B}{A}xy\right) + Cy^2 \qquad = Ax^2 + C\left(y^2 + \frac{B}{C}yx\right)$$

$$= A\underbrace{\left(x + \frac{B}{2A}y\right)}_{\bar{x}}{}^2 + \left(C - \frac{B^2}{4A}\right)\underbrace{y}_{\bar{y}}{}^2 \qquad = \left(A - \frac{B^2}{4C}\right)\underbrace{x}_{\bar{\bar{x}}}{}^2 + C\underbrace{\left(y + \frac{B}{2C}x\right)}_{\bar{\bar{y}}}{}^2$$

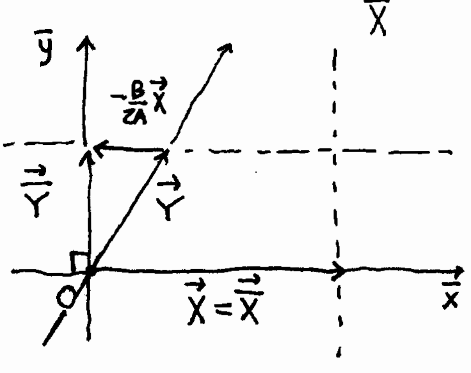
$\bar{x} = x + \frac{B}{2A}y$	$x = \bar{x} - \frac{B}{2A}\bar{y}$
$\bar{y} = y$	$y = \bar{y}$

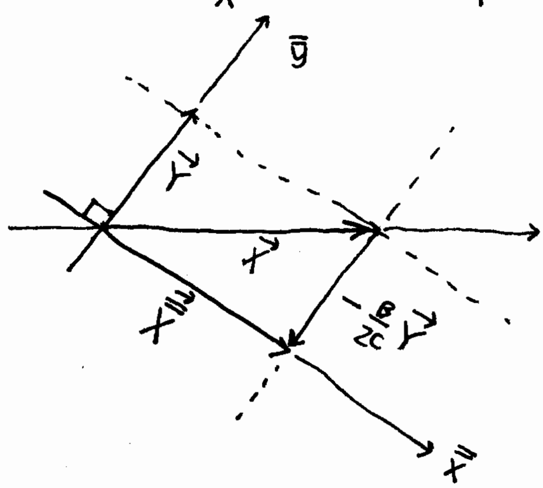
$\bar{\bar{x}} = x$	$x = \bar{\bar{x}}$
$\bar{\bar{y}} = y + \frac{B}{2C}x$	$y = \bar{\bar{y}} - \frac{B}{2C}\bar{\bar{x}}$

To find the corresponding basis vectors:

$$\vec{r} = x\vec{X} + y\vec{Y} = \left(\bar{x} - \frac{B}{2A}\bar{y}\right)\vec{X} + \bar{y}\vec{Y} \qquad = \bar{\bar{x}}\vec{X} + \left(\bar{\bar{y}} - \frac{B}{2C}\bar{\bar{x}}\right)\vec{Y}$$

$$= \bar{x}\underbrace{\vec{X}}_{\vec{\bar{X}}} + \bar{y}\underbrace{\left(\vec{Y} - \frac{B}{2A}\vec{X}\right)}_{\vec{\bar{Y}}} \qquad = \bar{\bar{x}}\underbrace{\left(\vec{X} - \frac{B}{2C}\vec{Y}\right)}_{\vec{\bar{\bar{X}}}} + \bar{\bar{y}}\underbrace{\vec{Y}}_{\vec{\bar{\bar{Y}}}}$$



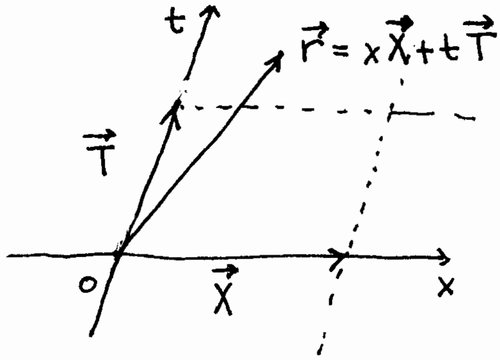


In each case the new basis vector is obtained by orthogonal projection of one of the old ones, a process complementary to completing the square on the corresponding coordinates.

EXAMPLE PART II : The Lorentz Plane case (indefinite quadratic form)

Although the algebra is unchanged in the indefinite case when $B^2-4AC>0$, the double barred picture changes and it is also convenient to use the coordinate names $\{x,t\}$ instead of $\{x,y\}$.

[Note $A=\vec X\cdot\vec X>0$, $C=\vec T\cdot\vec T<0$, $B^2-4AC>0$
Choose $B=2\vec X\cdot\vec T>0$ for diagram]



$$\|\vec r\|^2 = Ax^2+Bxt+Ct^2$$

$$\|\vec r\|^2 = A\left(x^2+\frac{B}{A}xt\right)+Ct^2 \qquad = Ax^2 + C\left(t^2+\frac{B}{C}tx\right)$$

$$= A\underbrace{\left(x+\frac{B}{2A}t\right)}_{\bar x}^2+\left(C-\frac{B^2}{4A}\right)\underbrace{t}_{\bar t}^2 \qquad = \left(A-\frac{B^2}{4C}\right)\underbrace{x}_{\bar{\bar x}}^2 + C\underbrace{\left(t+\frac{B}{2C}x\right)}_{\bar{\bar t}}^2$$

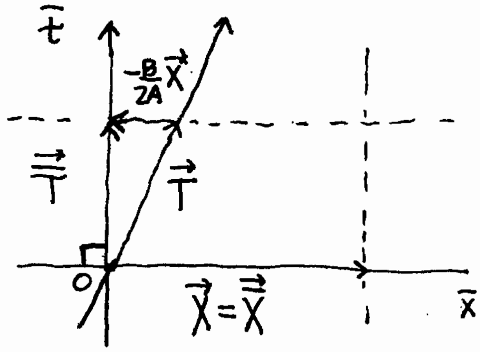
$\bar x = x+\frac{B}{2A}t$	$x=\bar x-\frac{B}{2A}\bar t$
$\bar t = t$	$t=\bar t$

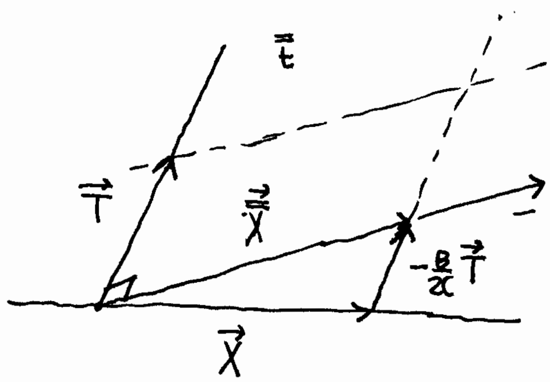
$\bar{\bar x} = x$	$x = \bar{\bar x}$
$\bar{\bar t} = t+\frac{B}{2C}x$	$t=\bar{\bar t}-\frac{B}{2C}\bar{\bar x}$

and the corresponding basis vectors:

$$\vec r = x\vec X + t\vec T = \left(\bar x-\frac{B}{2A}\bar t\right)\vec X+\bar t\vec T \qquad = \bar{\bar x}\vec X+\left(\bar{\bar t}-\frac{B}{2C}\bar{\bar x}\right)\vec T$$

$$= \bar x\underbrace{\vec X}_{\vec{\bar X}}+\bar t\underbrace{\left(\vec T-\frac{B}{2A}\vec X\right)}_{\vec{\bar T}} \qquad = \bar{\bar x}\underbrace{\left(\vec X-\frac{B}{2C}\vec T\right)}_{\vec{\bar{\bar X}}}+\bar{\bar t}\underbrace{\vec T}_{\vec{\bar{\bar T}}}$$

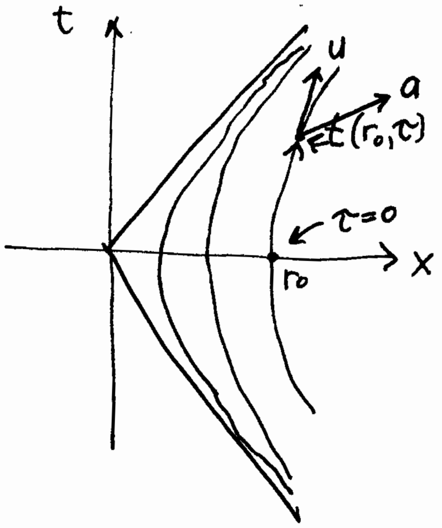




$\vec{\bar T}$ is the local time direction for an observer whose local rest space is along $\vec X$, and is the orthogonal projection of $\vec T$ with respect to $\vec X$.

$\vec{\bar{\bar X}}$ is the local rest space direction for an observer moving along the direction $\vec T$, and is the orthogonal projection of $\vec X$ with respect to $\vec T$.

"Rindler spacetime": uniformly accelerated observers



consider the following family of worldlines on 2-D Minkowski space (only outside the lightcone for $x>0$)

$$\begin{cases} t = r_0 \sinh(\tau/r_0) \\ x = r_0 \cosh(\tau/r_0) \end{cases}$$

$$\left.\begin{cases} u^t = \dot{t} = \cosh(\tau/r_0) \\ u^x = \dot{x} = \sinh(\tau/r_0) \end{cases}\right\} \text{4-velocity}$$

$$-(u^t)^2 + (u^x)^2 = 1.$$

so these are propertime parametrized worldlines

$$\left.\begin{cases} a^t = \dot{u}^t = \frac{1}{r_0}\sinh(\tau/r_0) \\ a^x = \dot{u}^x = \frac{1}{r_0}\cosh(\tau/r_0) \end{cases}\right\} \text{4-acceleration}$$

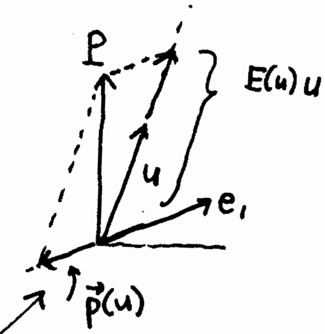
$$-(a^t)^2 + (a^x)^2 = \frac{1}{r_0^2} \rightarrow \|a\| = \frac{1}{r_0}$$, spacelike vector (constant along each worldline)

$$-u^t a^t + u^x a^x = 0 \leftarrow$$ "spatial vector w.r.t. u"

"constant acceleration"

In fact $e_0 = u$, $e_1 = \frac{a}{\|a\|} \leftrightarrow (\sinh(\tau/r_0), \cosh(\tau/r_0))$ is an ON frame on this part of the spacetime, with e_1 spanning to local rest space of the observer u.

A particle at rest in the original inertial coordinates has 4-momentum
$P^t = E$, $P^x = 0$.



$$P = E(u)\,u + \vec{p}(u)$$

$$-u_\alpha P^\alpha = +u^t P^t - u^x P^x = \underbrace{+E\cosh(\tau/r_0)}$$

energy observed by observer u. (changes with proper time of observer)

$$\vec{p}(u) = P + (u \cdot P)\,u$$

lorentz projection changes signs $\left(-\frac{u \cdot P}{u \cdot u} = +u \cdot P\right)$

$$u_\alpha P^\alpha = -E\cosh(\tau/r_0)$$

$$p(u)^x = \underset{0}{P^x} - E\cosh(\tau/r_0)\sinh(\tau/r_0) = -E\cosh(\tau/r_0)\sinh(\tau/r_0)$$

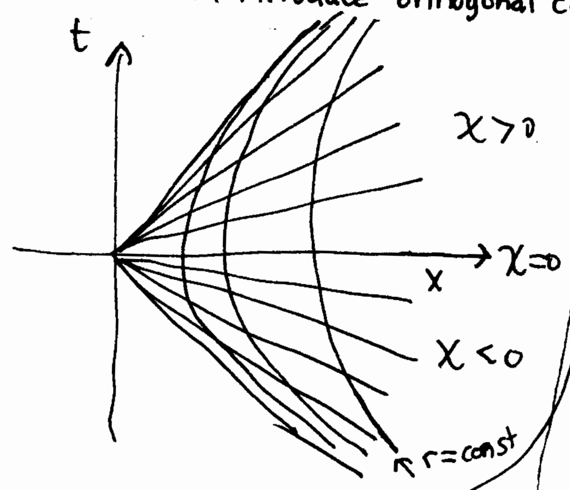
$$p(u)^t = P^t - E\cosh(\tau/r_0)\cosh(\tau/r_0) = E(1-\cosh^2(\tau/r_0)) = -E\sinh^2(\tau/r_0)$$

$$\vec{p}(u) : \underbrace{-E\sinh(\tau/r_0)}_{\|\vec{p}(u)\|}\ \underbrace{(\sinh(\tau/r_0), \cosh(\tau/r_0))}_{e_1}$$

reverse direction from e_1

So $E(u)^2 - \|\vec{p}(u)\|^2 = E^2(\cosh^2(\tau/r_0) - \sinh^2(\tau/r_0)) = E^2$. ✓

we can introduce orthogonal coordinates adapted to this family of observers.



$$ds^2=-dt^2+dx^2$$
$$=\dots\ \text{(calculate)}$$
$$=-r^2d\chi^2+dr^2$$

$$t=r\sinh\chi \qquad r=\sqrt{x^2-t^2}$$
$$x=r\cosh\chi \qquad \chi=\tanh^{-1}t/x$$

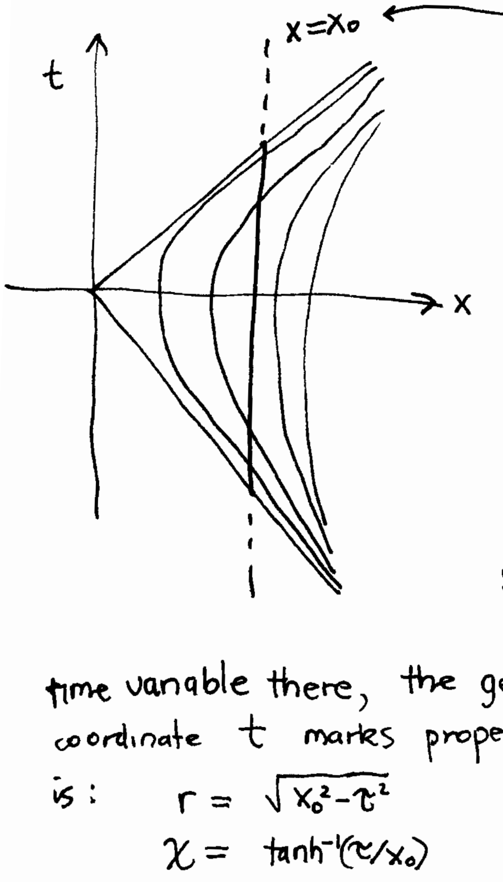
The worldlines then become:

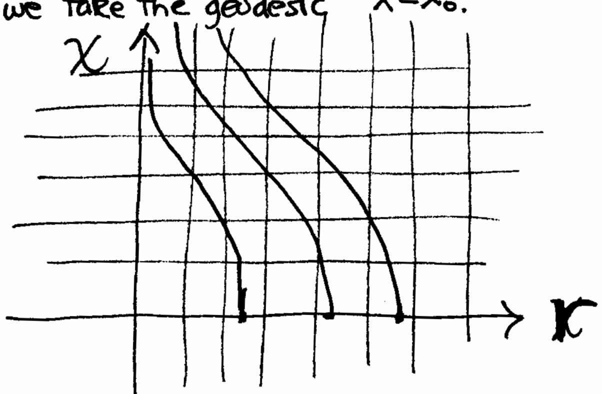
$$\chi=\tau/r_0$$
$$r=r_0$$

In other words the "time" lines (coordinate X=const lines) coincide with the observer worldlines but require reparametrization to get propertime parametrization.

These coordinates have a singularity on the null cone where they break down.

Suppose we take the geodesic $x=x_0$.





In the new coordinates it will start out in the χ direction but move towards smaller values of r, with χ going to ∞ as $t\to x_0$ at the light cone.

Because of the coordinate singularity which stretches the time variable there, the geodesic motion appears to freeze there. The coordinate t marks proper time on this geodesic, so the proper time parametrization is:

$$r=\sqrt{x_0^2-\tau^2} \qquad u^r=\frac12\frac{-2\tau}{\sqrt{x_0^2-\tau^2}}=-\frac{\tau}{r} \qquad u^{\hat r}=u^r=-\tau/r$$

$$\chi=\tanh^{-1}(\tau/x_0) \qquad u^{\chi}=\frac{1}{1-\tau^2/x_0^2}\frac{1}{x_0}=\frac{x_0}{r^2} \qquad u^{\hat\chi}=ru^{\chi}=\frac{x_0}{r}$$

$$\frac{u^{\hat r}}{u^{\hat\chi}}=-\frac{\tau}{x_0}=\beta\,,\quad \gamma=1/\sqrt{1-\beta^2}=\frac{1}{\sqrt{1-\tau^2/x_0^2}}=x_0/r$$

3-~~velocity~~ (increases to -1 as $\tau\to x_0$) gamma factor

$$\dot u^r=\ddot r=\frac{\dot r}{r^2}\tau-\frac1r=-\frac{1}{r^3}\tau^2-\frac1r=-\frac1r\left(\frac{\tau^2}{r^2}+1\right)=-\frac{x_0^2}{r^3}$$

If we define $g^r=-\frac{\partial}{\partial r}\ln|g_{\chi\chi}|=-\frac1r$, then $\gamma^2g^r=-\frac{x_0^2}{r^3}$ $\Big\}$ $\dot u^r=\ddot r=\underbrace{\gamma^2 g^r}$

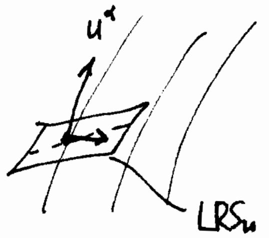
as though a gravitational force is present

we need to learn how to make this more precise.

SPLITTING: 1) algebra of projecting tensors and derivatives (g-forces)
2) relation to adapted coordinate systems (g-potentials)

EX observer congruence itself. $u^\alpha{}_{;\beta}$ is a tensor — can be split.

u^α



If had ON frame $u = e_0$, $\{e_\alpha\}$:

$u_{0;0}$; $u_{0;i}$, $u_{i;0}$, $u_{i;j}$

apart from sign $u^{\alpha;\beta} u_\alpha u_\beta$ $\quad u^\beta u_{\beta;\gamma} P^\gamma{}_{\alpha}$ $\quad P^\beta{}_\alpha u_{\beta;\gamma} u^\gamma$ $\quad P^\gamma{}_\alpha P^\delta{}_\beta u_{\gamma;\delta}$

BUT CAN DO SPLITTING WITHOUT ANY FRAME JUST USING THE PROJECTION FORMALISM:

$$\delta^\alpha{}_\beta = \underbrace{-u^\alpha u_\beta}_{\frac{u^\alpha u_\beta}{u^\gamma u^\gamma}} + \underbrace{\delta^\alpha{}_\beta + u^\alpha u_\beta}_{\delta^\alpha_\beta - \frac{u^\alpha u_\beta}{u^\gamma u^\gamma}} = \underbrace{T(u)^\alpha{}_\beta}_{\text{temporal projection}} + \underbrace{P(u)^\alpha{}_\beta}_{\text{spatial projection}}$$

$\nabla_u u^\alpha = a^\alpha$ ← acceleration is spatial

$\quad\| $

$u^\alpha{}_{;\beta} u^\beta$

$u_\alpha a^\alpha = u_\alpha u^\alpha{}_{;\beta} u^\beta$

$= \frac{1}{2}(u_\alpha u^\alpha)_{;\beta} u^\beta$

$= 0$

$\frac{1}{2}(u_\alpha u^\alpha = -1)_{;\beta}$

$u_\alpha u^\alpha{}_{;\beta} = 0$

$$u_{\alpha;\beta} = \delta^\gamma{}_\alpha \delta^\delta{}_\beta u_{\gamma;\delta}$$

$$= (-u^\gamma u_\alpha + P^\gamma{}_\alpha)(-u^\delta u_\beta + P^\delta{}_\beta)\, u_{\gamma;\delta}$$

$= u^\gamma u^\delta\, [u_{\gamma;\delta} u^\gamma u^\delta] \to 0$ ↔ scalar vanishes

$- u_\alpha [u^\gamma P^\delta{}_\beta u_{\gamma;\delta}]$ ↔ one of two vectors vanish

$- u_\beta [u^\delta P^\gamma{}_\alpha u_{\gamma;\delta}]$ — a_α (automatically spatial)

$+ P^\gamma{}_\alpha P^\delta{}_\beta u_{\gamma;\delta}$ ↔ spatial tensor

So

$$u_{\alpha;\beta} = -\underbrace{a_\alpha}_{\text{acceleration}} u_\beta + \underbrace{P^\gamma{}_\alpha P^\delta{}_\beta u_{\gamma;\delta}}_{\text{spatial tensor.}}$$

$\qquad\qquad\qquad\|$

$P^\gamma{}_{(\alpha} P^\delta{}_{\beta)} u_{\gamma;\delta} = \Theta_{\alpha\beta}$ (sym) expansion tensor

$+ P^\gamma{}_{[\alpha} P^\delta{}_{\beta]} u_{\gamma;\delta} = -\omega_{\alpha\beta}$ (antisym) rotation tensor.

↔ $(a_\alpha, \underbrace{\Theta_{\alpha\beta}, \omega_{\alpha\beta}}_{\text{describe relative motion of observers}})$ kinematical quantities of observer congruence.

↳ acceleration of each observer

EX Lorentz force measured by observers (in rest space, acceleration only due to electric field)

$a^\alpha = \nabla_u u^\alpha = \frac{q}{m} F^\alpha{}_\beta u^\beta = E^\alpha$ electric field

(set = 1 for convenience)

automatically spatial:
$\left(E^\alpha u_\alpha = F^\alpha{}_\beta u^\beta u_\alpha = F_{\alpha\beta} u^\alpha u^\beta = 0 \right)$

otherwise $a^\alpha = \frac{q}{m} E^\alpha$ etc

$F_{\alpha\beta} = \delta^\gamma{}_\alpha \delta^\delta{}_\beta F_{\gamma\delta} = \dots = u_\alpha u_\beta \cancel{F_{\gamma\delta} u^\gamma u^\delta}_0 = \pm(u_\alpha E_\beta - u_\beta E_\alpha) + B_{\alpha\beta}$

$\pm\, u_\alpha\, P^\gamma{}_\beta F_{\gamma\delta} u^\delta$ — E_β

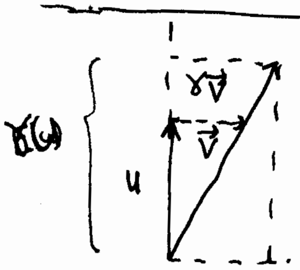
$\mp\, u_\beta\, P^\gamma{}_\alpha F_{\gamma\delta} u^\delta$ — E_α

$+\, P^\gamma{}_\alpha P^\delta{}_\beta F_{\gamma\delta}$ — $B_{\alpha\beta}$ → "spatial dual" defines magnetic field observed by observer u

check sign

↳ $(E_\alpha, B_{\alpha\beta})$

Ex. Lorentz Force on a particle in relative motion



$$\mathbb{U} = \gamma(u)[u + \vec{v}]$$

$-u_\alpha \mathbb{U}^\alpha$ $\qquad v^\alpha = \dfrac{P^\alpha{}_\beta \mathbb{U}^\beta}{-u_\gamma \mathbb{U}^\gamma}$

(in an ON frame as in SR; $\mathbb{U}^\alpha \leftrightarrow \gamma(1, v^i)$; $v^i = u^i/u^0$)

quotes, since the covariant derivative usually acts on a field defined on spacetime, not just on a single curve, but in fact if one extends the field off the curve, the result is independent of the extension, which is how the absolute derivative along a curve is defined.

$$\frac{D\mathbb{U}^\alpha}{d\tau} = \text{"}\underbrace{\nabla_{\mathbb{U}} \mathbb{U}^\alpha}_{\text{4-vector}}\text{"} = \left(\frac{q}{m}\right) F^\alpha{}_\beta \mathbb{U}^\beta = F^\alpha{}_\beta(\gamma)(u^\beta + v^\beta)$$

$$= \gamma \underbrace{F^\alpha{}_\beta u^\beta}_{E^\alpha} + F^\alpha{}_\beta v^\beta$$

$\frac{q}{m}$: set = 1 for convenience

SPLIT: temp, spatial $\longrightarrow$ force equation:

$$P^\alpha{}_\beta \mathbb{U}^\alpha{}_{;\gamma} \mathbb{U}^\gamma = \gamma P^\alpha{}_\beta (E^\beta + F^\beta{}_\gamma v^\gamma)$$

$$= \gamma(E^\alpha + \underbrace{[P(u) F]^\alpha{}_\beta v^\beta}_{B^\alpha{}_\beta v^\beta \,\to\, [v \times B]^\alpha})$$

(requires definition of spatial dual)

how to express projection of $\dfrac{D\mathbb{U}^\alpha}{d\tau}$.

$$P^\alpha{}_\beta \left(\frac{D\mathbb{U}^\beta}{d\tau}\right) = \gamma(E + v \times B)^\alpha$$

familiar form of Lorentz force

difference between proper time of particle and observer time

define $\gamma\, d\tau = d\tau_u$ as in S.R. along the single worldline.

$\cancel{\dfrac{d\mathbb{U}^\alpha}{d\tau}}$

$$\gamma^{-1} P^\alpha{}_\beta \left(\frac{D\mathbb{U}^\beta}{d\tau}\right) \underset{\substack{\uparrow \\ \text{corresponds to}}}{\sim} \frac{dp^i}{dt} = E^i + (v \times B)^i \quad \text{in S.R. inertial coords.}$$

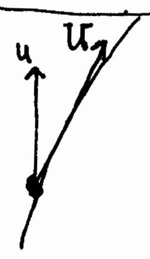
one must introduce projected derivatives when splitting covariant derivative expressions

(spatial) Fermi Walker derivative along $\mathbb{U}$: $\qquad \dfrac{D(u)\mathbb{U}^\alpha}{d\tau} = P^\alpha{}_\beta \dfrac{D\mathbb{U}^\beta}{d\tau}$ $\qquad$ (only for tensors along worldline of $\mathbb{U}$)

spatial cov der wrt u: $\qquad \nabla(u)_\alpha X^\beta = P^\gamma{}_\alpha P^\beta{}_\delta \nabla_\gamma X^\delta$

Fermi-Walker derivative along u: $\qquad \nabla_{fw(u)} X^\alpha = P^\alpha{}_\beta \nabla_u X^\beta = P^\alpha{}_\beta X^\beta{}_{;\gamma} u^\gamma$

if $u = \mathbb{U}$ related as above



One can also introduce a new parameter on the single particle worldline of $\mathbb{U}$ which corresponds at each point to the observer (u) proper time of the observer at that point, in the same relationship as the Minkowski spacetime global inertial time coordinate and a worldline proper time:

$$d\tau_u = \gamma\, d\tau \quad \text{or} \quad \frac{d\tau_u}{d\tau} = \gamma$$

$$\frac{D(u)U^\alpha}{d\tau_u} = \frac{D(u)}{d\tau_u}\left[\gamma(u^\alpha+v^\alpha)\right] = \underbrace{\frac{D(u)}{d\tau_u}\left[\gamma u^\alpha\right]} + \frac{D(u)}{d\tau_u}\big(\underbrace{\gamma v^\alpha}_{p(u)^\alpha}\big)$$

$p(u)^\alpha$ ← (set $m=1$ for simplicity or interpret $p(u)$ as momentum per unit mass)

$$= \underbrace{\gamma \frac{D(u)u^\alpha}{d\tau_u}}_{\equiv -\gamma F^{(G)\alpha}_{(fw)}\ \text{definition.}} + \underbrace{P^\alpha{}_\beta u^\beta}_{=0}\frac{D(u)\gamma}{d\tau_u} + \frac{D(u)p(u)^\alpha}{d\tau_u}$$

$$\underbrace{\frac{D(u)}{d\tau_u}p(u)^\alpha}_{\substack{\text{spatial rate of change of}\\ \text{spatial momentum along } U \text{ with}\\ \text{respect to } u \text{ proper time}}} = \underbrace{\gamma F^{(G)\alpha}_{(fw)}}_{\substack{\text{noninertial}\\ \text{gravitational}\\ \text{force}}} + \underbrace{E^\alpha + [v\times B]^\alpha}_{\text{EM force observed by } u}\qquad \left(\frac{q}{m}=1\right)$$

$$F^{(G)\alpha}_{(fw)} = -\frac{D(u)u^\alpha}{d\tau_u} = -\gamma^{-1}\frac{D(u)u^\alpha}{d\tau} = \text{"}-\gamma^{-1}P^\alpha{}_\beta \nabla_U u^\alpha\text{"}$$

$$= -\gamma^{-1}P^\alpha{}_\beta u^\beta{}_{;\gamma}U^\gamma = -P^\alpha{}_\beta u^\beta{}_{;\gamma}(u^\gamma+v^\gamma)$$

$$= -P^\alpha{}_\beta u^\beta{}_{;\gamma}u^\gamma + -P^\alpha{}_\beta u^\beta{}_{;\gamma}\underbrace{v^\gamma}_{P^\gamma{}_\delta v^\delta}$$

$$= \underbrace{-a^\alpha}_{g^\alpha} - \underbrace{(\theta^\alpha{}_\beta - \omega^\alpha{}_\beta)v^\beta}_{+H^\alpha{}_\beta = \omega^\alpha{}_\beta - \theta^\alpha{}_\beta}$$

$$\frac{D(u)p(u)^\alpha}{d\tau_{(u)}} = \gamma(\overset{\text{GE field}}{g^\alpha} + \overset{\text{GM field}}{H^\alpha{}_\beta v^\beta}) + E^\alpha + (v\times B)^\alpha$$

$H^\alpha{}_\beta v^\beta \to (\vec v\times\vec H)^\alpha - \theta^\alpha{}_\beta v^\beta$

extra factor of γ in gravitational case, with observer time parametrization — if we return to $d\tau$, get extra factor of γ

$$\frac{D(u)}{d\tau}p(u)^\alpha = \gamma^2(g+\cdots)^\alpha + \gamma(E+v\times B)^\alpha$$

as in Rindler spacetime example on page 7: orthogonal coords give simple spatial projection

$$\dot u^r = \gamma^2 g^r \qquad \left(\frac{d}{d\tau}\right)$$

Aside on spatial duall

unit alternating tensor η (totally antisymmetric):

1) in an orthonormal frame $\eta_{\alpha\beta\gamma\delta} = \underbrace{\epsilon_{\alpha\beta\gamma\delta}}_{\text{totally antisymmetric Levi-Civita symbol}}$, $\epsilon_{0123} = 1$

2) in a coordinate system or nonorthonormal frame:

$$\eta_{\alpha\beta\gamma\delta} = |\det(g_{\alpha\beta})|^{1/2}\, \epsilon_{\alpha\beta\gamma\delta}$$

Measurement of η: in splitting this tensor, for the same reason that at most one index can be zero in an orthonormal frame for a nonzero value $\eta^{00ij} = 0 = \eta^{000i} = \eta^{0000}$, at most one temporal projection can act on η without yielding zero ($\eta_{\alpha\beta\gamma\delta} U^\gamma U^\delta = 0$ by symmetry)

so

$$\eta(u)_{\alpha\beta\gamma} = U^\delta \eta_{\delta\alpha\beta\gamma} \quad \text{is automatically spatial}$$

and represents the unit alternating tensor of the local rest space.
In an adapted O.N. frame with $e_0 = u$, one has $\eta(u)_{123} = 1$.

spatial dual

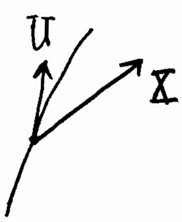
For an antisymmetric spatial tensor like the magnetic field $B_{\alpha\beta}$ or the vorticity (rotation) tensor $\omega_{\alpha\beta}$, one can define a spatial vector in the following way:

$$B_\gamma = \tfrac{1}{2}\eta(u)_{\gamma\alpha\beta} B^{\alpha\beta}, \quad B^{\alpha\beta} = \eta(u)^{\alpha\beta\gamma} B_\gamma$$

In the first case we get the magnetic field vector and in the second, the ~~angular velocity~~ rotation vector.
Also the spatial "cross product" $\times$ is defined by

$$B^\alpha{}_\beta V^\beta = \eta(u)^{\alpha\beta\gamma} V_\beta B_\gamma = (V \times B)^\alpha.$$

Aside on absolute derivative along a worldline with unit tangent U^α (4-velocity) parametrized by the proper time τ

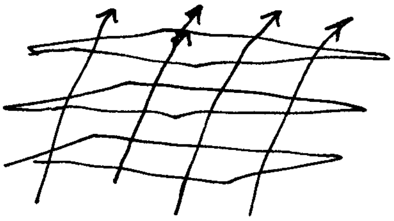


If X^α is defined along the worldline only, extend to vector field defined off the worldline, then in coordinate components:

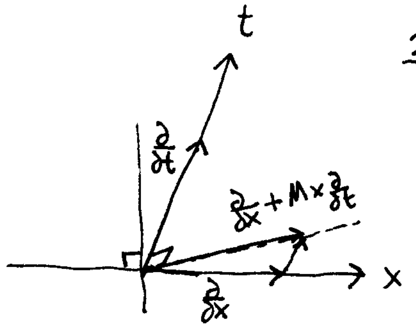
$$\frac{DX^\alpha}{d\tau} = \text{"}\nabla_U X^\alpha\text{"} = \text{"}X^\alpha{}_{;\beta} U^\beta\text{"} \underbrace{= \dots =}_{\text{calculation}} \underbrace{\frac{dX^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} U^\beta X^\gamma}_{\text{defined only on worldline}}$$

quotes for extended vector field.

$$\begin{bmatrix} (X^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma} X^\gamma) U^\beta \\ = X^\alpha{}_{,\beta} U^\beta + \dots \\ = \frac{dX^\alpha}{d\tau} + \dots \end{bmatrix}$$



In general a rotating family of observers $(\omega(u) \neq 0)$ does not admit an orthogonal family of spatial hypersurfaces, so to adapt coordinates to them, the best one can do is have the time coordinate lines be the observer worldlines and let the hypersurfaces of constant coordinate time be arbitrary. However, one can also introduce a new family of observers for each choice of hypersurface family, whose worldlines are the orthogonal trajectories. If there is a preferred family of hypersurfaces, one obtains a preferred family of (locally) nonrotating observers, in terms of which the original nonorthogonal coordinate system can still be used. For example in the stationary case, the time lines can follow the worldlines of the timelike Killing vector, while in the stationary axially symmetric case, a preferred family of hypersurfaces exist — leading to a nonorthogonal coordinate system adapted to both ~~for~~ observer families as in the black hole case.



$\boxed{\text{THREADING POINT OF VIEW}}$

2-D example ($\rightarrow$ 4-D by replacing x by x^i)

$$ds^2 = g_{tt}\,dt^2 + 2g_{tx}\,dt\,dx + g_{xx}\,dx^2$$

$$= g_{tt}\left(dt^2 + 2\frac{g_{tx}}{g_{tt}}dx\right) + g_{xx}\,dx^2$$

$$= \underbrace{g_{tt}}_{\equiv -M^2}\left(dt + \underbrace{\frac{g_{tx}}{g_{tt}}}_{\equiv -M_x}dx\right)^2 + \underbrace{\left(g_{xx} - \frac{g_{tx}^2}{g_{tt}}\right)}_{\equiv \gamma_{xx}}dx^2$$

$$= -M^2(dt - M_x dx)^2 + \gamma_{xx}\,dx^2$$

completing the square on dt corresponds to projecting the coordinate frame vector $\partial/\partial x$ orthogonal to the time coordinate lines, which are interpreted as the worldlines of an observer family with 4-velocity $u \equiv m \equiv M^{-1}\,\partial/\partial t$. [In the 4-D case the local rest spaces for which $\{\partial/\partial x^i + M_i\,\partial/\partial t\}$ are a basis are not "integrable" if u has nonzero rotation — i.e., there is no family of spacelike hypersurfaces orthogonal to the observers]. If we let $\epsilon_x = \partial/\partial x + M_x\,\partial/\partial t$ be the projected spatial direction, $\gamma_{xx} = \epsilon_x \cdot \epsilon_x$ is the square of its length.

One can then evaluate the gravitoelectric and gravitomagnetic fields in terms of this coordinate system. One finds

$$g_x = -\underbrace{\epsilon_x}(\ln M) = -(\mathrm{grad}_m)_x(\ln M) - \frac{\partial}{\partial t}M_x$$

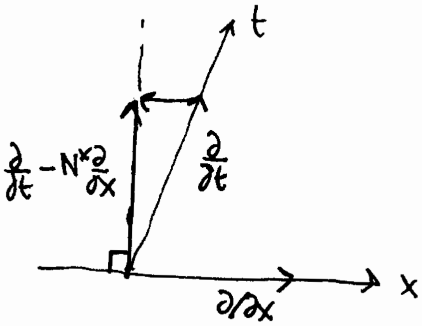
spatial derivative along LRS_u direction

= coordinate component of the spatial gradient

For the gravitomagnetic field one must have 4-D in order to define a spatial curl as in 3-D curvilinear coordinates but using the projected spatial derivatives $\partial/\partial x^i + M_i\,\partial/\partial t$

Compare with $E_i = -\partial_i\phi - \frac{\partial}{\partial t}A_i$ in Minkowski spacetime

One can interpret M and M_i as scalar and vector potentials for the GE and GM fields. The appropriately defined spatial curl of the spatial covector field $M_i dx^i$ (or 1-form) gives the GM vector field $H^i \left(\frac{\partial}{\partial x^i} + M_i \frac{\partial}{\partial t}\right)$ which is the spatial vector associated with the rotation vector field of the observer congruence.



SLICING POINT OF VIEW

$$
\begin{aligned}
ds^2 &= g_{tt}\, dt^2 + 2 g_{tx}\, dt\,dx + g_{xx}\, dx^2 \\
&= g_{tt}\, dt^2 + g_{xx}\left(dx^2 + 2\frac{g_{tx}}{g_{xx}}\, dx\,dt\right) \\
&= \underbrace{\left(g_{tt} - \frac{g_{tx}^2}{g_{xx}}\right)}_{-N^2} dt^2 + g_{xx}\left(dx + \underbrace{\frac{g_{tx}}{g_{xx}}}_{N^x} dt\right)^2 \\
&= -N^2\, dt^2 + g_{xx}\left(dx + N^x dt\right)^2
\end{aligned}
$$

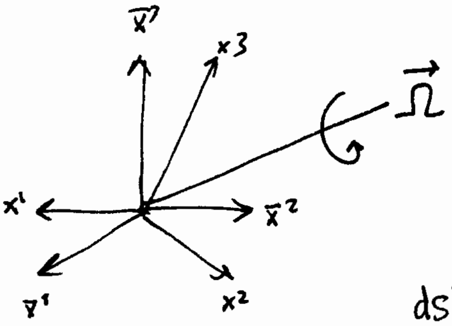
In this case one can take $u \equiv n \equiv N^{-1}\left(\frac{\partial}{\partial t} - N^x \frac{\partial}{\partial x}\right)$ as the observer 4-velocity, treating the time "hypersurfaces" as more important, but still keeping the original coordinate system, in terms of which the observer family is in relative motion.

The spatial direction is now along $\partial/\partial x$, but the temporal direction is not along $\partial/\partial t$.

For this observer family one can calculate the GE (and in the 4-D case the GM field), finding for example.

$$ g^{x}_{x} = -\underbrace{\frac{\partial}{\partial x}}_{\text{spatial gradient.}} (\ln N) \qquad \left(H^i \text{ "} \sim N \operatorname{curl}_n \vec{N} \text{ "} \right) $$

FLAT SPACETIME IN ROTATING COORDINATES



(x^1, x^2, x^3) rotating with fixed angular velocity $\vec{\Omega} = (\Omega^1, \Omega^2, \Omega^3)$.

~~con~~ [choose $\vec{\Omega} = (0,0,\Omega)$]

compared to space fixed coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$.

so that the metric takes the form:

$$ds^2 = -dt^2 + (d\vec{x} + \vec{\Omega}\times\vec{x}\, dt)\cdot(d\vec{x} + \vec{\Omega}\times\vec{x}\, dt)$$
$$= -dt^2 + \delta_{ij}(dx^i + \epsilon^{ilm}\Omega^l x^m dt)(dx^j + \epsilon^{jnp}\Omega^n x^p dt)$$

If one adapts polar coordinates to the axis of rotation, say $x^3 = \bar{x}^3$ for simplicity, the metric becomes (Landau and Lifshitz, Classical Theory of Fields)

$$ds^2 = -dt^2 + d\rho^2 + \rho^2(d\phi + \Omega dt)^2 + dz^2 \qquad \text{(slicing)}$$

$$= -\gamma^{-2}(dt - \gamma^2\rho^2\Omega\, d\phi)^2 + d\rho^2 + \gamma^2\rho^2 d\phi^2 + dz^2 \qquad \text{(threading)}$$

where $v = \rho\Omega$ (assume $\Omega > 0$) is the relative speed of a rotating observer and $\gamma = (1-v^2)^{-1/2}$ is the gamma factor.

If one calculates the GE & GM fields in the Cartesian coord threading point of view one finds the equations of motion of a point fixed in space (with respect to the nonrotating axes) compare to the familiar classical mechanical expression in a way that differs only by a proportionality factor involving the change to proper time parametrization

class mechanical: $\dfrac{d^2\vec{x}}{dt^2} = \underbrace{\vec{g}}_{\text{centrifugal}} + \underbrace{\dot{\vec{x}}\times\vec{H}}_{\text{coriolis}}$ $\qquad \vec{g} = -\vec{\Omega}\times(\vec{\Omega}\times\vec{x})$, $\vec{H} = 2\vec{\Omega}$

threading: $\dfrac{D^2\vec{x}}{dt^2} = -(2\vec{g}\cdot\dot{\vec{x}})\,\dot{\vec{x}} + \vec{g} + \dot{\vec{x}}\times\vec{H}$

extra term in D compared to d cancels $(\dot{x})$ quadratic term on right

threading fields are proportional (multiply by γ^2)

JUST A ROUGH SKETCH;

OUT OF TIME