Special Relativity Notes

by bob jantzen [January 1983?]

University of Rome "La Sapienza", Dipartimento di Fisica, Corso di Fisica Teorico (Ruffini)

A short guest course on special relativity, with a brief introduction and working some problems from the first edition of Taylor and Wheeler, but the second edition has been totally reorganized into essentially a completely new book and some problems dropped:

- Taylor and Wheeler, Spacetime Physics, First Edition, 1963 [google, amazon]
- Taylor and Wheeler, Spacetime Physics, Second Edition, 1992 [amazon]

Some problems have familiar names. We give the exercise numbers from the first edition and when retained, in the second edition.

GEOMETRICAL UNITS: both time and length are measured in cm:

$$t = c\, t_{CGS} \qquad c = 3\times10^{10}\ \text{cm/sec}$$

velocity is dimensionless in geometrical units:

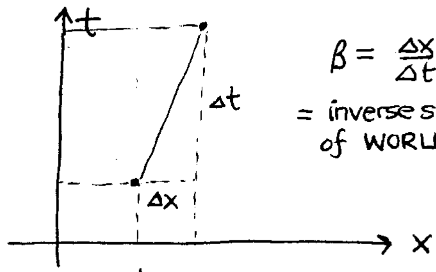
$$\beta = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{c\,\Delta t_{CGS}} = \frac{v}{c}$$

1 cm of time = time required for light to travel 1 cm $\approx 3\times10^{-11}$ sec

velocity of light: $|\beta| = 1$

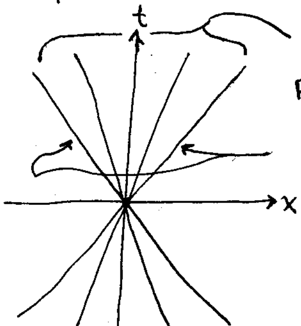
massive particles must have $|\beta| < 1$

ONE-DIMENSIONAL MOTION → TWO-DIMENSIONAL **SPACETIME DIAGRAM**



$$\beta = \frac{\Delta x}{\Delta t}$$

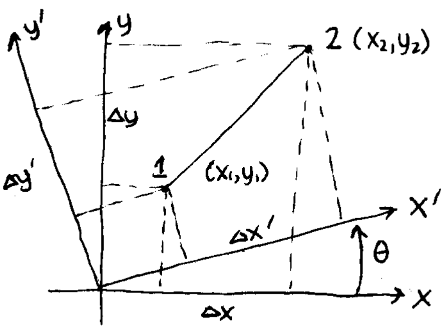
= inverse slope of WORLDLINE



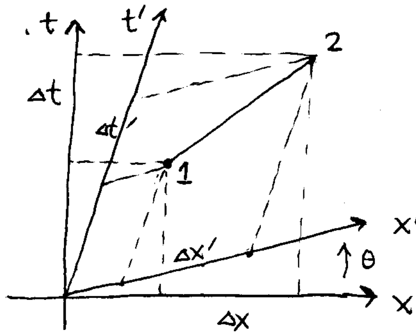
worldlines of particles $|\beta| < 1$

worldlines of light rays $|\beta| = 1$

ANALOGY BETWEEN ROTATIONS AND LORENTZ TRANSFORMATIONS IN 2-DIMENSIONS



positions of points



positions and times of spacetime "events"

$$d^2 = (\Delta x)^2 + (\Delta y)^2$$

distance

$\boxed{\text{INDEPENDENT OF COORDINATES}}$

$$s^2 = |(\Delta x)^2 - (\Delta t)^2|$$

interval

special coordinates only

rotation of coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\cos^2\theta + \sin^2\theta = 1$$

inverse:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Lorentz transformation of coordinates

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix}$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

inverse:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

worldline of origin of x' axis satisfies $x' = 0$ in spacetime diagram

$$\left.\begin{aligned} x &= \sinh\theta\ t' \\ t &= \cosh\theta\ t' \end{aligned}\right\} \quad \beta = \frac{x}{t} = \tanh\theta$$

= velocity of moving x'-axis with respect to the fixed x-axis (see next page)

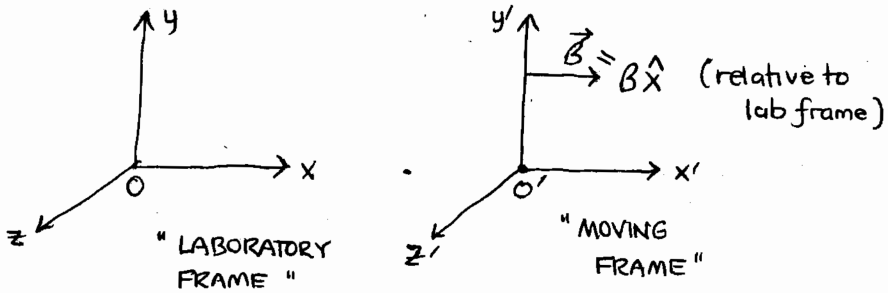
IDENTITIES:

$$\cosh\theta = \frac{1}{(1-\tanh^2\theta)^{1/2}} \equiv \gamma \qquad \sinh\theta = \frac{\tanh\theta}{(1-\tanh^2\theta)^{1/2}} \equiv \beta\gamma$$

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}} = \frac{1}{(1-\frac{v^2}{c^2})^{1/2}} \geq 1$$

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix} \qquad \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

② PHYSICAL SITUATION (4-dimensions)

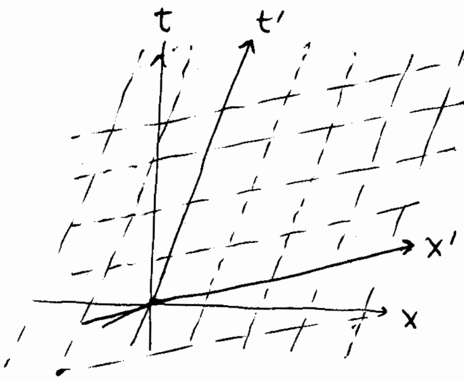


origins O and O' coincide at $t = 0 = t'$

moving frame moves with constant velocity β along x axis

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

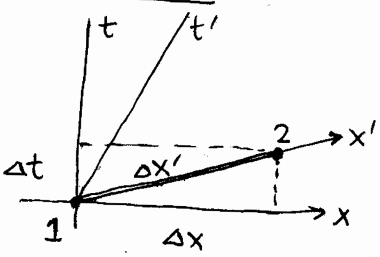
$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y' \\ z' \end{pmatrix}$$



x' coordinate lines (parallel to t'axis) represent worldlines of particles at rest in the moving frame.

t' coordinate lines (parallel to x'axis) represent events occuring at the same time as observed in the moving frame

SIMULTANEITY

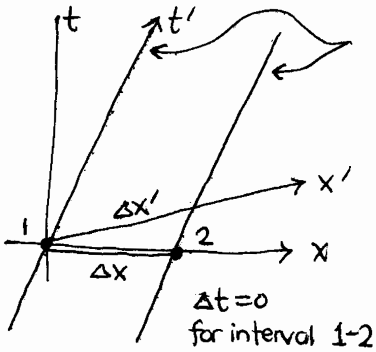


Two events considered simultaneous in the moving frame have $\Delta t' = 0$ (line segment between them is parallel to x' axis)

$$\left.\begin{aligned} \Delta t &= \beta\gamma\Delta x' + \gamma\underbrace{\Delta t'}_{0} = \beta\gamma\Delta x' \\ \Delta x &= \gamma\Delta x' + \beta\gamma\Delta t' = \gamma\Delta x' \end{aligned}\right\}$$

so time difference Δt as observed in lab frame.

LENGTH CONTRACTION



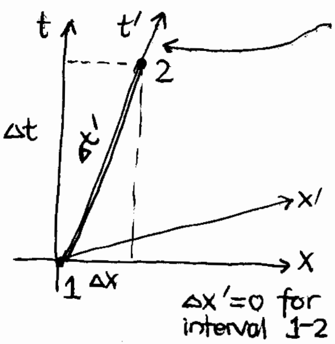
worldlines of ends of ruler at $x'=0$ and $x'=\Delta x'$ fixed in the moving frame

at $t=0$ in lab frame, ends of ruler are at $x=0$ and $x=\Delta x$:

$$\Delta x' = \gamma\Delta x - \beta\gamma\underbrace{\Delta t}_{=0} = \gamma\Delta x \rightarrow \Delta x = \gamma^{-1}\Delta x' < \Delta x$$

so length of ruler as measured in lab frame is less than measured in moving frame where it is at rest

TIME DILATION

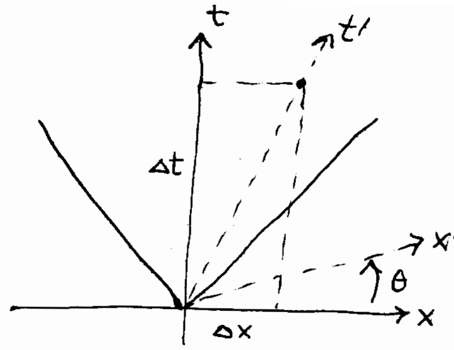


worldline of clock at rest at $x'=0$ in moving frame.

$$\Delta t = \gamma\Delta t' + \beta\gamma\Delta x' = \gamma\Delta t'$$
$$\Delta x = \beta\gamma\Delta t' + \gamma\Delta x' = \beta\Delta t$$

When clocktime $\Delta t'$ has elapsed, clock has moved distance $\Delta x = \beta\Delta t$ during time $\Delta t = \gamma\Delta t' > \Delta t'$ as observed in lab frame.

CAUSALITY



Suppose event 2 lies in the lightcone of event 1. Then one can transform to a moving frame in which the two events occur at the same place.

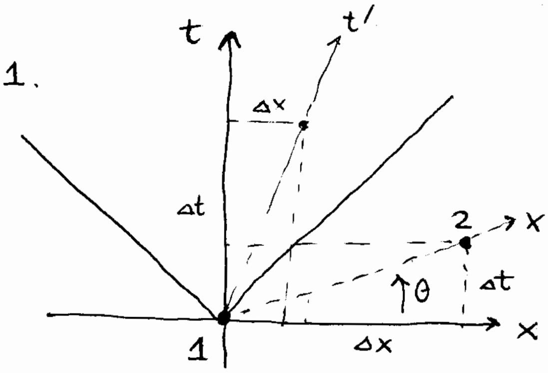
$$|\Delta x| < |\Delta t| \qquad \text{(light cone condition)}$$

Define $\quad \beta = \dfrac{\Delta x}{\Delta t} = \tanh\theta.$

β is the parameter associated with the desired moving frame relative to the given frame in which the event coordinates are given.

$$S_{21}^2 = |(\Delta x)^2 - (\Delta t)^2| = |\overset{0}{(\Delta x')^2} - (\Delta t')^2| = (\Delta t')^2$$

The interval between 1 and 2 is the time difference between the events in the moving frame. $\quad |\Delta t'| = \sqrt{1-\beta^2}\,\Delta t$



Suppose event 2 lies outside the light cone of event 1. Then one can transform to a moving frame in which the events occur at the same time.

$$|\Delta x| > |\Delta t| \qquad \text{(light cone condition)}$$

Define $\quad \beta = \dfrac{\Delta t}{\Delta x} = \tanh\theta.$

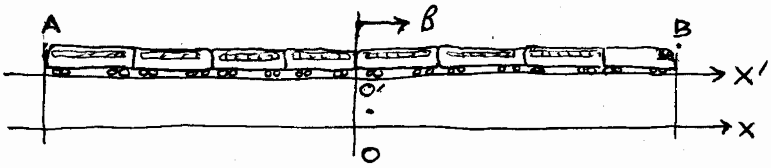
β is the parameter associated with the desired moving frame relative to the frame in which the event coordinates are given.

The interval between 1 and 2 is the distance between the events in this moving frame:

$$S_{21}^2 = |(\Delta x)^2 - (\Delta t)|^2 = |(\Delta x)^2 - \overset{0}{(\Delta t')^2}| = (\Delta x')^2$$

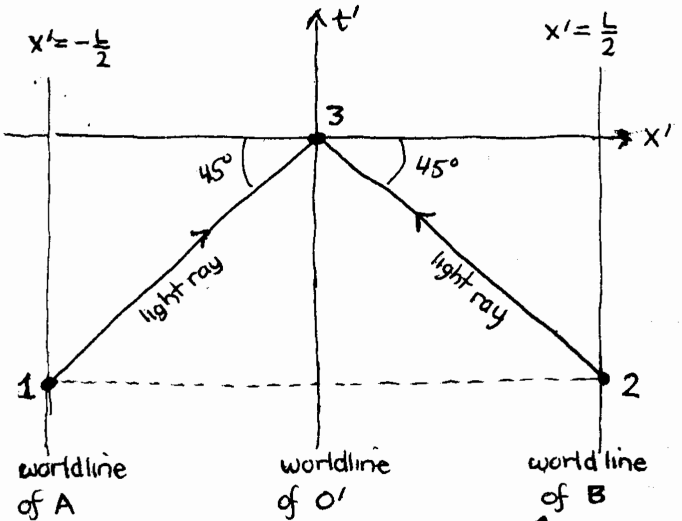
$$|\Delta x'| = \sqrt{(1-\beta^2)}\;\Delta x\,.$$

TRAIN PARADOX Spacetime Physics Ex. #23



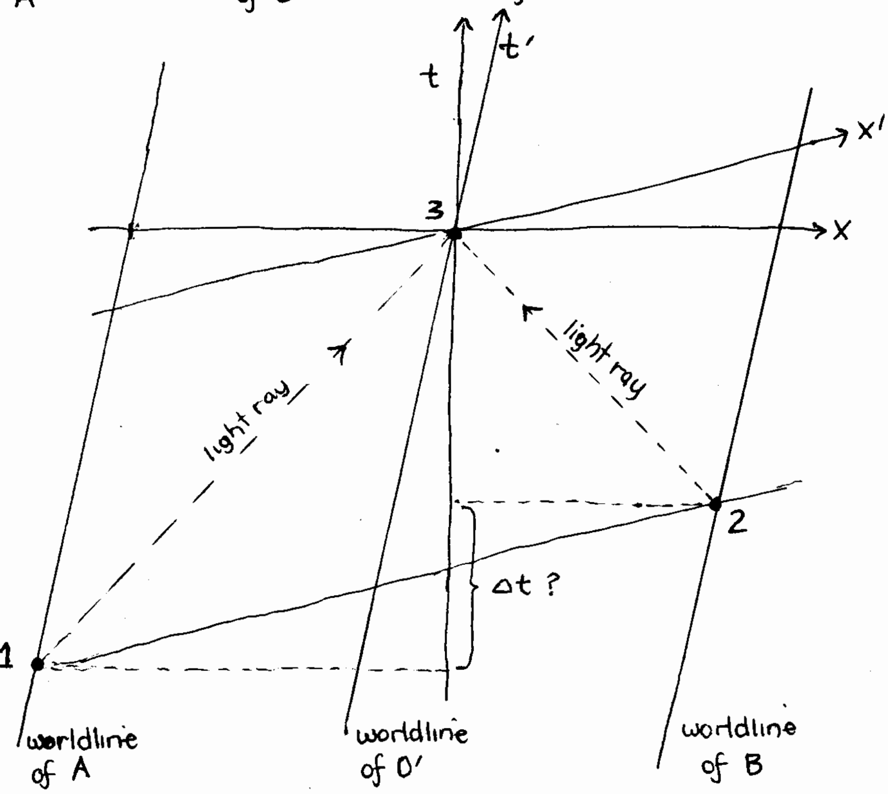
Flashes of light from both ends of train arrive simultaneously at $O = O'$ at $t = 0 = t'$.

In which frame were the flashes emitted simultaneously and what is the time difference between emissions in the other frame?



(1) In the moving frame (rest frame of train) the light rays each had to travel a distance $\frac{L}{2}$ (L = length of train in its rest frame), so if they arrived simultaneously at the center, each had to be emitted at the same time, namely $t' = -\frac{L}{2}$:

$$\boxed{\begin{aligned} \Delta x' &= x_2' - x_1' = L \\ \Delta t' &= t_2' - t_1' = 0 \end{aligned}}$$



(2) Using the Lorentz transformation one can evaluate $(\Delta x, \Delta t)$ for the two events 1 and 2 :

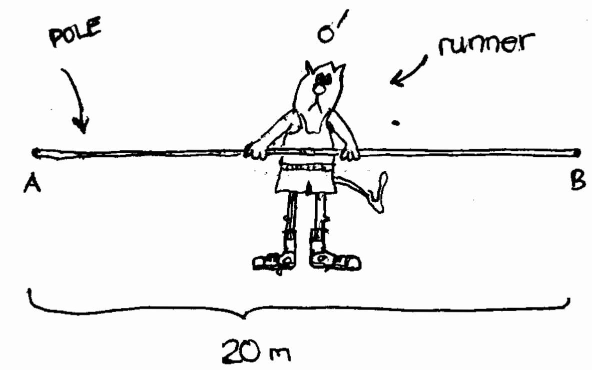
$$\Delta t = \beta\gamma\,\Delta x' + \gamma\underbrace{\Delta t'}_{0} = \beta\gamma L = t_2 - t_1 .$$

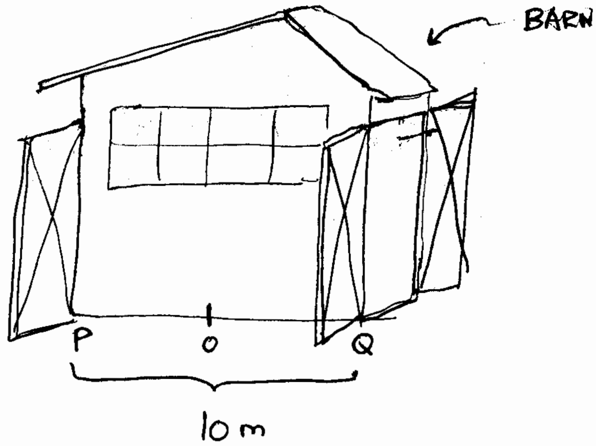
Therefore the flash from the end of the train (event 1), was emitted at a time $\beta\gamma L$ earlier than the flash from the front.

[In fact since $\begin{pmatrix} x_2 \\ t_2 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} x_2' \\ t_2' \end{pmatrix} = \gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} L/2 \\ -L/2 \end{pmatrix} = \frac{\gamma L}{2}(1-\beta)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

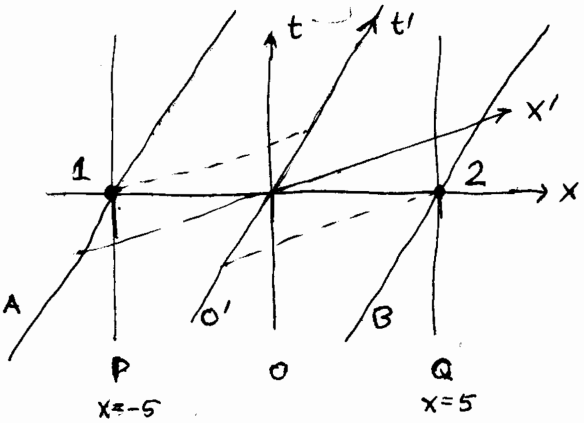
$t_2 = -\frac{1}{2}\gamma L(1-\beta)$ and $t_1 = t_2 - \Delta t = -\frac{1}{2}\gamma L(1-\beta) - \beta\gamma L = -\frac{1}{2}\gamma L(1+\beta)$.]

(5) **POLE AND BARN PARADOX** Spacetime Physics Ex. 25





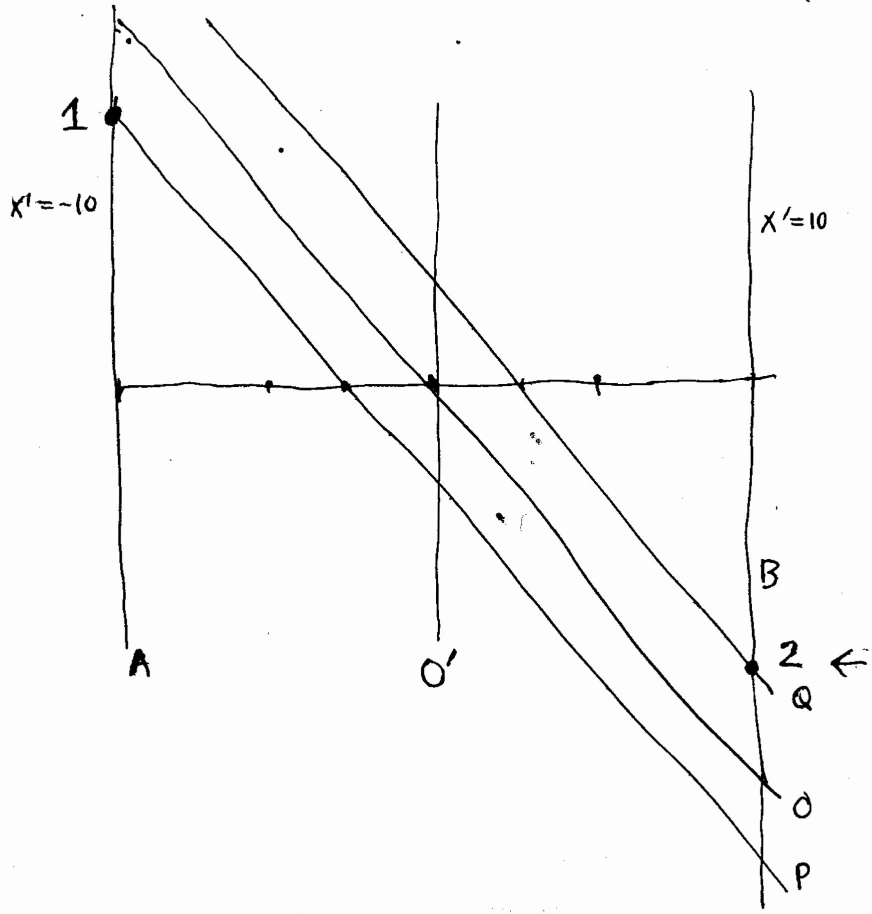
A runner carries a 20m pole so fast in the direction of its length that appears to be only 10m long in the lab frame. Therefore at some instant the pole will appear to be enclosed in a 10m barn.
BUT in the rest frame of the runner the barn appears to be contracted to half its length. How can a 20m pole fit in a 5 m barn?



$\gamma = 2$

$\beta = (1-\gamma^{-2})^{1/2} = \frac{\sqrt{3}}{2}$

Answer. In the rest frame of runner the coincidences of A and P (1) and of B and Q (2) are not simultaneous but event (1) occurs after event (2).



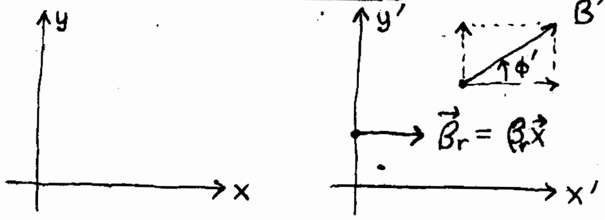
$$\begin{pmatrix} x_1' \\ t_1' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ t_1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -5\gamma \\ 5\beta\gamma \end{pmatrix} = \begin{pmatrix} -10 \\ 10\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{pmatrix} x_2' \\ t_2' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x_2' \\ t_2' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 10 \\ -10\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\Delta t_2' = t_1' - t_2' = 20\frac{\sqrt{3}}{2} = \beta \cdot 20$$

2 ← First the right barn door coincides with B, then much later the left barn door coincides with A.

⑥ "ADDITION" OF VELOCITIES



$\vec{\beta}' = (\beta^{x'}, \beta^{y'}) = \beta'(\cos\phi', \sin\phi')$

SP# 21

CALCULATE (β^x, β^y):

$$\beta^x = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' + \beta_r \Delta t')}{\gamma(\Delta t' + \beta_r \Delta x')} = \frac{\gamma\left(\frac{\Delta x'}{\Delta t'} + \beta_r\right)}{\left(1 + \beta_r \frac{\Delta x'}{\Delta t'}\right)} = \frac{\beta^{x'} + \beta_r}{1 + \beta_r \beta^{x'}}$$

$$\beta^y = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma(\Delta t' + \beta_r \Delta x')} = \cdots = \frac{\gamma^{-1}\beta^{y'}}{1 + \beta_r \beta^{x'}}$$

inverse:

$$\left(\begin{array}{l} \beta^{x'} = \dfrac{\beta^x - \beta_r}{1 - \beta_r \beta^x} \\[2ex] \beta^{y'} = \dfrac{\gamma^{-1}\beta^y}{1 - \beta_r \beta^x} \end{array}\right)$$

CALCULATE β^2:

$$\beta^2 = \frac{\beta^{x'2} + \beta_r^2 + 2\beta^{x'}\beta_r + (1-\beta_r^2)(\beta'^2 - \beta^{x'2})}{(1 + \beta_r \beta^{x'})^2} = \frac{\beta'^2 + (1-\beta'^2)\beta_r^2 + 2\beta^{x'}\beta_r + \beta^{x'2}\beta_r^2}{1 + 2\beta^{x'}\beta_r + \beta^{x'2}\beta_r^2}$$

For a light ray, $\beta' = 1$ and hence $\beta = 1$ also (invariance of the speed of light). In this case:

$$\cos\phi = \beta^x = \frac{\cos\phi' + \beta_r}{1 + \beta_r \cos\phi'}$$

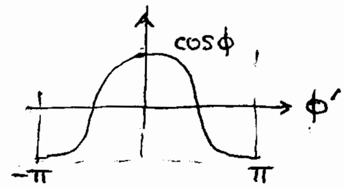
SP # 22

Note that:
$$\cos\phi = \frac{\cos\phi'(1 + \beta_r\cos\phi') - \beta_r\cos^2\phi' + \beta_r}{1 + \beta_r\cos\phi'} = \cos\phi' + \frac{\beta_r \sin^2\phi'}{1 + \beta_r\cos\phi'}$$

Assume $\beta_r > 0$. Since $\beta_r < 1$, $1 + \beta_r\cos\phi'$ is positive for all ϕ' and therefore the additional term is positive (or zero if $\phi' = 0$)

so $\cos\phi > \cos\phi'$.

For $\phi \in (-\pi, \pi)$, $|\phi| < |\phi'|$.



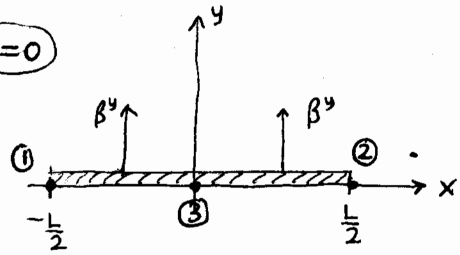
In other words, light rays are rotated towards the direction of motion.

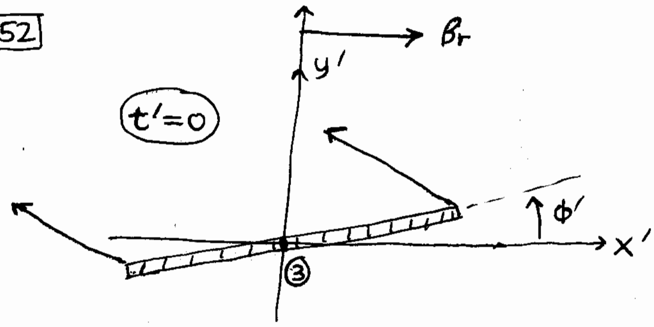
Now consider a particle at rest in the moving frame that emits light uniformly in all directions. Consider the 50 percent that goes into the forward hemisphere ($|\phi'| \leq \frac{\pi}{2}$). For $|\phi'| = \frac{\pi}{2}$, $\cos\phi' = 0$ and $\cos\phi = \beta_r$.

In the limit $\beta_r \to 1$, $\cos\phi \to 1$ and hence $|\phi| \to 0$.

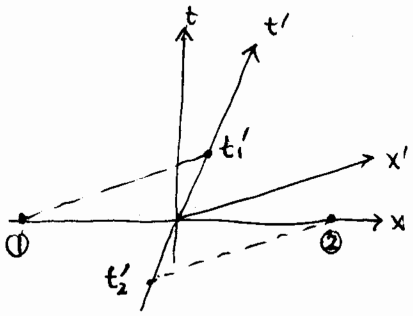
Therefore this light is concentrated in a very narrow cone about the direction of motion.

⑦ TILTED METER STICK $\boxed{SP\#52}$





A stick lies parallel to the x-axis and moves in the y direction with speed $\beta^y > 0$. In the moving frame, the stick is tilted upward in the positive x' direction. Why?



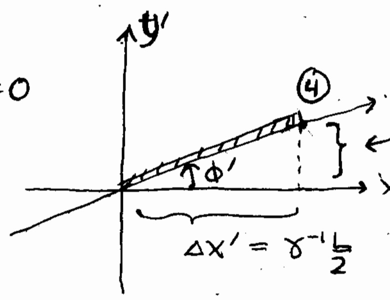
Because events 1 (left end crosses $y=0$) and 2 (right end crosses $y=0$) are not simultaneous in the moving frame but 2 occurs before 1 and since the stick also has a component $\beta_y' > 0$ in the moving frame, the right end crosses $y'=0$ first.

$$x_2' = \gamma(x_2 - \beta_r \underset{}{\overset{0}{t_2}}) = \gamma \tfrac{L}{2}$$

$$t_2' = \gamma(\underset{0}{t_2} - \beta_r \underset{\frac{L}{2}}{x_2}) = -\gamma\beta \tfrac{L}{2} \qquad\qquad t_1' = \gamma(\underset{0}{t_1} - \beta_r \underset{-\frac{L}{2}}{x_1}) = \gamma\beta_r \tfrac{L}{2}$$

$$\beta^{y\prime} = \frac{\gamma^{-1}\beta^y}{1 - \beta_r \underset{0}{\beta^x}} = \gamma^{-1}\beta^y$$

t'=0



$$\Delta y' = \beta^{y\prime} \underbrace{\Delta t}_{\gamma\beta_r\frac{L}{2}} = \beta_r \beta^y \tfrac{L}{2}$$

$$\tan\phi' = \frac{\Delta y'}{\Delta x'} = \frac{\frac{L}{2}\beta_r\beta^y}{\gamma^{-1}\frac{L}{2}} = \gamma\beta_r\beta^y$$

In CGS units: $\dfrac{\gamma v_r v^y}{c^2}$

i.e. usually a very small effect

At time $t_2' = -\gamma\beta\frac{L}{2}$, the right end crosses $y'=0$ at $x_2' = \gamma\frac{L}{2}$. After a time $\Delta t' = \gamma\beta\frac{L}{2}$, the x' coordinate is:

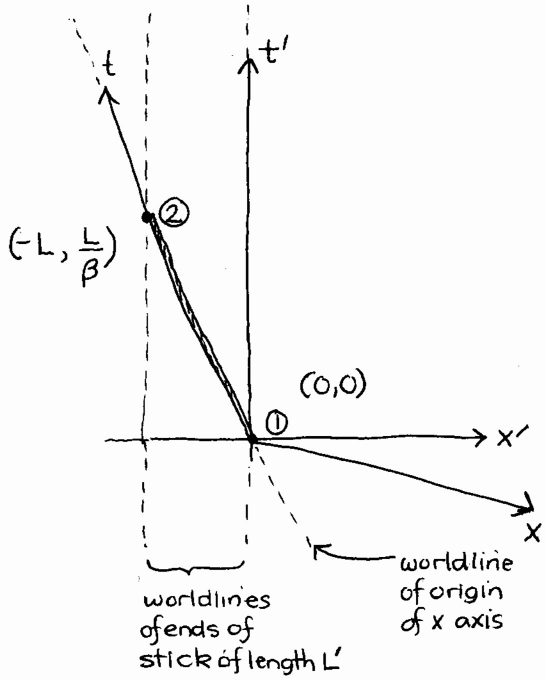
$$\Delta x = x_4' = x_2' - \beta_r \Delta t' = \gamma\tfrac{L}{2} - \beta_r(\gamma\beta_r\tfrac{L}{2})$$
$$= \gamma\tfrac{L}{2}(1-\beta_r^2) = \gamma^{-1}\tfrac{L}{2}$$

This is just Lorentz contraction of the dimensions along the x-axis

(I hope) (Please check)

⑧ LORENTZ CONTRACTION II [SP-13]

A meter stick lies along the x' axis and at rest in the rocket frame. Show that an observer in the laboratory frame will conclude that the meter stick has undergone Lorentz contraction if he measures how long it takes the meter stick to pass one of his clocks and multiplies this result by the relative velocity of the two frames.



DRAW THE SPACETIME DIAGRAM !!

It takes time $\Delta t' = t_2' - t_1' = \frac{L'}{\beta}$ for the origin of the x-axis to pass from one end of the stick to the other. $\Delta x' = -L' = x_2' - x_1'$ because origin of x-axis travels in negative x' direction, so:

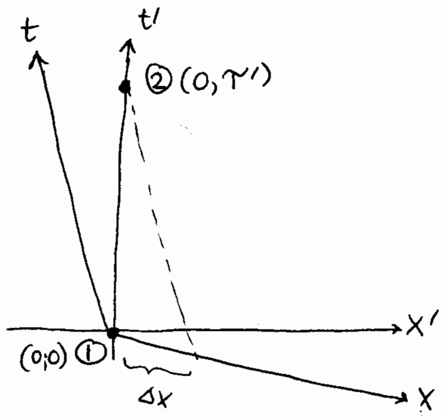
$$\Delta t = \gamma(\Delta t' + \beta \Delta x') = \gamma\left(\frac{L'}{\beta} - \beta L'\right)$$

$$L = \beta \Delta t = \gamma(1-\beta^2) L' = \gamma^{-1} L'$$

THE TWO RELEVANT EVENTS ARE WHERE THE ORIGIN OF THE X-AXIS COINCIDES WITH THE ENDS OF THE STICK.

TIME DILATION II [SP-14]

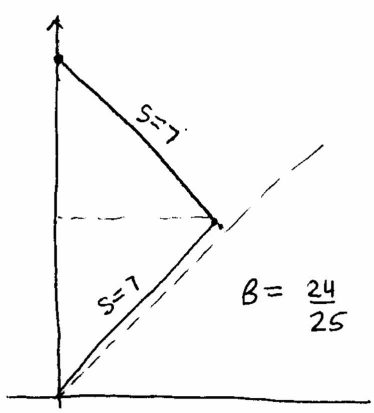
Two events occur at the same place but at different times in the rocket frame. Show that an observer in the laboratory frame will conclude that the time between the two events has been dilated if he measures the distance between them in the laboratory frame and divides this distance by the relative velocity of the two frames.



$$\Delta x = \gamma(\underbrace{\Delta x'}_{0} + \beta \underbrace{\Delta t'}_{\tau'}) = \gamma\beta\tau'$$

$$\tau = \frac{\Delta x}{\beta} = \gamma\tau'$$

[SP-27] CLOCK PARADOX Solution only.



$\beta = \frac{24}{25}$ $\gamma = \frac{1}{\sqrt{1-\left(\frac{24}{25}\right)^2}} = \frac{25}{7}$

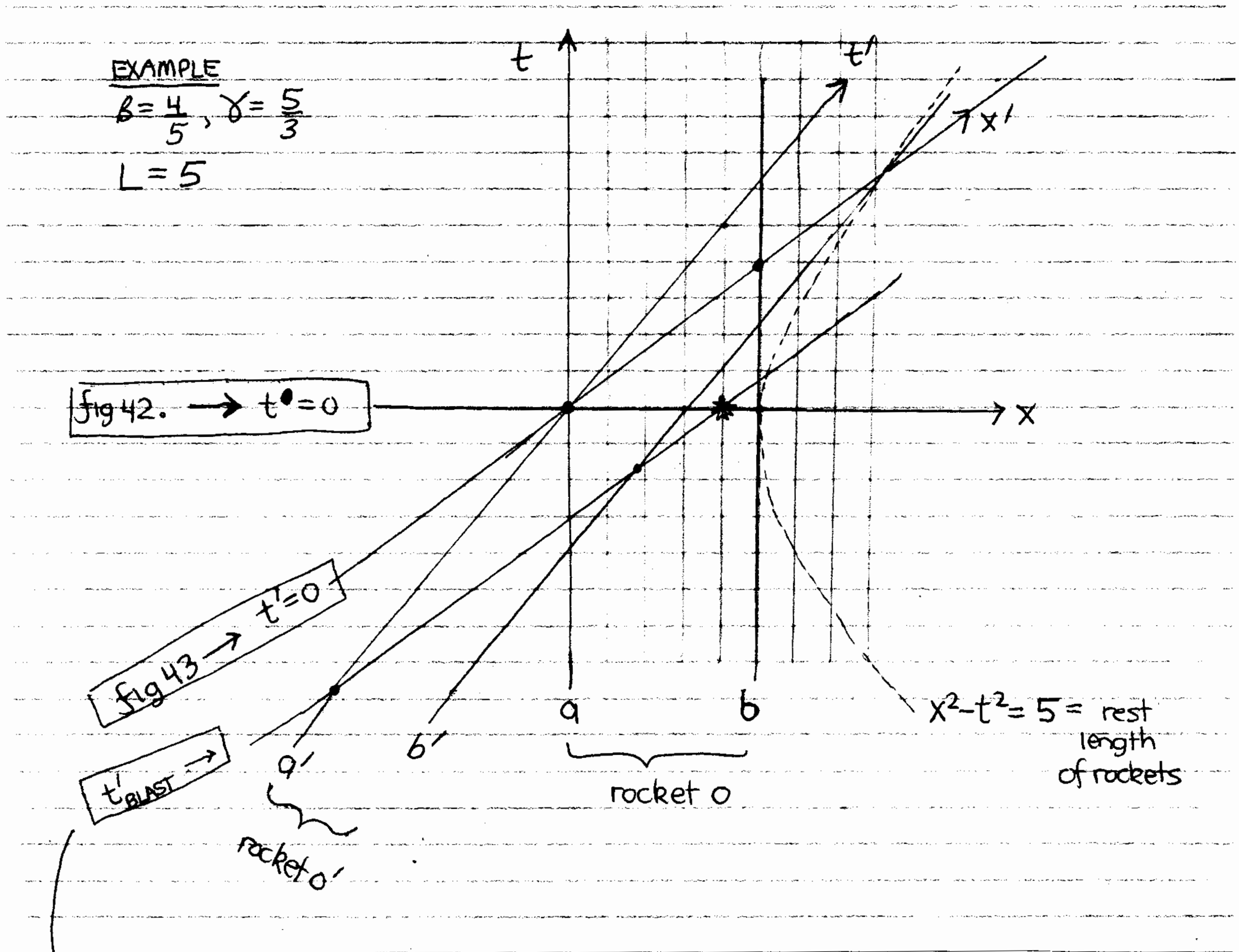
a) $21 + 14 = 35$

c) $\gamma \cdot 14 = 50$

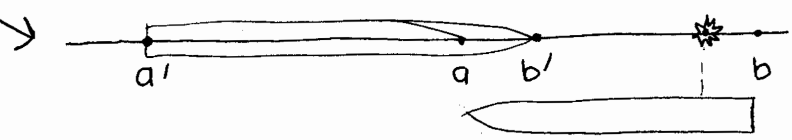
$21 + 50 = 71$

(9) $\boxed{\text{SP-26}}$ SPACE WAR. Solution only

In fig. 42 the coincidence of a and a′ is taken to be simultaneous with the firing of the bullet at b; therefore in fig. 43 where again a and a′ coincide, the firing of the bullet cannot appear because it cannot be simultaneous in the ~~other~~ second frame, therefore the bullet had to be fired at a different time. DRAW THE SPACETIME DIAGRAM.

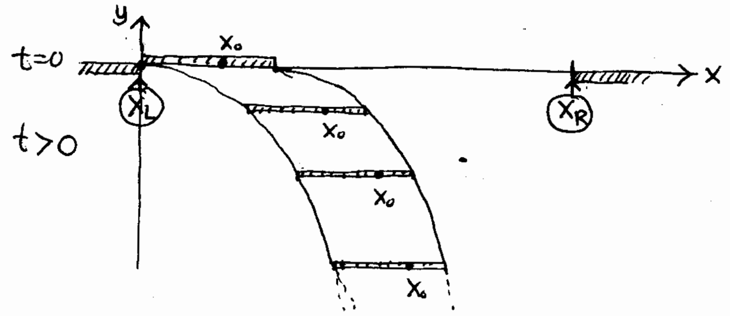


Obviously the bullet was fired before a and a′ coincided in the rocket frame.



The bullet missed because it was fired too soon.

⑩ $\boxed{\text{SP-54}}$ Solution only.



Let $x_0 \in [0, L]$ label the points on the stick, which moves to the right with velocity β. The left edge of the hole has $x_L = 0$, the right edge $x_R = \gamma L \equiv L'$.

The spacetime coordinates (x_s, y_s, t_s) of the point x_0 are given by:

$s \le 0$: $x_s = \beta s + x_0$, $y_s = 0$, $t_s = s$

$s \ge 0$: $x_s = \beta s + x_0$, $y_s = -\frac{1}{2} g s^2$, $t_s = s$

Using the Lorentz transformation to pass to the rest frame of the stick:

$$x_s' = \gamma(x_s - \beta t_s) = \gamma(\beta s + x_0 - \beta s) = \gamma x_0 \equiv x_0'$$

$$y_s' = y_s = -\tfrac{1}{2} g s^2 \quad (s \ge 0)$$

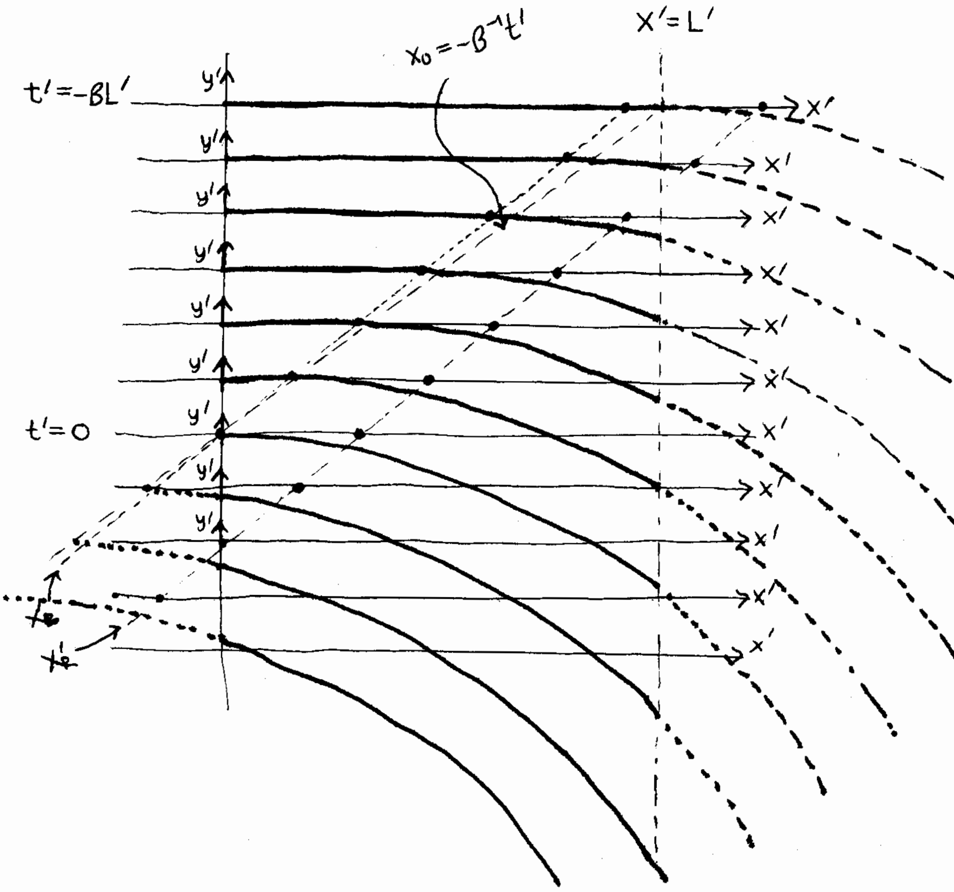
$$t_s' = \gamma(t_s - \beta x_s) = \gamma(s - \beta(\beta s + x_0)) = \gamma(\underbrace{(1-\beta^2)}_{\gamma^{-2}} s - \beta x_0) = \gamma^{-1} s - \beta x_0'$$

Solving for s: $s = \gamma(t_s' + \beta x_0')$

$$\boxed{\begin{array}{ll} y_s' = -\tfrac{1}{2} g \gamma^2 (t' + \beta x_0')^2 = -\tfrac{1}{2} g (\beta\gamma)^2 (x_0' + t'/\beta)^2 & t' \ge -\beta x_0' \\ y_s' = 0 & t' \le -\beta x_0' \end{array}}$$

Since the hole has width $\gamma^{-1} L'$ and moves to the left with velocity β in this frame:

$x_L' = -\beta t'$, $x_R' = \gamma^{-1} L' - \beta t'$.



For $t' \le -\beta x_0$, the point x_0 has $y' = 0$, and begins to drop at $t' = -\beta x_0$ or $x_0 = -\beta^{-1} t'$. Notice that the bending point moves to the left with velocity $\beta^{-1} > 1$.

The diagram shows the situation at invervals $\Delta t' = \beta L'/6$. The fixed shape parabola moves to the left with speed β^{-1}!

Notice that the stick begins bending just after the left end of the hole passes by