# Rough Notes Written on a Moving Train

## A Second Look at Tensors, Covariant Differentiation and Curvature


### by bob jantzen [March 1990]

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A vector space and its dual, with basis and dual basis and modern tensor notation are introduced and then tied to the tangent space of ordinary space in Cartesian coordinates, then transformed to spherical coordiinates. In this context parallel transport and covariant differentiation are introduced, leading to the vanishing curvature tensor of flat space.

- sl1990.pdf: 14 pages, 585 MEG

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## A Second Look at tensors, covariant differentiation and curvature

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correction last line on page 5:

$$C(t) = (x_{10}, \ldots, x_{\alpha 0}+t, \ldots, x_{30})$$

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Giancarlo
Antonella
Gianni

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* not related to Einstein's train

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$V$ n-dimensional vector space with basis $\{e_\alpha\}$, elements $X = X^\alpha e_\alpha$ called vectors

Ex. $\mathbb{R}^n = \{(x^1, \dots, x^n) \mid x^\alpha \in \mathbb{R}\}$, $\{e_\alpha\}$ standard basis: $e_1 = (1,0,\dots,0), \dots, e_n = (0,\dots,1)$.

$V^*$ dual space of $V$ = n-dim vectorspace of realvalued linear functions on $V$

linearity: $f(X) = f(X^\alpha e_\alpha) = X^\alpha f(e_\alpha)$ evaluation commutes with linear combination

elements called covectors.

Define basis of $V^*$ dual to basis $\{e_\alpha\}$ of $V$ by

$\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta$, i.e. $\omega^\alpha(X) = \omega^\alpha(X^\beta e_\beta) = \omega^\alpha(e_\beta) X^\beta = \delta^\alpha{}_\beta X^\beta = X^\alpha$

These are just the linear functions which give the $\alpha$th component with respect to the given basis.

Given $f \in V^*$, then define $f_\alpha = f(e_\alpha)$, so from above:

$$f(X) = X^\alpha f_\alpha = f_\alpha \omega^\alpha(X) \quad \rightarrow \quad f = f_\alpha \omega^\alpha$$

$(V^*)^*$ can be identified with $V$ since the evaluation function $I(f, X) \equiv f(X)$ is a linear function of the argument $f$ for fixed $X$. One can therefore associate with each $X$, a linear function on $V^*$. Note $I(\omega^\alpha, e_\beta) = \omega^\alpha(e_\beta) = \delta^\alpha{}_\beta$.

Define $X(f) = f(X)$ to be the value of this linear function $I(\ , X)$ on $f$. Note that $X(\omega^\alpha) = \omega^\alpha(X) = X^\alpha$.

$I$ is an example of a multilinear function on $V^* \times V$, ie a function with one covector argument and one vector argument which is linear in each argument alone.

$$I(f, X) = I(f_\beta \omega^\beta, X^\alpha e_\alpha) = I(\omega^\beta, e_\alpha) f_\beta X^\alpha = I(\omega^\beta, e_\alpha) f(e_\beta) X(\omega^\alpha)$$
$$= I(\omega^\beta, e_\alpha)\, e_\beta(f)\, \omega^\alpha(X)$$

Here we need some notation for the linear function which is the product of the linear functions $e_\beta$ and $\omega^\alpha$. Let $e_\beta \otimes \omega^\alpha$ be the multilinear function with one covector argument and one vector argument.

$$(e_\beta \otimes \omega^\alpha)(f, X) \equiv e_\beta(f)\, \omega^\alpha(X) \qquad \text{"tensor product"}$$

Then $\quad I(f, X) = I(\omega^\beta, e_\alpha)(e_\beta \otimes \omega^\alpha)(f, X)$

and $\quad I = \underbrace{I(\omega^\beta, e_\alpha)}\, e_\beta \otimes \omega^\alpha$

$I^\beta{}_\alpha = \delta^\beta{}_\alpha$ from above. The components of the evaluation function (1-form on a vector) are the Kronecker delta array.



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A ~~n~~ $\binom{p}{q}$-tensor over $V$ is just a multi~~li~~near (real-valued) function accepting $p$ covector arguments and ~~the~~ $q$ vector arguments, ordered, with the covector arguments first.

$$T(\sigma_{(1)},\ldots,\sigma_{(p)},X_{(1)},\ldots,X_{(q)}) = \underbrace{\sigma_{(1)\alpha_1}\cdots\sigma_{(p)\alpha_p}X_{(1)}^{\beta_1}\cdots X_{(q)}^{\beta_q}}\ \underbrace{T(\omega^{\alpha_1},\ldots,\omega^{\alpha_p},e_{\beta_1},\ldots,e_{\beta_q})}$$

$$\underbrace{e_{\alpha_1}(\sigma_{(1)})\cdots e_{\alpha_p}(\sigma_{(p)})\,\omega^{\beta_1}(X_{(1)})\cdots\omega^{\beta_q}(X_{(q)})} \qquad \equiv T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p}$$

$$(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_q})(\sigma_{(1)},\ldots,\sigma_{(p)},X_{(1)},\ldots,X_{(q)})$$

so

$$T = T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}\,\underbrace{e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_q}}$$

basis of the $n^{p+q}$-dimensional space of $\binom{p}{q}$-tensors over $V$.

EX. $\mathbb{R}^3$. Define $g(X,Y) = \delta_{\alpha\beta}X^\alpha Y^\beta = X\cdot Y$ in ~~the~~ the natural basis

$g(e_\alpha,e_\beta) = \delta_{\alpha\beta}$

$g = \delta_{\alpha\beta}\,\omega^\alpha\otimes\omega^\beta.$

Define $\det(X,Y,Z) = X\cdot(Y\times Z)$, a ~~n~~ $\binom{0}{3}$-tensor.

$\det(e_\alpha,e_\beta,e_\gamma) = e_\alpha\cdot(e_\beta\times e_\gamma) = \epsilon_{\alpha\beta\gamma} = \begin{cases}\text{sign}\begin{pmatrix}1&2&3\\ \alpha&\beta&\gamma\end{pmatrix} & \text{if } (\alpha,\beta,\gamma) = \text{perm of } (1,2,3)\\ 0 & \text{otherwise}\end{cases}$

$\det = \epsilon_{\alpha\beta\gamma}\,\omega^\alpha\otimes\omega^\beta\otimes\omega^\gamma$

Define $T(X,Y,Z,W) = (X\times Y)\cdot(Z\times W)$ a $\binom{0}{4}$-tensor.

$T(e_\alpha,e_\beta,e_\gamma,e_\delta) = \underbrace{(e_\alpha\times e_\beta)}\cdot\underbrace{(e_\gamma\times e_\delta)} \qquad = \epsilon_{\alpha\beta\lambda}\delta^{\lambda\nu}\epsilon_{\gamma\delta\rho}\delta^{\rho\sigma}\delta_{\nu\sigma}$

$\epsilon_{\alpha\beta\lambda}\delta^{\lambda\nu}\,\underbrace{e_\nu\ \ \epsilon_{\gamma\delta\rho}\delta^{\rho\sigma}e_\sigma}_{\delta_{\nu\sigma}}$

$T = \ldots$

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under a change of basis

$$\bar{e}_\alpha = e_\beta A^{-1\beta}{}_\alpha \quad \rightarrow \quad \bar{\omega}^\alpha = A^\alpha{}_\beta \omega^\beta$$

$$\left[ \begin{aligned} \bar{\omega}^\alpha(\bar{e}_\beta) &= A^\alpha{}_\gamma \omega^\gamma (A^{-1\delta}{}_\beta e_\delta) \\ &= A^\alpha{}_\gamma A^{-1\delta}{}_\beta \delta^\gamma{}_\delta = A^{\alpha}{}_\gamma A^{-1\gamma}{}_\beta = \delta^\alpha{}_\beta \end{aligned} \right]$$

then the components of a tensor change in an analogous way for each index:

$$\bar{T}^{\alpha_1 \ldots}{}_{\beta_1 \ldots} = \cancel{A^{\alpha_1}{}_{\gamma_1} \ldots A^{-1\delta_1}{}_{\beta_1} \ldots T^{\gamma_1 \ldots}{}_{\delta_1 \ldots}}$$

$$= T(\bar{\omega}^{\alpha_1}, \ldots, \bar{e}_{\beta_1}, \ldots) = A^{\alpha_1}{}_{\gamma_1} \ldots A^{-1\delta_1}{}_{\beta_1} \ldots T(\omega^{\alpha_1}, \ldots, e_{\beta_1}, \ldots)$$

$$= A^{\alpha_1}{}_{\gamma_1} \ldots A^{-1\delta_1}{}_{\beta_1} \ldots T^{\alpha_1 \ldots}{}_{\beta_1 \ldots}$$

" tensor transformation law" (tensor doesn't change – components do).



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In calculus on $\mathbb{R}^3$ with standard coordinates $\{x^1, x^2, x^3\} \equiv \{\omega^1, \omega^2, \omega^3\}$ one introduces at each point ~~new cartesian coordinates~~ $P_0$ with coords $(X_0^1, X_0^2, X_0^3)$ new cartesian coords $\{dx^1|_{P_0}, dx^2|_{P_0}, dx^3|_{P_0}\}$ with origin at $P_0$

$$dx^\alpha|_{P_0} \equiv X^\alpha - X^\alpha_0$$

These functions give the ~~vec~~ components of difference vectors in $\mathbb{R}^3$ with initial point $P_0$ and an arbitrary terminal point.

~~One can call the tangent space at $P_0$ the 3-dim vector space of all~~ such difference vectors. Let $\boxed{e_\alpha|_{P_0}}$ be the translations of the standard basis vectors to the point $P_0$. Then $\boxed{dx^\alpha|_{P_0}}$ is the dual basis to the basis $\{e_\alpha|_{P_0}\}$ of the tangent space at $P_0$. Notation $T\mathbb{R}^3_{P_0}$.

At each point $P_0$ one has a copy of $\mathbb{R}^3$ and one can consider the tensor algebra over this 3-dim vector space $T\mathbb{R}^3_{P_0}$

~~A $\binom{p}{q}$-tensor field on $\mathbb{R}^3$ is a function on $\mathbb{R}^3$ whose value at $P_0$ is a $\binom{p}{q}$-tensor over $T\mathbb{R}^3_{P_0}$.~~

~~EX. Choose the dot product tensor in each tangent space~~

$$g_{P_0} = \delta_{\alpha\beta}\, dx^\alpha|_{P_0} \otimes dx^\beta|_{P_0} \quad \text{or} \quad g = \delta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$$

[just gives dot products of diff vectors at each pt]

OR the evaluation tensor

$$I_{P_0} = \delta^\alpha{}_\beta\, e_\alpha|_{P_0} \otimes dx^\beta|_{P_0} \quad \text{or} \quad I = \delta^\alpha{}_\beta\, e_\alpha \otimes dx^\beta$$

[here we must understand the vector field $e_\alpha$]

$g_{P_0}$ tells us the inner products of "tangent vectors" at $P_0$.

WHAT IS WRONG WITH THIS PICTURE?

It depends crucially on the fact that $\mathbb{R}^3$ is a vector space so that one has a space of difference vectors at each point to represent the tangent space.

If we start with a space without linear structure we can't repeat this construction. Instead we must generalize some other mathematical structure of $\mathbb{R}^3$.



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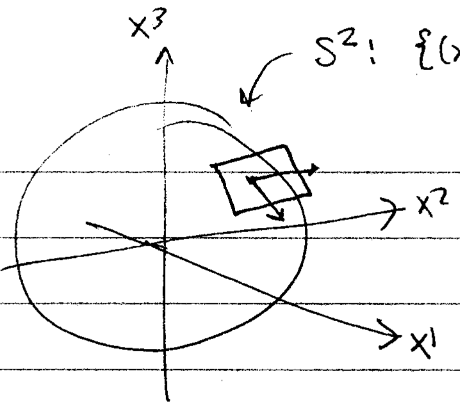
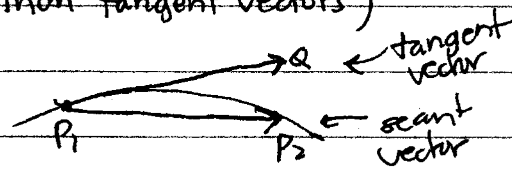
EX.




For points on the unit sphere in $\mathbb{R}^3$ the tangent spaces are well-defined as the difference vectors in $\mathbb{R}^3$ which lie on the plane tangent to the given point.

These are not related to the difference vectors in $\mathbb{R}^3$ that result from the differences of points on $S^2$ (secant vectors rather than tangent vectors)




If we describe the 2-sphere in terms of spherical coordinates, we dont even have the linear structure of $\mathbb{R}^3$ to use in describing the tangent space.

The answer is to define the tangent space as the space of tangent vectors to all possible curves through the point, where the tangent vector itself is chosen not as a difference vector but as the derivative operator associated with the chain rule.

Given a parametrized curve $C(t) = (C^1(t), C^2(t), C^3(t))$ in $\mathbb{R}^3$ with $C(0) = (x_0^1, x_0^2, x_0^3)$

Then if $f$ is any real valued function on $\mathbb{R}^3$ define the tangent vector $X = C'(0)$ by

$$X f = C'(0) f \equiv \frac{d}{dt}\Big|_0 f(C(t)) = \frac{\partial f}{\partial x^\alpha}(C(0)) \underbrace{\frac{dx^\alpha(t)}{dt}}_{C^{\alpha\prime}(0)}\Big|_{t=0}$$

$$= \underbrace{C^{\alpha\prime}(0)\frac{\partial}{\partial x^\alpha}\Big|_{P_0}}_{\vec{C}'(0)\cdot\nabla|_{P_0} \text{ in the classical notation.}} f$$

Thus $\left\{ \frac{\partial}{\partial x^\alpha}\Big|_{P_0} \right\}$ is a basis of this newly defined tangent space and the components of this tangent vector are just the components of the usual difference vector tangent vector of the first definition.

$\frac{\partial}{\partial x^\alpha}\Big|_{P_0}$ is the tangent to the simple curve $c(t) = (x_{10}, \ldots, x_{\alpha 0}+t, \ldots, x_{30})$



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The map $\left\{ \frac{\partial}{\partial x^\alpha}\Big|_{p_0} \to \frac{\partial f}{\partial x^\alpha}\Big|_{p_0} \right\}$ or $\left\{ X \in T\mathbb{R}^3_{p_0} \to Xf \right\}$ for fixed $f$ is a linear ~~map~~ real-valued function called the differential of $f$ at $p_0$: $\boxed{df|_{p_0}\left(\frac{\partial}{\partial x^\alpha}\Big|_{p_0}\right) = \frac{\partial f}{\partial x^\alpha}\Big|_{p_0}}$, therefore an element of the dual space to the tangent space at $p_0$. The dual basis is $dx^\alpha|_{p_0}$ since

$$dx^\beta|_{p_0}\left(\frac{\partial}{\partial x^\alpha}\Big|_{p_0}\right) = \frac{\partial x^\beta}{\partial x^\alpha}\Big|_{p_0} = \delta^\beta{}_\alpha .$$

Now $dx^\alpha|_{p_0}$ is being interpreted as a linear function on the space of tangent vectors ~~interpreted~~ defined as derivative operators on functions.

Now $g = \delta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ defines the Euclidean metric tensor field on $\mathbb{R}^3$ in this sense.

Given another coordinate system on $\mathbb{R}^3$, like spherical coordinates, one obtains a new basis of each tangent space, corresponding to the tangents ~~along the~~ of translation along the coordinate curves.

~~The chain rule gives the transformation of the basis~~

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial \bar{x}^\beta}{\partial x^\alpha}\frac{\partial}{\partial \bar{x}^\beta} \equiv A^\beta{}_\alpha \frac{\partial}{\partial \bar{x}^\beta} \quad \text{or} \quad \frac{\partial}{\partial \bar{x}^\alpha} = \frac{\partial x^\beta}{\partial \bar{x}^\alpha}\frac{\partial}{\partial x^\beta} = A^{-1\,\beta}{}_\alpha \frac{\partial}{\partial x^\beta}$$

~~and for the differentials~~

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\beta}\, dx^\beta = A^\alpha{}_\beta\, dx^\beta$$

~~The components of a $\binom{p}{q}$ tensor transform exactly as above in terms of the~~ matrix of the transformation $A^\alpha{}_\beta$, leading to the classical "tensor transformation law".

But one can choose any nonsingular matrix-valued function on $\mathbb{R}^3$ to define a new basis of each tangent space

$$\left.\begin{aligned} \bar{e}_\alpha &\equiv A^{-1\,\beta}{}_\alpha(x)\cdot \frac{\partial}{\partial x^\beta} \\ \bar{\omega}^\alpha &= A^\alpha{}_\beta(x)\, dx^\beta \end{aligned}\right\} \quad \text{"non-coordinate frame"}$$

(frame: basis-valued function on $\mathbb{R}^3$)

The new components of any tensor again obey the same transformation law.



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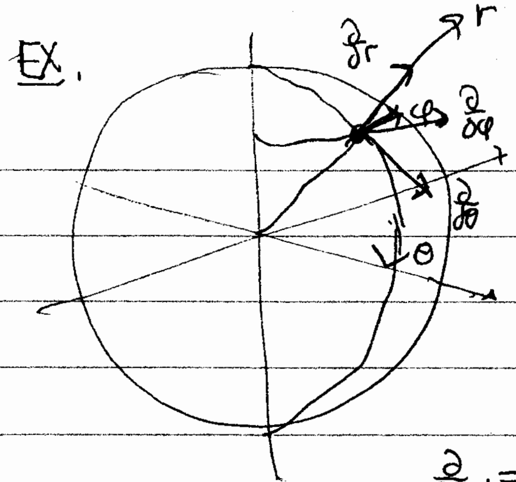
EX.




spherical coords on $\mathbb{R}^3$: $\{\bar{x}^\alpha\} = \{r, \theta, \varphi\}$

$$x^1 = r\sin\theta\sin\varphi$$
$$x^2 = r\sin\theta\cos\varphi$$
$$x^3 = r\cos\theta$$

The original basis of the tangent space $\{\frac{\partial}{\partial x^\alpha}\}$ may be identified with the standard basis vectors $\{e_\alpha\}$ translated from the origin. The new basis

$$\frac{\partial}{\partial \bar{x}^\alpha} = \frac{\partial x^\beta}{\partial \bar{x}^\alpha}\frac{\partial}{\partial x^\beta} = \bar{A}^\beta{}_\alpha \frac{\partial}{\partial x^\beta}$$

may be visualized as the vectors $\bar{A}^\beta{}_\alpha(x)\, e_\beta$ translated to the point with coords $\{x^1, x^2, x^3\}$.

The Euclidean metric

$$g = \delta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta = \delta_{\alpha\beta}\, \bar{A}^\alpha{}_\gamma \bar{A}^\beta{}_\delta\, d\bar{x}^\gamma \otimes d\bar{x}^\delta$$
$$= \dots = \underbrace{dr \otimes dr}_{1 = g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})} + \underbrace{r^2}_{= g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta})} d\theta \otimes d\theta + \underbrace{r^2 \sin^2\theta}_{= g(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi})}\, d\varphi \otimes d\varphi$$

tells us that $\frac{\partial}{\partial r}$ is a unit vector, $\frac{\partial}{\partial \theta}$ has length $r$, $\frac{\partial}{\partial \varphi}$ has length $r\sin\theta$.

[ ORTHONORMAL FRAME: $\hat{e}_1 = \frac{\partial}{\partial r}$, $\hat{e}_2 = \frac{1}{r}\frac{\partial}{\partial \theta}$, $\hat{e}_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$ ]

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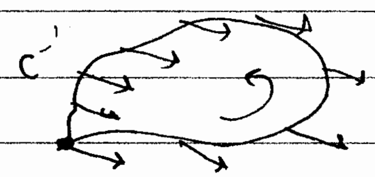
The translation on $\mathbb{R}^3$ defines a global parallel transport.




If you take a tangent vector at a point, you can translate it to each point around a closed curve ~~starting at the point~~.

In fact given a tangent vector at a point, you can translate it all over $\mathbb{R}^3$ to define a constant vector field. Any vector field $X = X^\alpha \frac{\partial}{\partial x^\alpha}$, with constant components in the cartesian coord system, i.e. $\frac{\partial X^\alpha}{\partial x^\beta} = 0$, is such a constant vector field.

One can introduce the covariant derivative of any tensor as the new tensor whose cartesian components are

$$\nabla_\gamma T^{\alpha\cdots}{}_{\beta\cdots} \equiv \partial_\gamma T^{\alpha\cdots}{}_{\beta\cdots} \qquad \partial_\gamma \equiv \frac{\partial}{\partial x^\gamma}.$$

If it's zero, the tensor field is covariant constant, i.e. "constant" in the usual sense. The metric, for example, is covariant constant:

$$\nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} = \partial_\gamma \delta_{\alpha\beta} = 0.$$

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The basis $\{\frac{\partial}{\partial x^\alpha}\}$ itself consists of constant vector fields, and so have zero covariant derivatives

$$\nabla_\gamma\left(\frac{\partial}{\partial x^\alpha}\right) = 0.$$

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In the spherical coord system, a covariant constant vector field obviously has components which are not constant, since the coordinate frame vector fields are not constant

$$\bar\nabla_{\bar\alpha}\left(\bar X^\beta \frac{\partial}{\partial \bar x^\beta}\right) = \underbrace{(\bar\nabla_{\bar\alpha} \bar X^\beta)}_{\partial_{\bar\alpha}\bar X^\beta} \frac{\partial}{\partial \bar x^\beta} + \bar X^\beta \underbrace{\left(\bar\nabla_{\bar\alpha} \frac{\partial}{\partial \bar x^\beta}\right)}_{\bar\Gamma^\gamma{}_{\alpha\beta}\frac{\partial}{\partial \bar x^\gamma}}$$

nonzero derivatives define new vector fields which may be expressed in terms of the frame

$$= \left(\partial_{\bar\alpha}\bar X^\gamma + \bar\Gamma^\gamma{}_{\alpha\beta}\bar X^\beta\right)\frac{\partial}{\partial \bar x^\gamma}$$

↑ correction term that comes from covariant derivatives of frame vectors themselves

$$\bar\nabla_\alpha \frac{\partial}{\partial \bar x^\beta} = \bar\Gamma^\gamma{}_{\alpha\beta}\frac{\partial}{\partial \bar x^\gamma}$$ = cov der of $\beta^{th}$ framevector along $\alpha^{th}$ direction.

TWO WAYS TO CALCULATE CORRECTION TERMS

1) explicitly: $$\bar\nabla_\alpha\left(\frac{\partial}{\partial \bar x^\beta}\right) = \bar\nabla_\alpha\left(A^{-1\gamma}{}_\beta \frac{\partial}{\partial x^\gamma}\right) = \underbrace{(\bar\nabla_\alpha A^{-1\gamma}{}_\beta)}_{\partial_{\bar\alpha} A^{-1\gamma}{}_\beta} \underbrace{\frac{\partial}{\partial x^\gamma}}_{A^\delta{}_\gamma \frac{\partial}{\partial \bar x^\delta}}$$

(just ordinary derivatives of cartesian components)

$$\bar\Gamma^\delta{}_{\alpha\beta} = (\partial_{\bar\alpha} A^{-1\gamma}{}_\beta) A^\delta{}_\gamma = \frac{\partial^2 x^\gamma}{\partial \bar x^\alpha \partial \bar x^\beta}\frac{\partial \bar x^\delta}{\partial x^\gamma} = \bar\Gamma^\delta{}_{\beta\alpha}$$

symmetric in lower indices

2) implicitly: The metric is covariant constant.

$$0 = \bar\nabla_\gamma(\bar g_{\alpha\beta} d\bar x^\alpha \otimes d\bar x^\beta) = (\partial_{\bar\gamma}\bar g_{\alpha\beta})\, d\bar x^\alpha \otimes d\bar x^\beta + \bar g_{\alpha\beta}\, \bar\nabla_\gamma(d\bar x^\alpha)\otimes d\bar x^\beta + \bar g_{\alpha\beta}\, d\bar x^\alpha \otimes (\bar\nabla_\gamma d\bar x^\beta)$$

But $\bar\nabla_\gamma\left[d\bar x^\alpha\left(\frac{\partial}{\partial \bar x^\beta}\right) = 0\right]$ $$\underbrace{(\bar\nabla_\gamma d\bar x^\alpha)}_{\text{covector}}\left(\frac{\partial}{\partial \bar x^\beta}\right) + d\bar x^\alpha\left(\underbrace{\bar\nabla_\gamma \frac{\partial}{\partial \bar x^\beta}}_{\bar\Gamma^\delta{}_{\gamma\beta}\frac{\partial}{\partial \bar x^\delta}}\right) = 0$$

$\beta^{th}$ component ↓ $\delta^\alpha{}_\delta$ → $\bar\Gamma^\alpha{}_{\gamma\beta}$

$$\text{so } \bar\nabla_\gamma d\bar x^\alpha = -\bar\Gamma^\alpha{}_{\gamma\beta}\, d\bar x^\beta$$

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componentwise:

$$0 = \text{"}\nabla_\gamma \bar{g}_{\alpha\beta}\text{"} = \partial_{\bar\gamma} \bar{g}_{\alpha\beta} - \bar\Gamma^{\delta}{}_{\gamma\alpha} \bar{g}_{\delta\beta} - \bar\Gamma^{\delta}{}_{\gamma\beta} \bar{g}_{\alpha\delta}$$

~~But~~ or $0 = \partial_{\bar\gamma} \bar{g}_{\alpha\beta} - \bar\Gamma_{\beta\gamma\alpha} - \bar\Gamma_{\alpha\gamma\beta}$

and $0 = -\partial_{\bar\alpha} \bar{g}_{\beta\gamma} + \Gamma_{\gamma\alpha\beta} + \Gamma_{\beta\alpha\gamma}$

$0 = \partial_{\bar\beta} \bar{g}_{\gamma\alpha} - \Gamma_{\alpha\beta\gamma} - \Gamma_{\gamma\beta\alpha}$

Adding $0 = [\partial_{\bar\gamma} \bar{g}_{\alpha\beta} - \partial_{\bar\alpha} \bar{g}_{\beta\gamma} + \partial_{\bar\beta} \bar{g}_{\gamma\alpha}] - (\Gamma_{\alpha\beta\gamma} + \Gamma_{\alpha\gamma\beta})$
$+ (\Gamma_{\beta\alpha\gamma} - \Gamma_{\beta\gamma\alpha}] + [\Gamma_{\gamma\alpha\beta} - \Gamma_{\gamma\beta\alpha}]$

If assume $\Gamma^{\gamma}{}_{\alpha\beta} = \Gamma^{\gamma}{}_{\beta\alpha}$ (SYMMETRIC COV DER)

then $\bar\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\partial_{\bar\gamma} \bar{g}_{\alpha\beta} - \partial_{\bar\alpha} \bar{g}_{\beta\gamma} + \partial_{\bar\beta} \bar{g}_{\gamma\alpha})$

$\bar\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} \bar{g}^{\alpha\delta} (\partial_{\bar\gamma} \bar{g}_{\delta\beta} - \partial_{\bar\delta} \bar{g}_{\beta\gamma} + \partial_{\bar\beta} \bar{g}_{\gamma\delta})$

equivalent to previous expression using

$$\begin{cases} \bar{g}_{\alpha\beta} = \delta_{\gamma\delta} A^{-1\gamma}{}_{\alpha} A^{-1\delta}{}_{\beta} \\ \bar{g}^{\alpha\beta} = \delta^{\gamma\delta} A^{\alpha}{}_{\gamma} A^{\beta}{}_{\delta} \end{cases}$$

---

similarly for any tensor

$$T = \bar{T}^{\alpha\cdots}{}_{\beta\cdots} \frac{\partial}{\partial \bar{x}^{\alpha}} \otimes \cdots \otimes d\bar{x}^{\beta} \otimes \cdots$$

$$\bar\nabla_\gamma T = (\nabla_\gamma \bar{T}^{\alpha\cdots}{}_{\beta\cdots}) \frac{\partial}{\partial \bar{x}^{\alpha}} \otimes \cdots \otimes d\bar{x}^{\beta}$$
$$+ \bar{T}^{\alpha\cdots}{}_{\beta\cdots} (\bar\nabla_\gamma \frac{\partial}{\partial \bar{x}^{\alpha}}) \otimes \cdots d\bar{x}^{\beta} + \cdots$$
$$+ T^{\alpha\cdots}{}_{\beta\cdots} \frac{\partial}{\partial \bar{x}^{\alpha}} \otimes \cdots \otimes (\bar\nabla_\gamma d\bar{x}^{\beta}) + \cdots$$
$$= (\partial_{\bar\gamma} \bar{T}^{\alpha\cdots}{}_{\beta\cdots} + \bar\Gamma^{\alpha}{}_{\gamma\delta} \bar{T}^{\delta\cdots}{}_{\beta\cdots} + \cdots - \bar\Gamma^{\delta}{}_{\gamma\beta} \bar{T}^{\alpha\cdots}{}_{\delta\cdots} - \cdots) \frac{\partial}{\partial \bar{x}^{\alpha}} \otimes \cdots \otimes d\bar{x}^{\beta} \otimes \cdots$$

Note: $\nabla T = (\bar\nabla_\gamma T) \otimes dx^\gamma$ tensor with one more covariant index.

while $\bar\nabla_\gamma T = \nabla_{(\frac{\partial}{\partial \bar{x}^\gamma})} T$ = covariant derivative of $T$ along vector field $\frac{\partial}{\partial \bar{x}^\gamma}$.

In general $\nabla_{\bar{X}} T = \bar{X}^\gamma \bar\nabla_\gamma T$. for the covariant derivative of $T$ along $\bar{X}$.

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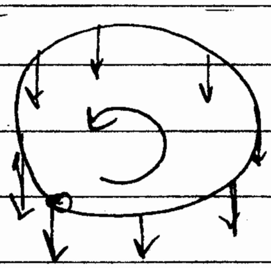


Now in spherical coordinates if we insist that the covariant derivative along a curve of a tangent vector be zero:

$$\frac{D\overline{X}^\alpha}{dt} = \overline{\nabla}_{C'(t)} \overline{X}^\alpha = \text{"} \nabla_\beta \overline{X}^\alpha \, \overline{C}'(t)^\beta \text{"}$$

$$= \frac{d\overline{X}^\alpha}{dt} + \overline{\Gamma}^\alpha{}_{\beta\gamma} \overline{X}^\gamma \overline{C}'(t)^\beta = 0$$

we get diff eqs to transport tangent vector around curve so that it is "translated" in the usual sense. This "translation" preserves all lengths & relative angles $\left( \text{since } \frac{D\overline{g}_{\alpha\beta}}{dt} = 0 \right)$

Given any metric ~~$\overline{g}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ on any space, one has this same~~ parallel transport defined · but in general when a vector is transported around a closed curve, its direction will change.

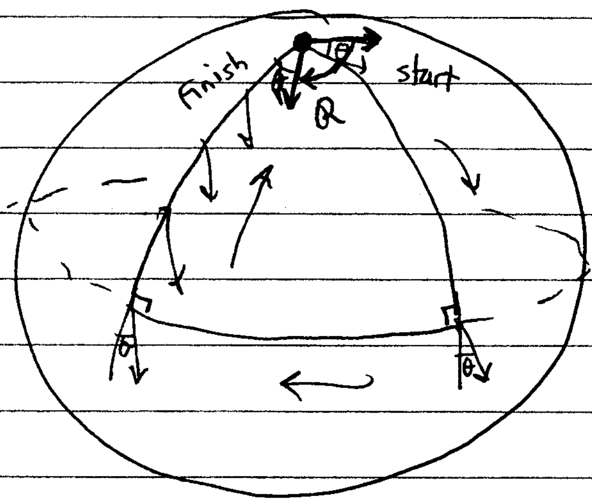
EX. $r = a$ describes a sphere of radius $a$ in spherical coords

$dr = 0 \rightarrow \quad g \mapsto \underbrace{a^2}_{g_{\theta\theta}} d\theta \otimes d\theta + \underbrace{a^2 \sin^2}_{g_{\varphi\varphi}} d\varphi \otimes d\varphi$

can compute $\Gamma^\alpha{}_{\beta\gamma}$ $\quad \alpha, \beta, \gamma = 2, 3$

but already know how parallel transport works.

Maintains angles & lengths so




parallel transport around a closed loop ~~rotates a vector~~:

$X^\alpha \longrightarrow \underbrace{R^\alpha{}_\beta(C)}_{\text{rotation.}} X^\beta$

This is the result of <u>curvature</u>.

on $\mathbb{R}^3$ $\quad [\partial_\alpha, \partial_\beta] X^\gamma \equiv 0$ since partial derivatives commute

~~so the transfor~~ so we have $\quad [\nabla_\alpha, \nabla_\beta] X^\gamma = 0$. for the covariant derivatives of any vector field.

One can evaluate this in the spherical coordinates

$$\bar\nabla_\beta \bar X^\gamma = \partial_{\bar\beta} \bar X^\gamma + \bar\Gamma^\gamma{}_{\beta\delta} \bar X^\delta$$

$$\bar\nabla_\alpha \bar\nabla_\beta \bar X^\gamma = \partial_{\bar\alpha} (\bar\nabla_\beta \bar X^\gamma) + \bar\Gamma^\gamma{}_{\alpha\epsilon} \bar\nabla_\beta \bar X^\epsilon - \bar\Gamma^\epsilon{}_{\alpha\beta} \bar\nabla_\epsilon \bar X^\gamma$$

$$= \partial_{\bar\alpha} (\partial_{\bar\beta} \bar X^\gamma + \bar\Gamma^\gamma{}_{\beta\delta} \bar X^\delta) + \bar\Gamma^\gamma{}_{\alpha\epsilon} (\partial_{\bar\beta} \bar X^\epsilon + \bar\Gamma^\epsilon{}_{\beta\delta} \bar X^\delta)$$
$$- \bar\Gamma^\epsilon{}_{\alpha\beta} (\partial_{\bar\epsilon} \bar X^\gamma + \bar\Gamma^\gamma{}_{\epsilon\delta} \bar X^\delta)$$

$$[\bar\nabla_\alpha, \bar\nabla_\beta] \bar X^\gamma = \dots = \underbrace{\left(\partial_{\bar\alpha} \bar\Gamma^\gamma{}_{\beta\delta} - \partial_{\bar\beta} \bar\Gamma^\gamma{}_{\alpha\delta} + \bar\Gamma^\gamma{}_{\alpha\epsilon} \bar\Gamma^\epsilon{}_{\beta\delta} - \bar\Gamma^{\cdot\cdot}_{\cdot\cdot} \bar\Gamma^{\cdot\cdot}_{\cdot\cdot}\right)}_{\bar R^\gamma{}_{\delta\alpha\beta} \equiv 0} \bar X^\delta$$

This tells us cov.der's commute. &

reflects the "integrability" of the parallel transport on flat space.

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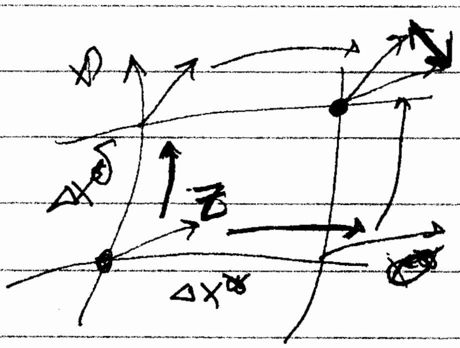
On a nonflat space this quantity is nonzero




[coordinate mesh loop in surface of coordinates $x^\gamma$ and $x^\delta$]

~~for small increments of the coordinates~~

$\Delta Z^\alpha \sim R^\alpha{}_{\beta\gamma\delta}\, \Delta x^\gamma \Delta x^\delta Z^\delta$

& describes "nonintegrability" of parallel transport.

for some more details:
Geometrical Methods of (Mathematical?) Physics
B. Schutz.

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