Notes on the Lifshitz Perturbation Analysis

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A brief overview of the foundations of the mathematical Lifshitz perturbation analysis, followed by the definition of the Lie derivative and gauge transformations of the metric, with a discussion of the available gauge freedom in the perturbed metric and fluid variables. Harmonic analysis is developed using Cartesian tensors, and the tensor harmonics and their covariant derivatives are introduced and calculated, together with the Ricci identities needed for their manipulation. THe Lie derivative pages have been used repeatedly in later notes.

- lpa1984.pdf: 27 pages, 750K

Notes on the Lifshitz Perturbation Analysis:

Linearized Einstein Equations at FRW Background and Harmonic Expansions on Spaces of Constant Curvature

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① Perturbations of FRW cosmologies

MTW Gravitation p. 964
Lifshitz & Khalatnikov
Adv. Phys. 12, 185 (1963)

1) Linearization of Einstein Eq (Dynamics)
2) Harmonic decomposition of linearized fields (symmetry)
fourier analysis on E^3 ($k=0$)
tensor, vector, scalar harmonics on S^3, H^3 ($k=1,-1$)

I: $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \lambda h_{\mu\nu}$ λ = expansion parameter, curve of solutions in space of metrics

$\equiv \delta g_{\mu\nu}$ $h_{\mu\nu} = \frac{d}{d\lambda}\Big|_{\lambda=0} g_{\mu\nu}$ tangent to curve at $g^{(0)}$

$\delta \equiv \lambda\left(\frac{d}{d\lambda}\Big|_{\lambda=0}\right)$ variation operator. $\boxed{\text{Set } \lambda=1 \text{ get } g_{\mu\nu} - g^{(0)}_{\mu\nu} = h_{\mu\nu}}$

Assume $G_{\mu\nu} = \kappa T_{\mu\nu}$, for all λ. Then first at $\lambda=0$: $G^{(0)}_{\mu\nu} = \kappa T^{(0)}_{\mu\nu}$

~~Thed~~ $\underbrace{\delta G_{\mu\nu}} = \kappa\, \delta T_{\mu\nu}$.

How to calculate?

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}$$

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\mu\alpha} \qquad (= R^\alpha{}_{\mu\alpha\nu})$$

$$\Gamma^\alpha{}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\delta}(g_{\delta\mu,\nu} + g_{\delta\nu,\mu} - g_{\mu\nu,\delta})$$

(i) $\delta g^{\alpha\beta} = -\, g^{(0)\alpha\gamma} g^{(0)\beta\delta}\, \delta g_{\gamma\delta}$
$$\left(\begin{array}{l} g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma \\ \delta g^{\alpha\beta} g^{(0)}_{\beta\gamma} + g^{(0)\alpha\beta}\delta g_{\beta\gamma} = 0 \\ \delta g^{\alpha\beta} = -\, g^{\alpha\delta} g^{\beta\gamma}\delta g_{\gamma\delta} \end{array}\right)$$ Take $\lambda\left(\frac{d}{d\lambda}\Big|_{\lambda=0}\right)$

(ii) $$\delta\Gamma^\alpha{}_{\mu\nu} = -\tfrac{1}{2}\, g^{(0)\alpha\epsilon} \underbrace{g^{(0)\delta\rho}(g^{(0)}_{\delta\mu,\nu} + g^{(0)}_{\delta\nu,\mu} - g^{(0)}_{\mu\nu,\delta})}_{\Gamma^\rho{}_{\mu\nu}} \delta g_{\epsilon\rho} + \tfrac{1}{2} g^{(0)\alpha\delta}(\delta g_{\delta\mu,\nu} \ldots)$$

$$= -\tfrac{1}{2} g^{\alpha\delta}\Gamma^\rho{}_{\mu\nu}\delta g_{\delta\rho} + \tfrac{1}{2} g^{\alpha\delta}(\delta g_{\delta\mu,\nu} + \delta g_{\delta\nu,\mu} - \delta g_{\mu\nu,\delta})$$

$$\overset{\text{exercise}}{=} \tfrac{1}{2} g^{\alpha\delta}(\delta g_{\delta\mu;\nu} + \delta g_{\delta\nu;\mu} - \delta g_{\mu\nu;\delta}) \qquad \text{L\&F (I.3)}$$

$g^{(0)}_{\mu\nu}$ covariant derivative

(iii) $$\delta R_{\mu\nu} = \delta\Gamma^\alpha{}_{\mu\nu,\alpha} - \delta\Gamma^\alpha{}_{\mu\alpha,\nu} + \delta\Gamma^\alpha{}_{\beta\alpha}\Gamma^{(0)\beta}{}_{\mu\nu} + \Gamma^{(0)\alpha}{}_{\beta\alpha}\delta\Gamma^\beta{}_{\mu\nu} - \ldots$$

see L&K appendix I.

$$\overset{\text{exercise}}{=} \tfrac{1}{2}\left[-(g^{\gamma\delta}\delta g_{\gamma\delta})_{;\mu\nu} - \delta g_{\mu\nu;\alpha}{}^{;\alpha} + 2\,\delta g_{\alpha(\mu;\nu)}{}^{;\alpha}\right]$$

$$\delta R_{\mu\nu}\Big|_{\lambda=1} = \tfrac{1}{2}\left[-h_{;\mu\nu} - h_{\mu\nu;\alpha}{}^{;\alpha} + 2\, h_{\alpha(\mu;\nu)}{}^{;\alpha}\right] \qquad \text{(MTW 35.58a)}$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \tfrac{1}{2} R\,\delta g_{\mu\nu} - \tfrac{1}{2}\delta R\, g_{\mu\nu} = \ldots$$

(2)

same formulas used to describe gravitational radiation in flat spacetime or in the field of a black hole, etc.

$\boxed{\text{Background quantities}}$ $g^{(0)}_{\mu\nu}$:

$$ds^{2\,(0)} = -dt^2 + a(t)^2\left[d\chi^2 + \begin{pmatrix}\sin^2\chi\\ \chi^2\\ \sinh^2\chi\end{pmatrix} d\Omega^2\right]$$

$$= a(\eta)^2\left[-d\eta^2 + d\chi^2 + \begin{pmatrix}\sin^2\chi\\ \chi^2\\ \sinh^2\chi\end{pmatrix} d\Omega^2\right]$$

$dt = a\,d\eta$

conformal time.

$$\begin{cases} g^{(0)}_{00} = -a^2 \\ g^{(0)}_{ij} = a^2\delta_{ij}\end{cases}$$

$\boxed{i,j = 1,2,3}$

(decompose everything into space + time components $\underline{\underline{3+1}}$)

$$\Gamma^{(0)\alpha}{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}\left(g^{(0)}_{\delta\beta,\gamma} + g^{(0)}_{\delta\gamma,\beta} - g^{(0)}_{\beta\gamma,\delta}\right)$$

$$\Gamma^{(0)0}{}_{00} = \frac{\dot a}{a}$$

$$\Gamma^{(0)0}{}_{ij} = -\frac{\dot a}{a}\delta_{ij}$$

$$\Gamma^{(0)i}{}_{0j} = \tfrac{1}{2} g^{ik}(g_{kj,0}) = \frac{\dot a}{a}'$$

$$\Gamma^{i}{}_{jk} = \text{"}\Gamma(g^{(0)}_{ij})\text{"} \quad \text{3-connection.}$$

$u^\alpha = a^{-1}\delta^\alpha_0$

$$T_{\alpha\beta} = (\rho+p)u_\alpha u_\beta + p\,g_{\alpha\beta}$$

$$R^{(0)i}{}_{j} = a^{-4}(2a^2 + a'^2 + aa'')\delta^i_j$$

$$R^0{}_0 = 3a^{-4}(aa'' - a'^2)$$

$$R^0{}_\alpha = 0$$

$$R^i_i = 6a^{-3}(a + a'')$$

$$T^{(0)i}{}_j = p\,\delta^i_j$$

$$T^{(0)0}{}_0 = -\rho$$

$$T^{(0)0}{}_\alpha = 0$$

$p = (\gamma-1)\rho$

0 dust

$\frac{1}{3}$ radiation

E.E. $\begin{cases}\rho = 3a^{-4}(a^2 + a'^2)\\ \cancel{p = a^{-4}(a'^2 - 2aa'' - a^2)}\end{cases}$

	$k=1$	$\begin{pmatrix}\eta\to i\eta\\ \chi\to i\eta\\ a\to ia\end{pmatrix}$ $k=-1$	$k=0$
$\boxed{p=0,\ \rho=\rho_0 a^{-3}}$	$a = a_0(1-\cos\eta)$ $t = a_0(\eta - \sin\eta)$	$a = a_0(\cosh\eta - 1)$ $t = a_0(\sinh\eta - \eta)$	$\downarrow$
$\boxed{p=\frac{1}{3},\ \rho=\rho_0 a^{-4}}$	$a = b_0\sin\eta$ $t = b_0(1-\cos\eta)$	etc.	

3+1 decomposition of perturbation

$h_{\alpha\beta} \leftrightarrow h_{00}, h_{0i}, h_{ij}$, let $h = g^{(0)ij} h_{ij}$

"scalar" "vector" "tensor"

time + space derivatives

$h_{\alpha\beta;\gamma} \leftrightarrow h'_{\alpha\beta}, h_{\alpha\beta|i}$

3-connection $\Gamma^i{}_{jk}$

$$\delta R^i{}_j = \tfrac{1}{2}a^{-2}\left(h_j{}^k{}_{|}{}^i{}_k + h^i{}_{k|j}{}^{|k} - h^i{}_{j|k}{}^{|k} - h_{|j}{}^{|i}\right)$$
$$+ \frac{1}{2a^2} h^i{}_j{}'' + \frac{a'}{a^3} h^i{}_j{}' + \frac{a'}{2a^3} h'\delta^i_j \mp \frac{2}{a^2} h^i{}_j .$$

$$\delta R^0{}_0 = \frac{1}{2a}h'' + \frac{a'}{2a^3}h'$$

$$\delta R^0{}_i = \frac{1}{2a^2}\left(h'_{,i} - h^j{}_{i|j}{}'\right)$$

$$\delta R^i{}_i = \frac{1}{a^2}\left(h^j{}_j{}^{;k}{}_{;k} - h_{;k}{}^{;k}\right) + \frac{1}{a^2}h'' + \frac{3a'}{a^3}h' \pm \frac{2h}{a^3}$$

spatial curvature.
$+ \to k=1$
$- \quad k=-1$
0 if $k=0$

$h_{0\alpha} = 0$ linearized synchronous gauge.

$$\left\{\begin{aligned} g_{0\alpha} &= -a^2\delta^0_\alpha \\ g^{(0)}_{0\alpha} &= -a^2\delta^0_\alpha \\ h_{0\alpha} &= 0 \end{aligned}\right.$$

Indep of λ

almost synchronous gauge (conformal time gauge.)

(synchronous gauge $g_{0\alpha} = -\delta^0_\alpha$)

FLUID VARIATION

$g_{\alpha\beta} u^\alpha u^\beta = -1$

$g^{(0)}_{\alpha\beta} u^\alpha \delta u^\beta + h_{\alpha\beta}u^\alpha u^\beta = 0 \rightarrow -a^2 u^0 \delta u^0 + g_{ij}\cancel{u^i}\delta u^0 + h_{ij}u^i \cancel{u^j} = 0$

$\delta u^0 = 0$.

~~$(\alpha\beta)=(0,0)$~~ ~~$-a^2u^0\delta u^0 + h$~~

$T^\alpha{}_\beta = (\rho+p)u^\alpha u_\beta + p\delta^\alpha_\beta$

$$\left.\begin{aligned} \delta T^0{}_0 &= (\delta\rho+\delta p)(-1) + \delta p = -\delta\rho \\ \delta T^i{}_0 &= (\rho+p)\underbrace{u_0}_{-1}\delta u^i = -(\rho+p)\delta u^i \\ \delta T^i{}_j &= \delta p\,\delta^i_j \end{aligned}\right\}$$

$\delta p = \left(\frac{dp}{d\rho}\right)\delta\rho$

so $\delta T^i{}_j = -\delta^i{}_j \frac{dp}{d\rho}\delta T^0_0$

(4)

$$\delta G^i{}_j = 0 \quad i \neq j \quad (\delta T^\alpha{}_\beta = 0 \text{ if } \alpha \neq \beta)$$

$$\delta G^i{}_i = \kappa\, \delta T^i{}_i = -\kappa \frac{dp}{d\rho}\, \delta T^0{}_0 = -\kappa \underbrace{\frac{dp}{d\rho}}_{(\gamma-1) = \left\{ 0,\ \frac{1}{3} \right.} \delta G^0{}_0$$

independent of fluid variables
determine h_{ij}

$\downarrow$ then specify $\delta\rho$, δu^i

$$\delta G^0{}_0 = -\kappa\, \delta T^0{}_0 = +\kappa\, \delta\rho$$

$$\delta G^i{}_0 = \delta R^i{}_0 = \kappa\, \delta T^i{}_0 = \kappa(p+\rho)\, \delta u^i$$

$$\delta\rho = \kappa^{-1}\, \delta G^0{}_0$$

$$\delta u^i = -\frac{1}{\kappa(p+\rho)}\, \delta R^i{}_0$$

since EE are coordinate invariant

if $g_{\mu\nu}$ ~~is~~ solution, so is $\bar{g}_{\mu\nu}$

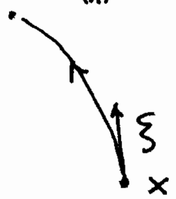
hence if $h_{\mu\nu}$ is a linearized solution, so is $h_{\mu\nu} - \mathcal{L}_{\xi}\, g^{(0)}_{\mu\nu}$.

can use this freedom to eliminate unimportant constants in solution of linearized EE.

Lie Derivative

$x^\mu \to \bar{x}^\mu = f^\mu_{(\lambda)}(x)$ — 1-parameter family of point transformations ("diffeomorphisms")

$\bar{x} = f_{(\lambda)}(x)$



$x = f^{-1}_{(\lambda)}(\bar{x})$

$f^\mu_{(0)}(x) = x^\mu$ identity transformation

Define
$$\begin{cases} \xi^\mu(x) \equiv \dfrac{df^\mu_{(0)}}{d\lambda}(x) & \text{vector field} \\[2ex] \xi = \xi^\mu \dfrac{\partial}{\partial x^\mu} & \text{1st order differential operator} \end{cases}$$

Then
$$\bar{x}^\mu = f^\mu_{(0)}(x) + \lambda \frac{df^\mu_{(0)}}{d\lambda}(x) + \tfrac{1}{2}\lambda^2 \frac{d^2 f^\mu_{(0)}}{dx^2}(x) + \dots \qquad \text{power series expansion}$$
$$= x^\mu + \lambda \xi^\mu(x) + \dots$$
$$\approx x^\mu + \lambda \xi^\mu(x) \qquad \text{for } \lambda \ll 1.$$

The inverse transformation $f^{-1}_{(\lambda)}$ satisfies: $f^{-1\,\mu}_{(\lambda)}(f_{(\lambda)}(x)) = x^\mu$

Using $f^{-1\,\mu}_{(0)}(x) = x^\mu$ and taking $\frac{d}{d\lambda}\Big|_{\lambda=0}$ of this yields

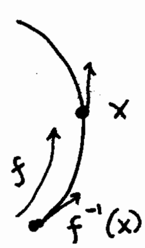
$$\frac{df^{-1\,\mu}_{(0)}}{d\lambda}(f_{(0)}(x)) + \frac{d}{d\lambda}\Big|_{\lambda=0} \underbrace{f^{-1\,\mu}_{(0)}(f_{(\lambda)}(x))}_{f^\mu_{(\lambda)}(x)} = 0 \quad \to \quad \frac{df^{-1\,\mu}_{(0)}}{d\lambda}(x) = -\frac{df^\mu_{(0)}}{d\lambda}(x) = -\xi^\mu(x)$$

so the inverse transformation: $$f^{-1\,\mu}_{(\lambda)}(x) = x^\mu - \lambda\xi^\mu(x) + \dots \underset{\lambda \ll 1}{\approx} x^\mu - \lambda \xi^\mu(x)$$

If $\phi(x)$ is a (scalar) function, let $\Phi_{(\lambda)}(x)$ be the function transformed by the point transformation:

$$\underset{\text{value at new point}}{\Phi_{(\lambda)}(\bar{x})} = \underset{\text{value at old point}}{\phi(x)} \qquad \text{or} \qquad \underbrace{\Phi_{(\lambda)}(x) = \phi(f^{-1}_{(\lambda)}(x))}$$

This definition moves the function in the direction of the point transformation

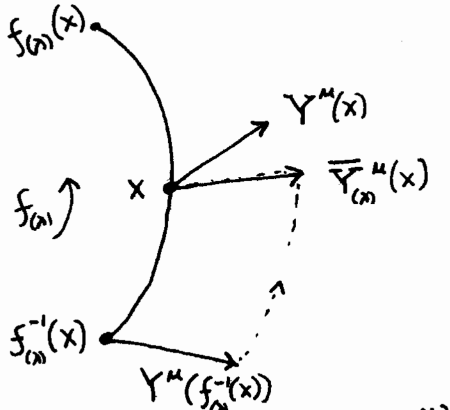


$$\Phi_{(\lambda)}(x) \;\Vert\; \phi(f^{-1}(x))$$

The rate of change of $\Phi_{(\lambda)}$ with respect to λ at $\lambda = 0$ tells how ϕ begins to change under the point transformation

$$\frac{d}{d\lambda}\Big|_{\lambda=0} \Phi_{(\lambda)}(x) = \frac{\partial \phi}{\partial x^\mu}(f^{-1}_{(0)}(x)) \frac{df^{-1\,\mu}_{(0)}}{d\lambda}(x) = -\xi^\mu(x) \frac{\partial \phi}{\partial x^\mu}(x) = -\xi(x)\phi$$
$$\equiv -(\pounds_\xi \phi)(x)$$

The Lie derivative of a scalar by a vector field ξ is just the directional derivative of the scalar along that vector field:
$$\pounds_\xi \phi = \xi^\mu \frac{\partial \phi}{\partial x^\mu} \equiv \phi_{,\mu} \xi^\mu$$



A vector field is transformed by the point transformation as follows:

$$\underbrace{\bar{Y}_{(\lambda)}{}^{\mu}(x)}_{\text{value at } x} = \frac{\partial f_{(\lambda)}^{\mu}}{\partial x^{\nu}}(f^{-1}_{(\lambda)}(x))\; \underbrace{Y^{\nu}(f^{-1}_{(\lambda)}(x))}_{\text{value at point mapped onto } x \text{ by } f_{(\lambda)}}$$

Now calculate its Lie derivative exactly as for the scalar:

$$(\mathcal{L}_\xi Y^\mu)(x) = -\frac{d}{d\lambda}\Big|_{\lambda=0} \bar{Y}_{(\lambda)}{}^{\mu}(x)$$

$$= -\Big[\underbrace{\frac{\partial f_{(0)}^\mu}{\partial x^\nu}(\underbrace{f_{(0)}^{-1}(x)}_{x})}_{\frac{\partial x^\mu}{\partial x^\nu}=\delta^\mu{}_\nu}\Big]\; \underbrace{\underbrace{\frac{\partial Y^\nu}{\partial x^\rho}(f^{-1}_{(0)}(x))\, \frac{d f^{-1\rho}_{(0)}}{d\lambda}(x)}_{-\xi^\rho(x)\frac{\partial Y^\nu}{\partial x^\rho}(x)}}_{\text{just like in scalar case}} - \frac{\partial}{\partial x^\nu}\Big[\underbrace{\frac{df_{(0)}^\mu}{d\lambda}}_{\xi^\mu}\Big](f^{-1}_{(0)}(x))\; Y^\nu(\underbrace{f^{-1}_{(0)}(x)}_{x})$$

$$= \Big[\underbrace{\xi^\rho \frac{\partial Y^\nu}{\partial x^\rho}}_{\text{directional derivative of components}} - \frac{\partial \xi^\mu}{\partial x^\nu}\, Y^\nu\Big](x)$$

$$\boxed{\mathcal{L}_\xi \phi = \phi_{,\rho}\,\xi^\rho}$$

$$\boxed{\mathcal{L}_\xi Y^\mu = Y^\mu{}_{,\rho}\,\xi^\rho - \xi^\mu{}_{,\rho}\,Y^\rho}$$

$$\boxed{\mathcal{L}_\xi Z_\mu = Z_{\mu,\rho}\,\xi^\rho + Z_\rho\, \xi^\rho{}_{,\mu}}$$

$$\boxed{\mathcal{L}_\xi g_{\mu\nu} = \underbrace{g_{\mu\nu,\rho}}_{\Gamma_{\nu\mu\rho}+\Gamma_{\mu\nu\rho}}\xi^\rho + g_{\rho\nu}\,\xi^\rho{}_{,\mu} + g_{\mu\rho}\,\xi^\rho{}_{,\nu}}$$

If we do the same thing for a covariant vector field

$$\bar{Z}_\mu^{(\lambda)} = \underbrace{\frac{\partial f^{-1\,\nu}_{(\lambda)}}{\partial x^\mu}(x)}\, Z_\nu(f^{-1}_{(\lambda)}(x))$$

the λ-derivative of this term leads instead to $-\xi^\nu{}_{,\mu} Z_\nu$ so we get

For the metric we get one of these terms for each covariant index, but always the first term is just the directional derivative of the components

The expression for the $\mathcal{L}_\xi g_{\mu\nu}$ can be rewritten using

to yield:

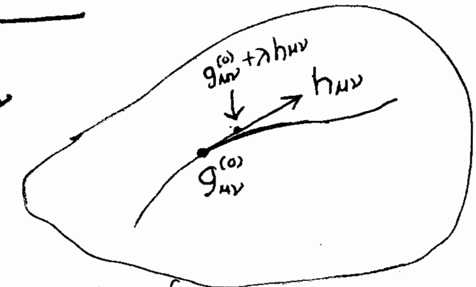
$$\mathcal{L}_\xi g_{\mu\nu} = g_{\rho\nu}\,\xi^\rho{}_{;\mu} + g_{\mu\rho}\,\xi^\rho{}_{;\nu} = (g_{\rho\nu}\xi^\rho)_{;\mu} + (g_{\mu\rho}\xi^\rho)_{;\nu}$$

$$= \xi_{\nu;\mu} + \xi_{\mu;\nu} \qquad (\text{since } g_{\rho\nu;\sigma}=0)$$

Gauge transformations of the gravitational field $g_{\mu\nu}$ and of the linearized field $h_{\mu\nu}$

$$\underbrace{g_{\mu\nu}}_{\text{solution of EE to first order in }\lambda} = \underbrace{g^{(0)}_{\mu\nu}}_{\text{solution of EE.}} + \lambda \underbrace{h_{\mu\nu}}_{\text{solution of linearized E.E.}} \,, \qquad h_{\mu\nu} = \frac{d}{d\lambda}\Big|_{\lambda=0} g_{\mu\nu}$$

(tangent to curve of solutions of EE.)



space of solutions of E.E.

Since the EE are invariant under point transformations, we are free to transform $g_{\mu\nu}$ by a point transformation, which may depend on λ:

$$\bar g^{(\lambda)}_{\mu\nu}(x) = \frac{\partial f^{-1\,\alpha}_{(\lambda)}(x)}{\partial x^\mu}\frac{\partial f^{-1\,\beta}_{(\lambda)}(x)}{\partial x^\nu}\, g_{\mu\nu}(f^{-1}(x))$$

$$= \bar g^{(0)(\lambda)}_{\mu\nu} + \lambda\, \bar h^{(\lambda)}_{\mu\nu}$$

The new linearized field is

$$\bar h_{\mu\nu} = \frac{d}{d\lambda}\Big|_{\lambda=0} \bar g^{(\lambda)}_{\mu\nu} = \underbrace{\bar h^{(0)}_{\mu\nu}}_{h_{\mu\nu}} + \underbrace{0\cdot\frac{d}{d\lambda}\Big|_{\lambda=0}\bar h^{(\lambda)}_{\mu\nu}}_{\text{zero}} + \underbrace{\frac{d}{d\lambda}\Big|_{\lambda=0}\bar g^{(0)(\lambda)}_{\mu\nu}}_{-\mathcal{L}_\xi g^{(0)}_{\mu\nu}}$$

identity transformation

$$\boxed{\bar h_{\mu\nu} = h_{\mu\nu} - \mathcal{L}_\xi g^{(0)}_{\mu\nu}} = h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu})$$

Thus the point transformation gauge freedom of the nonlinear EE leads to this additive gauge freedom of the linearized EE, i.e. if $h_{\mu\nu}$ is a soln of the linearized EE, so is $\bar h_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi g^{(0)}_{\mu\nu}$ (the sign doesnt matter). for <u>any</u> vector field ξ.

Gauge freedom of FRW linearized fields

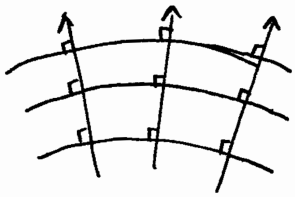
[Note Lifshitz uses the index 0 to refer to t, not η]

The metric $g^{(0)}_{\mu\nu}$ is : $ds^{2(0)} = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)\Big[-d\eta^2 + \underbrace{dx^2 + \begin{pmatrix}\sin^2\chi\\ \chi^2\\ \sinh^2\chi\end{pmatrix} d\Omega^2}_{\gamma_{ij}dx^i dx^j}\Big]$

i.e. $g^{(0)}_{00} = -a^2$, $g^{(0)}_{ij} = a^2\gamma_{ij}$

or $\boxed{g^{(0)}_{0\alpha} = -a^2\delta^0_\alpha}$ (almost synchronous gauge)

Here $a\,d\eta = dt$ relates η to the proper time t measured along the geodesics perpendicular to the space sections, which are the time lines.



If we impose the same gauge on $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ then we must have $h_{0\alpha} = 0$ (linearized synchronous gauge)

The spacetime with metric $g_{\mu\nu}$ has no natural time function since it has no symmetry like the background metric.

Not only is there the freedom to change the spatial coordinates on the given spatial hypersurfaces of constant η-time $x^0 \to x^0$, $x^i \to f^i(x^j)$ in a time independent way as we can on the background spacetime, but we can also deform the spatial hypersurfaces themselves $x^0 \to x^0 + f^0(x^i)$ in which case the time lines must change to remain perpendicular to the new space sections of constant η-time.

[See Landau & Lifshitz, Classical Theory of Fields, § Synchronous coordinates]

In the linearized problem, the gauge $h_{0\alpha} = 0$ is preserved as long as $\mathcal{L}_\xi g^{(0)}_{0\alpha} = 0$, $\quad \Big[\, h_{0\alpha} \to \underbrace{h_{0\alpha}}_{0} + \mathcal{L}_\xi g^{(0)}_{0\alpha} = 0 \,\Big]$

which may be considered as an equation determining those vector fields ξ^μ whose associated "infinitesimal" transformations preserve this gauge.

$$\mathcal{L}_\xi g^{(0)}_{\mu\nu} = g^{(0)}_{\mu\nu,\rho}\xi^\rho + g^{(0)}_{\rho\nu}\xi^\rho{}_{,\mu} + g^{(0)}_{\mu\rho}\xi^\rho{}_{,\nu} \overset{\text{or}}{=} \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

(i) $0 = \mathcal{L}_\xi g^{(0)}_{00} = g_{00,0}\xi^0 + 2\xi^0{}_{,0}g_{00} = -2aa' - 2a^2\xi^0{}_{,0} = -2a\,(a\xi^0)'$

(ii) $0 = \mathcal{L}_\xi g^{(0)}_{0i} = \xi^j{}_{,0}g^{(0)}_{ji} + \xi^0{}_{,i}g_{00} = -a^2\xi^0{}_{,i} + \xi^j{}_{,0}g^{(0)}_{ji}$

(iii) $\mathcal{L}_\xi g^{(0)}_{ij} = \underbrace{\xi_{i|j} + \xi_{j|i}}_{\text{spatial Lie derivative of spatial metric}} + \underbrace{\frac{2a'}{a}g^{(0)}_{ij}}_{=g^{(0)}_{ij,0}}\xi^0$

From (i) we get: $a\xi^0 = f^0(x^i)$.
Using this in (ii): $-a f^0{}_{,i} + \xi^j{}_{,0}\, a^2 \gamma_{ji} = 0 \rightarrow \xi^i{}_{,0} = \frac{1}{a} f^0{}_{,j} \gamma^{ji} \rightarrow$

$$\xi^i = \int \frac{d\eta}{a} f^0{}_{,j} \gamma^{ji} + f^i(x^j)$$

Thus the allowable $h_{0\alpha}=0$ gauge preserving transformations are determined by four functions of the spatial coordinates: $f^0(x^i)$, $f^k(x^i)$.
h_{ij} then changes by: $\mathcal{L}_\xi g^{(0)}_{ij} = f_{i|j} + f_{j|i} + \frac{2a'}{a^2} f^0 g^{(0)}_{ij}$

$$+ 2a^2 \int \frac{d\eta}{a} f^0{}_{|ij}$$

(Just evaluate (iii) using $\xi^0 = a^{-1} f^0$, $\xi^i = \ldots$)

Going back to the notation $\delta g_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu} = \lambda h_{\mu\nu}$ we have that under a gauge preserving point transformation, the linearized fields transform by:

$$\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} + \mathcal{L}_\xi g^{(0)}_{\mu\nu}$$

$$\delta\rho \rightarrow \delta\rho + \mathcal{L}_\xi \rho^{(0)} = \delta\rho + \xi^0 \rho^{(0)\prime} = \delta\rho + a^{-1} f^{(0)} \rho^{(0)\prime}$$

$$\delta u_\mu \rightarrow \delta u_\mu + \mathcal{L}_\xi u^{(0)}_\mu \Rightarrow \delta u_0 = 0$$

$$\delta u_i \rightarrow \delta u_i + u_0 \xi^0{}_{,i} = \delta u_i - f^0{}_{,i}$$

Notice that the energy density perturbations and the velocity perturbations are not gauge invariant.

HARMONIC ANALYSIS ON 3-SPACES OF CONSTANT CURVATURE

We begin with the positive curvature case of $S^{D-1}\subset E^{D}$ (D-dimensional Euclidean space)

Suppose f is an analytic function ... within some sphere of radius $r_0>1$ then f can be represented by a power series expansion about the origin:

$$f(x) = \sum_{M=0}^{\infty} \underbrace{f_{,a_1\cdots a_M}}_{\text{symmetric since partial derivatives commute}}(0)\; x^{a_1}\cdots x^{a_M}$$

For f to be harmonic ($0=\Delta f = -f_{,a}{}^{,a}$) it is sufficient that $f_{,a_1\cdots a_{M-2}a}{}^{a}=0$, i.e. $f_{,a_1\cdots a_{M-2}}$ is tracefree

since
$$f_{,a}{}^{,a} = \sum_{i,j}\sum f_{,a_1\cdots a_M}\, x^{a_1}\cdots\delta^{a_i}_{a}\cdots\delta^{a_j}_{a}\cdots x^{a_M}$$
$$= \sum f_{,a_1\cdots a\cdots a\cdots a_M}\, x^{a_1}\cdots\check{x}^{a_i}\cdots\check{x}^{a_j}\cdots x^{a_M} = 0.$$

$\boxed{\text{these factors not present}}$

We can introduce a new function $\hat f$ which coincides with f on the unit sphere but is independent of radius, i.e., is essentially a function on the unit sphere.

$$\hat f(x) = \sum_{M=0}^{\infty} f_{,a_1\cdots a_M}(0)\,\hat x^{a_1}\cdots\hat x^{a_M} = \sum_{M=0}^{\infty} f_{,a_1\cdots a_M}(0)\, x^{a_1}\cdots x^{a_M} r^{-M}$$

$$\hat x^{a}\equiv x^{a} r^{-1}\;;\quad \delta_{ab}\hat x^{a}\hat x^{b} = \delta_{ab}x^{a}x^{b}r^{-2}=1$$

use this

Note that if $f_{,a_1\cdots a_M}(0)$ is not tracefree, the coefficients of the expansion of a function $\hat f$ on the sphere in the polynomials $X^{a_1\cdots a_M} = \hat x^{a_1}\cdots\hat x^{a_M}$ is not unique

since $\left(f_{,a_1\cdots,a}{}^{,a}\;\delta_{a_{M-1}a_M} X^{a_1\cdots a_M} = f_{,a_1\cdots a_{M-2}a}{}^{,a}\right) X^{a_1\cdots a_{M-2}}$

the pure trace parts reduce to terms of order $M-2$:

In other words any nice function on the sphere can be expanded uniquely in terms of the tracefree basis:

$$X^{TF\,a_1\cdots a_M} = X^{a_1\cdots a_M} - \frac{1}{c}\, g^{(a_1 a_2}X^{a_3\cdots a_{M-2})a}{}_{a}$$
$$c = D + \frac{m(m-1)}{2} - 1$$

Note that $\Delta(r^{M} X^{TF\,a_1\cdots a_M})=0$, so these functions correspond to harmonic functions on E^{D}.

One can introduce spherical coordinates on E^D in a standard way

$$X^a = r\,\hat{X}^a \quad , \quad \hat{X}^a = \hat{X}^a(\underbrace{\ldots, \chi, \theta, \varphi}_{\theta^i}) \qquad a = 1, \ldots, D$$

θ^i angular coordinates, i.e. coord's on S^{D-1}

$i, j = 1, \ldots, D-1$

On E^4:

$$\begin{aligned} X^1 &= r \sin\chi \, \sin\theta \, \cos\varphi \\ X^2 &= r \sin\chi \, \sin\theta \, \sin\varphi \\ X^3 &= r \sin\chi \, \cos\theta \\ X^4 &= r \cos\chi \end{aligned}$$

φ: coord's on S^1

θ, φ: coord's on S^2

χ, θ, φ: coord's on S^3

$$ds^2 = dr^2 + r^2\underbrace{[d\chi^2 + \sin^2\chi \underbrace{[d\theta^2 + \sin^2\theta \underbrace{d\varphi^2}_{\text{metric on } S^1}]}_{\text{metric on } S^2}]}_{\text{metric on } S^3}$$

$$g^{\frac{1}{2}} = \underbrace{(r\sin^2\chi)\ \underbrace{(r\sin\theta)\ \underbrace{(r)}_{E^2}}_{E^3}}_{E^4} = r^3 \sin^2\chi \sin\theta = r^{D-1}\ {}^{(D-1)}g^{\frac{1}{2}}$$

Recall $\quad X^a{}_{;a} = g^{-\frac{1}{2}}(g^{\frac{1}{2}} X^a)_{,a}$ $\qquad$ Let $\quad g_{ij} = r^2 \underbrace{{}^{D-1}g_{ij}}_{\text{metric on } S^{D-1}}$

So

$$-{}^{D}\!\Delta\phi \equiv \phi_{;a}{}^{;a} = g^{-\frac{1}{2}}(g^{\frac{1}{2}} g^{ab}\phi_{,b})_{,a}$$

$$= r^{-(D-1)}(r^{D-1}\phi_{,r})_{,r} + \frac{1}{r^2}\underbrace{\phi_{|i}{}^{|i}}_{\equiv -{}^{D-1}\!\Delta}$$

Laplacian on S^{D-1}, index raised with ${}^{D-1}g_{ij}$

$$\phi_{|i}{}^{|i} = \underbrace{\frac{1}{\sin^2\chi}\Big[(\sin^2\chi\,\phi_{,\chi})_{,\chi} + \underbrace{\frac{1}{\sin\theta}\{(\sin\theta\,\phi_{,\theta})_{,\theta} + \frac{1}{\sin\theta}\underbrace{(\phi_{,\varphi\varphi})}_{-{}^{1}\!\Delta}\}}_{-{}^{2}\!\Delta}\Big]}_{-{}^{3}\!\Delta}$$

If $\phi = \phi(\theta^i)$, then $\phi_{,a}{}^{,a} = \phi_{;a}{}^{;a} = r^{-2}\phi_{|i}{}^{|i}$.

This enables us to calculate the S^3 Laplacian of the purely angular functions $\mathcal{X}^{TF\,a_1 \ldots a_m}$.

To do this calculation let $A_{a_1\cdots a_M}$ be a constant symmetric tracefree tensor

so $A_{a_1\cdots a_M} X^{TF\, a_1\cdots a_M} = A_{a_1\cdots a_M} X^{a_1}\cdots X^{a_M} r^{-M} \equiv Q^{(n)}$, $n \equiv M+1$

where $M = N-1$ is the rank of this tensor and the degree of the homogeneous polynomial $Q^{(n)}$. Recall $\partial_a r^M = M X^a r^{M-2}$ since $r = (\delta_{ab} X^a X^b)^{1/2}$.

So $\quad Q^{(n)} = A_{abc\ldots}\, X^a X^b X^c \ldots\, r^{-M} \qquad A_{abc\ldots} = A_{(abc\ldots)},\ A^b{}_{bc\ldots} = 0$

$$\partial_d Q^{(n)} = A_{abc\ldots}\left[\delta^a{}_d X^b X^c\ldots + X^a \delta^b{}_d X^c\ldots + X^a X^b \delta^c{}_d \ldots + \ldots\right] r^{-M}$$
$$+ A_{abc\ldots}\, X^a X^b X^c\ldots \left[-M X^d r^{-(M+2)}\right]$$

$$\partial^d \partial_d Q^{(n)} = A_{abc\ldots}\left[\delta^a{}_d(\delta^{db}X^c\ldots + X^b \delta^{cd}\ldots)\ldots + \delta^b{}_d(\delta^{ad}X^c\ldots + X^a\delta^{cd}\ldots + \ldots)\right.$$
$$\left. + \delta^c{}_d(\delta^{ad}X^b\ldots + X^a\delta^{bd}\ldots + \ldots) + \ldots\right] r^{-M}$$
$$+ A_{abc\ldots}\big[\underbrace{\delta^a{}_d X^b X^c\ldots + X^a\delta^b{}_d X^c\ldots + X^a X^b \delta^c{}_d\ldots + \ldots}_{M \text{ terms}}\big]\left[-M X^d r^{-(M+2)}\right]$$
$$- M X^d A_{abc\ldots}\left[\overbrace{\delta^{ad}X^b X^c\ldots + X^a\delta^{bd}X^c\ldots + X^a X^b\delta^{cd}\ldots + \ldots}\right] X^{-(M+2)}$$
$$- M\, \partial_d X^d \quad A_{abc\ldots} X^a X^b X^c\ldots\, r^{-M+2} \qquad \left[\partial_d X^d = \delta^d{}_d = D\right]$$
$$+ M(M+2)\underbrace{X^d X_d}_{r^2}\, r^{-(M+4)}\quad A_{abc\ldots}\, X^a X^b X^c\ldots$$

$$= A_{abc\ldots}\left[(\delta^{ab}X^c\ldots + X^b\delta^{ac}\ldots) + \ldots\right] r^{-M} \to 0 \quad (\text{traces} = 0)$$

$$-2M^2 A_{abc} X^a X^b X^c\ldots r^{-M}\ r^{-2} \qquad (\text{from next two lines})$$

$$- MD\ A_{abc} X^a X^b X^c\ldots r^{-M}\ r^{-2} \qquad (\text{from } 4^{\text{th}} \text{ line})$$

$$+ M(M+2)\ A_{abc} X^a X^b X^c\ldots r^{-M}\, r^{-2} \qquad (\text{from last line})$$

$$= -M[M+D-2]\quad Q^{(n)}\, r^{-2} \qquad \equiv Q^{(n)}{}_{|i}{}^{|i}\, r^{-2}$$

$$= -\big[\underbrace{n^2 - (D-3) + (D-4)n}_{\lambda^{(n)}}\big] Q^{(n)} r^{-2}$$

$$ {}^3\!\Delta Q^{(n)} = -Q^{(n)}{}_{|i}{}^{|i} = \lambda^{(n)} Q^{(n)}$$

D	$\lambda^{(n)}$	dim. eigenspace $\lambda^{(n)}$	sphere	harmonics
2	M^2	1	S^1	ordinary harmonics: $e^{im\varphi}$
3	$M(M+1)$	$2M+1 = 2n-1$	S^2	spherical harmonics: $Y_{\ell m}(\theta,\varphi) = \sum_{m=-\ell}^{\ell} c_{\ell m} P_{\ell m}(\theta)\, e^{im\varphi}$ $(\ell \equiv M)$
4	$M(M+2) = n^2-1$	$(M-1)^2 = n^2$	S^3	S^3 harmonics: $Q_{n\ell m}(\chi\theta\varphi) = \sum_{\ell=0}^{n-1} c_{n\ell m}\, \Pi_{n\ell}(\chi)\, Y_{\ell m}(\theta\varphi)$

SEPARATION OF VARIABLES, USING EXPRESSION FOR THE LAPLACIAN IN SPHERICAL COORDINATES:

$$\left(\frac{d}{d\varphi}\right)^2 e^{im\varphi} + m^2 e^{im\varphi} = 0, \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP_{\ell m}}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] P_{\ell m} = 0$$

$$\frac{1}{\sin^2\chi}\left(\sin^2\chi\, \Pi_{n\ell,\chi}\right)_{,\chi} + \left[n^2 - 1 - \frac{\ell(\ell+1)}{\sin^2\chi}\right]\Pi_{n\ell} = 0$$

vector harmonics

If T_a is a vectorfield on E^D, its radial and angular components are:

$$T_r = T_a \frac{\partial X^a}{\partial r} = T_a \hat{X}^a = T_a \frac{X^a}{r}$$

$$T_i = T_a \frac{\partial X^a}{\partial \theta^i} = r\, T_a \frac{\partial \hat{X}^a}{\partial \theta^i}$$

A vector field tangent to the spheres of radius r satisfies $0 = T_r$ or $0 = T_a X^a$

If T_i (or equivalently $r T_a$ since $\frac{\partial \hat{X}^a}{\partial \theta^i}$ is purely angular) does not depend on r, it is equivalent to a vector-valued function on the sphere, and we can expand its components in terms of the tracefree polynomials (= scalar harmonics)

So we can let

not derivative symbol

$$r S^{(n)}_a = B_{ab,cde\ldots} \underbrace{\overbrace{X^b X^c X^d X^e \ldots}^{M \text{ factors}}}_{\text{only scalar harmonics contribute}} r^{-M} \qquad n \equiv M+1$$

constant cartesian tensor
- antisymmetric in ab, so that $S^{(n)}_a X^a = 0$;
- tracefree in all indices;
- symmetric in indices $cde\ldots$;

$$S^{(n)}_a = B_{ab,cde\ldots} \underbrace{X^b X^c X^d X^e \ldots}_{M \text{ factors}} r^{-(M+1)}$$

We must first evaluate $S^{(n)}_{a,f}{}^{,f} = S^{(n)}_{a;f}{}^{;f} =$

in order to calculate the Laplacian of the vector harmonics.

The only difference is that the extra index "a" is carried along and the exponent of r is $M+1$ instead of M.

$$\partial_f S^{(n)}_a = B_{ab,cde\ldots}\Big\{\Big[\overbrace{\delta^b_f X^c X^d X^e \ldots + X^b[\delta^c_f X^d X^e + \cdots}^{M \text{ terms}}\,]\Big] r^{-(M+1)} - X^b X^c X^d X^e \ldots (M+1) X^f r^{-(M+3)}\Big\}$$

$$\partial^f \partial_f S^{(n)}_a = \ldots \qquad \text{same as before with degree} = M, \ \text{exponent} = M+1$$

$$= \begin{Bmatrix} -2M(M+1) \\ -(M+1)D \\ +(M+1)(M+3) \end{Bmatrix} S^{(n)}_a r^{-2}$$

$$= -(M+1)(M+D-3)\, S^{(n)}_a r^{-2}$$

$$-(n)(n+D-4)\, S^{(n)}_a r^{-2} \longrightarrow$$

$$\boxed{\text{So for } n=4 \text{ we have } \partial^f \partial_f S^{(n)}_a = -\frac{n^2}{r^2} S^{(n)}_a}$$

$$\partial^a S^{(n)}_a = B_{ab,cde\ldots}\Big\{\big[\delta^{ab} X^c X^d X^e \ldots + X^b[\delta^{ac} X^d X^e \ldots + \ldots]\big] r^{-(M+1)} - (M+1) \underbrace{X^a X^b}_{\text{symmetric in } ab} X^c X^d X^e r^{-(M+2)}\Big\}$$

all trace terms which therefore vanish

antisymmetric in ab so this vanishes

$$= 0$$

3+1 split of covariant derivatives (3 = angular coord's, 1 = radial coord)

In order to relate the Euclidean derivatives to S^3-covariant derivatives, we must decompose covariant derivatives in spherical coordinates, which are Gaussian normal coordinates. The difference is of course just the constant extrinsic curvature of S^3 and the decomposition is the Euclidean analog of the one which occurs in synchronous coordinates on FRW spacetimes (see MTW GRAVITATION).

$$a \sim (r, i) \qquad ds^2 = dr^2 + g_{ij}\, d\theta^i d\theta^j\,, \quad g_{ij} = r^2 \gamma_{ij}$$

We wish to reexpress E^4 derivatives of fields tangent to the spheres of constant radius in terms of the metric γ_{ij} on S^3.

$$g_{ij} = r^2\gamma_{ij} \qquad g_{ij,r} = \frac{2}{r} g_{ij} \qquad \Gamma_{rij} = -\frac{1}{r} g_{ij} \qquad \Gamma_{ijr} = \frac{1}{r} g_{ij}$$

$$\boxed{\begin{aligned} &\Gamma^r{}_{ij} = -\frac{1}{r} g_{ij} = -r\gamma_{ij} \\ &\Gamma^{ri}{}_i = -\frac{3}{r} \\ &\Gamma^i{}_{jr} = \frac{1}{r}\delta^i{}_j \qquad \Gamma^i{}_{ir} = \frac{3}{r} \end{aligned}}$$

Let us repeat the scalar case for practice:

$$Q_{;a}{}^{;a} = Q_{,a}{}^{,a} - \Gamma^{da}{}_a Q_{,d} = \underbrace{Q_{,r}{}^{,r} - \underbrace{\Gamma^{ri}{}_i}_{\frac{3}{r}} Q_{,r}}_{r^{-3}(r^3 Q_{,r})^{,r}} + \frac{1}{r^2} Q_{|i}{}^{|i}$$

$|i$ ← index raised with γ_{ij}

↑ covariant derivative of γ_{ij}

so we obtain again the previous formula.

Now let X_a be a covariant vector field with no radial component $X_r = 0$:

$$X_{a;b} = X_{a,b} - X_d \Gamma^d{}_{ba}$$

$$X_{a;b;c} = (X_{a,b} - X_d\Gamma^d{}_{ba})_{,c} - (X_{e,b} - X_d \Gamma^d{}_{be})\Gamma^e{}_{ca} - (X_{a,e} - X_d\Gamma^d{}_{ae})\Gamma^e{}_{cb}$$

$$X_{a;b}{}^{;b} = X_{a,b}{}^{,b} - X_{d,b}\Gamma^{db}{}_a - X_d \Gamma^{db}{}_{a,b} - (X_{e,b} - X_d\Gamma^d{}_{be})(\Gamma^{eb}{}_a) - (X_{a,e} - X_d\Gamma^d{}_{ae})\Gamma^{eb}{}_b$$

$$X_{i;b}{}^{;b} = X_{i,r}{}^{,r} - X_{k,r}\underbrace{\Gamma^{kr}{}_i}_{\frac{1}{r}\delta^k{}_i} - X_{k,r}\underbrace{\Gamma^{kr}{}_i}_{\frac{1}{r}\delta^k{}_i} - X_{i,r}\underbrace{\Gamma^{rj}{}_j}_{-\frac{3}{r}} - X_k\underbrace{\Gamma^{kr}{}_{i,r}}_{-\frac{1}{r^2}\delta^k{}_i} + 2X_k\underbrace{\Gamma^k{}_{rj}\Gamma^{jr}{}_i}_{-\frac{1}{r^2}\delta^k{}_i} + X_k\underbrace{\Gamma^k{}_{ir}\Gamma^{rj}{}_j}_{-\frac{3}{r^2}\delta^k{}_i}$$

$$+ \underbrace{\text{all indices } ijkl}_{= \frac{1}{r^2} X_{i|j}{}^{|j}}$$

← necessary since raised with γ_{ij}

$$= X_{i,r}{}^{,r} + \frac{1}{r} X_{i,r} - \frac{2}{r^2} X_i + \frac{1}{r^2} X_{i|j}{}^{|j}$$

so if $X_{i,r} = 0$, $X_{i;b}{}^{;b} = \frac{1}{r^2}(X_{i|j}{}^{|j} - 2X_i)$

But above we calculated the covariant divergence in cartesian coordinates of the r-independent (in spherical coordinates) covector field $S^{(n)}_a$:

$$S^{(n)}_{a;b}{}^{;b} = -\frac{n^2}{r^2} S^{(n)}_a \quad \longrightarrow \quad \boxed{S^{(n)}_{i|j}{}^{|j} = -(n^2 - 2) S^{(n)}_i}$$

Similarly since $S^{(n)}_a{}^{;a} = 0$ ($S^{(n)}_a{}^{,a} = 0$ in cartesian coordinates)

and $S^{(n)}_a{}^{;a} = \underbrace{S^{(n)}_r{}^{,r}}_{0} - \Gamma^{ri}{}_i \underbrace{S^{(n)}_r}_{0} + \frac{1}{r^2} S^{(n)}_i{}^{|i}$, $\quad \boxed{S^{(n)}_i{}^{|i} = 0}$

The symmetric tracefree second rank covariant tensor spherical harmonics

To get a symmetric tensor with no radial components, we now need two pairs of antisymmetric indices and two extra factors of r to convert the two tensor indices from cartesian to angular variables. The pairs of antisymmetric indices must be symmetric under pair exchange to make the tensor symmetric, plus one needs all indices to be tracefree

$$r^2 G^{(n)}_{ab} = \underbrace{C_{ac,bd,ef\ldots}}_{\text{constant cartesian tensor}} x^c x^d x^e x^f \ldots r^{-M} \quad \text{or} \quad G^{(n)}_{ab} = C_{ac,bd,ef\ldots} \underbrace{x^c x^d x^e x^f \ldots}_{M \text{ factors}} r^{-(M+2)}$$

not derivative symbol

The degree is still $M = n-1$, but the exponent is now $M+2$.

The calculation of cartesian derivatives is again the same:

$$\partial_g G^{(n)}_{ab} = C_{ac,bd,ef\ldots} \{ [\delta^c_g x^d x^e x^f \ldots + x^c \delta^d_g x^e x^f + \cdots] r^{-(M+2)} - (M+2) x^g (x^c x^d x^e x^f \ldots) r^{-(M+4)} \}$$

$$\partial^g \partial_g G^{(n)}_{ab} = \underset{\text{as before}}{\cdots} = \left\{ \begin{array}{l} -2M(M+2) \\ -(M+2)D \\ +(M+2)(M+4) \end{array} \right\} G^{(n)}_{ab} r^{-2} = -\underbrace{(M+2)(M+D-4)}_{(n+1)(n-1) = n^2-1} G^{(n)}_{ab} r^{-2}$$

$$\partial^a G^{(n)}_{ab} = \cdots \text{ as in vector case } \left[\begin{array}{l} \text{tracefree condition} \\ \text{antisymmetry in first 2 pairs} \end{array} \right] \rightarrow \text{leads to } = 0.$$

Next one needs to relate these to the S^3-covariant derivatives

One can repeat the vector calculation here splitting $T_{ab;c}{}^{;c}$ into radial and angular parts in spherical coordinates, assuming $T_{ra} = 0$.

This would be a good exercise.

The result is: $\qquad T^i{}_{j;k}{}^{;k} = T^i{}_{j,r}{}^{,r} + \frac{3}{r} T^i{}_{j,r} - \frac{2}{r^2} T^i{}_j + \frac{1}{r^2} T^i{}_{j|k}{}^{|k}$

for the mixed form of a symmetric tensor T_{ab} satisfying $T_{ra} = 0$; the index i here has been raised by γ_{ij} which is r-independent, so the same formula holds with i lowered.

Therefore using $G^{(n)}_{ij,r} = 0$; one gets $\quad G^{(n)\,i}{}_{j|k}{}^{|k} - G^{(n)\,i}{}_j = -(n^2-1) G^{(n)\,i}{}_j$

$$\boxed{G^{(n)}_{ij|k}{}^{|k} = -(n^2-3)\, G^{(n)}_{ij}}$$

Similarly $\quad G^{(n)}_{ab;}{}^{b} = 0 \rightarrow \boxed{G^{(n)}_{ij|}{}^{j} = 0} \quad \leftarrow$ Another useful exercise. (Repeat method of vector case)

Recall the tracefree property $\rightarrow \boxed{G^{(n)\,i}{}_i = 0}$

MORE HARMONICS

The scalar, vector and tensor harmonics we have defined have maximum angular momentum in the sense that in each case the eigenspaces for a given value of n have the maximum dimension. In the vector and tensor cases, other eigenspaces exist for each value of n which have nonzero divergence and may be obtained by performing algebraic and differential operations to the scalar and vector harmonics already described.

Let us suppress the index n to simplify expressions.

For each Q we can introduce a vector & two tensors (symmetric):

$$P_i = \frac{1}{n^2-1} Q_{,i} \qquad \rightarrow \quad P_i{}^{|i} = \frac{1}{n^2-1} Q_{|i}{}^{|i} = -Q$$

$$Q^i{}_j = \tfrac{1}{3}\delta^i{}_j Q \qquad \rightarrow \quad Q^i{}_i = Q \,/\, Q^i{}_{j|i} = \tfrac{1}{3} Q_{,j} \,/\, Q^i{}_{j|k}{}^{|k} = \frac{\delta^i_j}{3} Q_{|k}{}^{|k} = -\frac{(n^2-1)}{3} Q \delta^i_j$$

$$= -(n^2-1) Q^i{}_j$$

$$P^i{}_j = \frac{1}{n^2-1} Q^{|i}{}_{|j} + Q^i{}_j \rightarrow P^i{}_i = 0 .$$

For each S we can introduce a tensor (symmetric):

$$S_{ij} = \mathcal{L}_S \gamma_{ij} = S_{i|j} + S_{j|i} . \qquad \rightarrow \quad S^i{}_i = 2 S_i{}^{|i} = 0$$

However, to compute $\quad P^i{}_{j|i} = \frac{1}{n^2-1} \underbrace{Q^{|i}{}_{|ji}} + \frac{1}{3} Q_{|j}$,

we need to commute the covariant derivative in order to use $Q^{|i}{}_{|i|j} = -(n^2-1) Q_{|j}$.
Similar statements hold for many other derivatives which occur in the linearized Einstein equations. The commutation is accomplished using The RICCI IDENTITIES.

Note that $\quad P_{ij} - Q_{ij} = \frac{1}{n^2-1} Q_{|ij} = \frac{1}{2}(P_{i|j} + P_{j|i}) = \mathcal{L}_{\frac{1}{2}P_k} g_{ij} .$

RICCI IDENTITIES

A space of constant curvature (Riemannian case, g_{ij} positive definite) with radius of curvature r can be shown to have a Riemann tensor

$$R^{ij}{}_{k\ell} = \mathrm{sgn}(\text{curvature})\,\frac{1}{r^2}\,\delta^{ij}{}_{k\ell} \qquad \left(\delta^{ij}_{k\ell} \equiv \delta^i_k\delta^j_\ell - \delta^i_\ell\delta^j_k\right)$$

If one computes this for the metric $g_{ij} = r^2\gamma_{ij}$ on a sphere of radius r one finds:

$$R^{ij}{}_{k\ell} = \frac{1}{r^2}\,\delta^{ij}_{k\ell}\,, \quad R^i{}_k = R^{ij}{}_{kj} = \frac{1}{r^2}\left[\delta^i{}_k\delta^j{}_j - \delta^i{}_j\delta^j{}_k\right] = \frac{2}{r^2}\,\delta^i{}_j$$

On the unit sphere S^3 with metric γ_{ij} one has:

$$R^{ij}{}_{k\ell} = \delta^{ij}_{k\ell}\,, \quad R^i{}_j = 2\delta^i{}_j \qquad \left(R^{ij}{}_{k\ell|m} = 0 = R^i{}_{j|m}\right)$$

The Ricci identities on S^3 are:

$$T^{i\ldots}{}_{j\ldots|k\ell} = T^{i\ldots}{}_{j\ldots|\ell k} + R^i{}_{mk\ell}\,T^{m\ldots}{}_{j\ldots} + \cdots - T^{i\ldots}{}_{m\ldots}\,R^m{}_{jk\ell} - \cdots$$

For example: $$Q^{|i}{}_{|ji} = Q^{|i}{}_{|ij} + \underbrace{R^i{}_{mji}}_{-R_{mj}}Q^{|m} = -(n^2-1)\,Q_{|j} - 2Q_{|j}$$

so $$P^i{}_{j|i} = -Q_{|j} - \frac{2}{(n^2-1)}Q_{|j} + \frac{1}{3}Q_{|j} \qquad \text{(from previous page)}$$

$$= -\frac{2}{3}(n^2-1)\,P_j - 2P_j = -\frac{2}{3}(n^2-4)\,P_j$$

Proceeding in this way one can easily find:

$$P^i{}_{j|k}{}^{|k} = -(n^2-7)\,P^i{}_j$$
$$P^i{}_{k|j}{}^{|k} = -\frac{1}{3}(2n^2-17)\,P^i{}_j + \frac{2}{3}(n^2-4)\,Q^{|i}{}_{|j}$$
$$P^j{}_{i|j} = -\frac{2}{3}(n^2-4)\,P_i$$

$$S^i{}_{j|k}{}^{|k} = -(n^2-6)\,S^i{}_j$$
$$S^i{}_{k|j}{}^{|k} + S^i{}_{k|j}{}^{|k} = -(n^2-10)\,S^i{}_j$$
$$S^j{}_{i|j} = -(n^2-4)\,S_i$$

We have now computed all the relevant spatial derivatives of the harmonics which appear in the linearized Einstein equations.
All of these results may be obtained using the fact that S^3 is the group manifold of $SU(2)$ [J. Math. Phys. 19, 1163 (1978) Tensor Harmonics on the 3-sphere].

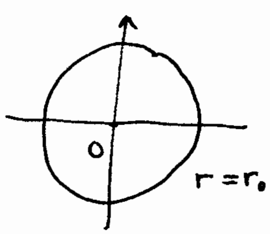
THE BIG PICTURE

Our goal is to study covariant wave equations on a spacetime with a great deal of symmetry. Examples of these wave equations are the Klein-Gordon equation for a scalar field (spin zero), Maxwell's equations (spin one), the Dirac equation (spin ½) and the linearized Einstein equations (spin 2).

These equations can be reduced to uncoupled ordinary differential equations or at least coupled only in space of a few variables by the technique of separation of variables in symmetry adapted coordinates. By expanding the field we are interested in (or collection of fields in the case of the linearized E.E.'s) in a complete set of harmonics on the space of constant curvature which occurs in the FRW models, the equation satisfied by the field can be reduced to ordinary differential equations for the expansion coefficients.

For the $k>0$ FRW models the space of constant curvature relevant to the problem is the 3-sphere which has the intrinsic geometry of a sphere of unit radius in 4-D Euclidean space (E^4). A trick then enables us to obtain the results for the $k<0$ FRW models.

EXAMPLE 1. $E^3,\ r \geq r_0.$ $\quad \phi(r,\theta,\varphi)$



Solve $\Delta\phi = -\phi_{;i}{}^{;i} = 0$ (elliptic equation)

with boundary conditions: $\begin{cases} \phi(r_0,\theta,\varphi) = \phi_0(\theta,\varphi) \\ \phi(\infty,\theta,\varphi) = 0 \end{cases}$

In spherical coordinates:

$$\left[r^{-2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{L^2}{r^2}\right]\phi = 0$$

where $\quad -L^2 = -\Delta_{S^2} = (\sin\theta)^{-1}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\varphi^2}.$

Consider spherical harmonics: $\quad Y_{\ell m}(\theta,\varphi)$

$$\begin{cases} L^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m} \\ L_3 Y_{\ell m} = m Y_{\ell m} \\ \int_{S^2} Y^*_{\ell m} Y_{\ell' m'}\, d\Omega = \delta_{\ell\ell'}\delta_{mm'} \end{cases} \qquad \left(L_3 \equiv i^{\pm 1}\frac{\partial}{\partial\varphi}\right)$$

Expand: $\quad \phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}(r)\, Y_{\ell m}(\theta,\varphi)$

$$\Delta\phi = \sum\sum \underbrace{\left[\left(-r^{-2}\partial_r r^2 \partial_r + r^{-2}\ell(\ell+1)\right) C_{\ell m}(r)\right]}_{\text{radial equation for coefficients}} Y_{\ell m}(\theta,\varphi)$$

solution: $\quad C_{\ell m}(r) = \underbrace{C_{\ell m} r^{\ell}}_{\text{Don't satisfy B.C.'s at } r=\infty} + D_{\ell m} r^{-(\ell+1)}$

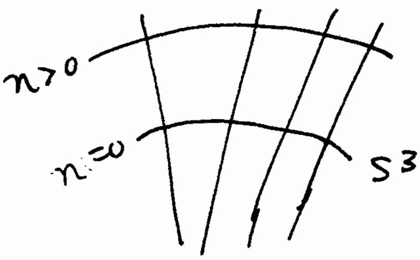
$$\phi = \sum\sum D_{\ell m}\, r^{-(\ell+1)} Y_{\ell m}(\theta,\varphi)$$

$$\phi(r_0,\theta,\varphi) = \sum\sum D_{\ell m}\, r_0^{-(\ell+1)} Y_{\ell m}(\theta,\varphi)$$

$$D_{\ell m} r_0^{-(\ell+1)} = \int_{S^2} Y_{\ell m}^*(\theta,\varphi)\, \Phi_0(\ ,\varphi)\, d\Omega$$ defines the constants $D_{\ell m}$ from the other B.C.

Problem solved.

<u>EXAMPLE 2</u> $\quad K>0$ FRW, $a(\eta)$ arbitrary $\quad \left[\begin{array}{l} ds^2 = a^2(-d\eta^2 + \gamma_{ij}\, d\theta^i d\theta^j) \\ \gamma_{ij}\, d\theta^i d\theta^j = d\chi^2 + \sin^2\chi\, d\ell^2 \end{array} \right.$



$\phi_{;\alpha}{}^{;\alpha} = 0$ (hyperbolic equation)

$\phi(\eta=0) = \phi_0 =$ function on S^3 (initial condition)

In "almost synchronous coordinates":

$\frac{d}{d\eta} = {}'$

$$\phi_{;\alpha}{}^{;\alpha} = ({}^4g)^{-1/2}\left({}^4g^{\frac{1}{2}}\, \phi_{,\alpha}\, g^{\alpha\beta}\right)_{,\beta} = -a^{-4}(a^2\phi')' + a^{-2}\gamma^{ij}\phi_{|ij}$$

Consider S^3 scalar harmonics in spherical coordinates (χ,θ,φ): $Q_{n\ell m}(\chi,\theta,\varphi)$

$$\left[\begin{array}{ll} {}^3\Delta Q_{n\ell m} = -Q_{n\ell m\,|ij}\gamma^{ij} = (n^2-1)\, Q_{n\ell m} & \\ L^2 Q_{n\ell m} = \ell(\ell+1)\, Q_{n\ell m} & \\ L_3 Q_{n\ell m} = m\, Q_{n\ell m} & \\ \int_{S^3} Q^*_{n\ell m}\, Q_{n'\ell'm'}\, dV_{S^3} = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'} & [\text{normalization condition}] \end{array} \right.$$

Expand: $$\phi = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} C_{n\ell m}(\eta)\, Q_{n\ell m}(\chi\theta\varphi)$$

Plug in: $$\sum\sum\sum \left[-a^{-4}(a^2 C_{n\ell m}')' - a^{-2}(n^2-1) C_{n\ell m}\right] Q_{n\ell m} = 0$$

$$\therefore\ a^{-2}(a^2 C_{n\ell m}')' + (n^2-1)\, C_{n\ell m} = 0$$ ← [uncoupled ordinary differential equations for expansion coefficients]

$$C_{n\ell m}(0) = \int_{S^3} Q^*_{n\ell m}\, \phi_0(\chi\theta\varphi)\, dV_{S^3}$$ ← (initial conditions for ODE's)

for an example take the Einstein universe: $a'=0$, $a=a_0$.

define $\omega_n^2 = n^2-1$; then we have the solution $C_{n\ell m}(\eta) = C_{n\ell m}(0)\, e^{i\omega_n\eta}$

Problem solved.

For the $p=(\gamma-1)\rho$ FRW spacetimes, one cannot explicitly solved these ODE's so one must analyse them qualitatively (or numerically).

$$\phi = \sum \sum D_{\ell m}\, r^{-(\ell+1)} Y_{\ell m}(\theta, \varphi)$$

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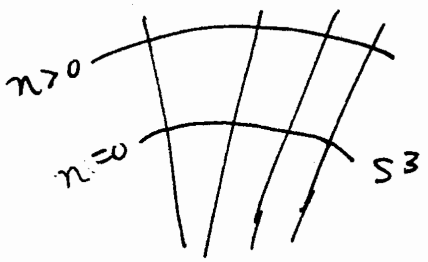

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Expand: $$\phi = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} C_{n\ell m}(\eta)\, Q_{n\ell m}(\chi\theta\varphi)$$

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Problem solved.

For the $p = (\gamma-1)\rho$ FRW spacetimes, one cannot explicitly solved these ODE'S so one must analyse them qualitatively (or numerically).

However,

I still have not explained where the eigenfunctions $Q_{n\ell m}(\chi\theta\varphi)$ came from. I only discussed the scalar harmonics in 4-D Euclidean coordinates:

$$Q^{(n)} = A_{a_1 \cdots a_M} x^{a_1} \ldots x^{a_M} r^{-M} \quad , \qquad A_{a_1 \cdots a_M} = A_{(a_1 \cdots a_M)}$$
$$A^{a}{}_{a a_3 \cdots a_M} = 0 \quad , \quad n \equiv M+1$$

I said that the linear space of such scalar harmonics had dimension n^2 and eigenvalue $\;{}^3\!\Delta Q^{(n)} = (n^2-1) Q^{(n)}$, but I did not explicitly construct the basis $Q_{n\ell m}$ using spherical coordinates.

Recall

$$-{}^3\!\Delta \phi = (\sin\chi)^{-2} \Big[(\sin^2\chi\, \phi_{,\chi})_{,\chi} + \underbrace{(\sin\theta)^{-1} \{ (\sin\theta\, \phi_{,\theta})_{,\theta} + (\sin\theta)^{-1} \underbrace{(\phi_{,\varphi\varphi})}_{-{}^1\!\Delta \equiv -(L_3)^2} \}}_{-{}^2\!\Delta \equiv -L^2} \Big]$$

If we let $\quad Q^{(n)} = \Sigma\, G_{n\ell m}(\chi)\, Y_{\ell m}(\theta, \varphi)$

then we obtain:

$$(\sin\chi)^{-2} (\sin^2\chi\, G_{n\ell m,\chi})_{,\chi} + \left[n^2 - 1 - \frac{\ell(\ell+1)}{\sin^2\chi} \right] G_{n\ell m} = 0$$

Let $\quad G_{n\ell m}(\chi) = c_{n\ell m} \Pi_{n\ell}(\chi)$ where

$$(\sin\chi)^{-2} (\sin^2\chi\, \Pi_{n\ell,\chi})_{,\chi} + \left[n^2 - 1 - \frac{(\ell)(\ell+1)}{\sin^2\chi} \right] = 0$$

and we get $\qquad Q^{(n)} = \Sigma\, c_{n\ell m}\, \Pi_{n\ell}(\chi)\, Y_{\ell m}(\theta, \varphi)$

Define $\qquad Q_{n\ell m} = \dfrac{\Pi_{n\ell}(\chi)\, Y_{\ell m}(\theta, \varphi)}{N_{n\ell m}}$

where $\qquad (N_{n\ell m})^2 = \displaystyle\int_{S^3} |\Pi_{n\ell} Y_{\ell m}|^2\, dV_{S^3}$

to obtain normalized spherical harmonics $Q_{n\ell m}$, in terms of which any regular function on S^3 may be expanded.

One can similarly define vector and tensor harmonic bases but their explicit form is not necessary if one is only interested in how the expansion coefficients evolve in time when discussing the solution of a particular wave equation.

$k=-1, 0$ cases : WEYL TRICK

We have assumed that $k = 1, 0, -1$ and have studied the case $k=1$ for which the spatial metric $\gamma_{ij} d\theta^i d\theta^j$ is that of S^3. Now let us put the parameter k back in to our results by the following transformation

$$(\eta, \chi, a, \gamma_{ij} d\theta^i d\theta^j, k, n)$$
$$= (\lambda\tilde{\eta}, \lambda\tilde{\chi}, \lambda^{-1}\tilde{a}, \lambda^{-2}\tilde{\gamma}_{ij} d\theta^i d\theta^j, \lambda^{-2}\tilde{k}, \lambda\tilde{n})$$

Then
$$ds^2 = d\tilde{s}^2 = \tilde{a}^2(-d\tilde{\eta}^2 + d\tilde{\chi}^2 + \lambda^{-2}\sin^2\lambda\tilde{\chi}\, d\Omega^2)$$
$$\tilde{\gamma}_{ij} d\tilde{\theta}^i d\tilde{\theta}^j = d\chi^2 + \lambda^{-2}\sin^2\lambda\tilde{\chi}\, d\Omega^2$$
$$\gamma^{ij} Q^{(n)}{}_{|ij} = -(n^2-1) Q^{(n)} \rightarrow \quad \tilde{\gamma}^{ij} Q^{(\tilde{n})}{}_{|ij} = -(\tilde{n}^2-\lambda^2) Q^{(\tilde{n})}$$

Setting $\lambda = i$ takes us from S^3 to H^3:
$$Q^{(\tilde{n})}{}_{|i}{}^{|i} = -(\tilde{n}^2+1) Q^{(n)} \quad \text{on } H^3 \quad (\tilde{k}=-1)$$

The other harmonic formulas change in the same way, namely $n^2 \to \tilde{n}^2$ but the constants $(1, 4, 7$ etc$)$ which follow change sign.

On S^3, the condition that $n = \lambda\tilde{n}$ $(\lambda$ real$)$ be an integer (which may be assumed nonnegative) follows from the fact that $\chi \in [0, \pi]$ and hence smooth functions on S^3 must satisfy a periodicity condition in this coordinate.

On H^3 no such condition must be met, so $\tilde{n} \in [0, \infty)$ and we have a continuous spectrum.

It is necessary to make the transformation $n = \lambda\tilde{n} = i\tilde{n}$ in going from S^3 to H^3 in order to obtain functions on H^3 which die off at $\chi \to \infty$ since on S^3:

$$\Pi_{n0} \sim \frac{\sin n\chi}{\sin\chi} \Rightarrow \underbrace{\frac{\sinh n\tilde{\chi}}{\sinh\tilde{\chi}}}_{\underbrace{\xrightarrow[\tilde{\chi}\to\infty]{} e^{(n-1)\tilde{\chi}}}_{\text{these blow up}}} = \underbrace{\frac{i\sin\tilde{n}\tilde{\chi}}{\sinh\tilde{\chi}}}_{\underbrace{\xrightarrow[\tilde{\chi}\to\infty]{} e^{-\tilde{\chi}}\sin\tilde{n}\tilde{\chi}}_{\text{these decay}}}$$

Since we wish to describe localized fields which are well behaved at $\chi \to \infty$, we choose the decaying eigenfunctions.

By taking the limit as $\lambda \to 0$ we pass from S^3 to flat E^3 in ordinary spherical coordinates $(r,\theta,\varphi) = (\tilde{\chi},\theta,\varphi)$

$$\tilde{\gamma}_{ij}\, d\tilde{\theta}^i d\tilde{\theta}^j = dr^2 + r^2 d\Omega^2$$

and we must solve the equation

$$\left[r^{-2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \tilde{n}^2 - \frac{\ell(\ell+1)}{r^2} \right] \Pi_{\tilde{n}\ell}(r) = 0$$

leading to the eigenfunctions $\quad Q_{\tilde{n}\ell m}(r,\theta,\varphi) = \Pi_{\tilde{n}\ell}(r)\, Y_{\ell m}(\theta,\varphi)$

which are used in spherical wave expansions in quantum mechanics.

[Choose solutions regular at $r=0$ and decaying at $r\to\infty$]

In this case we have

$$Q^{(\tilde{n})}{}_{|i}{}^{|i} = -\tilde{n}^2 Q^{(\tilde{n})}$$

and all the other harmonic formulas change in the same way, namely $n^2 \to \tilde{n}^2$ and the constants which follow (due to the nonzero curvature $\tilde{k}$ in the other cases) vanish.

The quantity $\tilde{\lambda}_{wave} = \dfrac{\tilde{a}}{\tilde{n}}$ corresponds to the "wavelength" of the $(\tilde{n})$-harmonics.

For fixed $\tilde{a}$, the large $\tilde{n}$ limit corresponds to the small wavelength limit in which the effects of spatial curvature become negligible.

OTHER HARMONICS

We are all familiar with plane wave expansions on E^3 using Fourier analysis in cartesian coordinates:

$$Q_{\vec{k}}(x^1,x^2,x^3) = e^{i\vec{k}\cdot\vec{x}}, \quad \vec{k} \in R^3$$

These "scalar harmonics" on E^3 are adapted to the translation symmetry rather than the spherical symmetry of E^3, and are not normalizable. Instead one has a delta function normalization. Any L^2-function on E^3 can be expanded in plane waves:

$$\phi(x) = \frac{1}{(2\pi)^3}\int d^3k\, \phi(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}, \quad \phi(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}}\phi(x)\, d^3x$$

I may have got conventional signs & factors wrong since I am without books now.

The same thing may be done on S^3 and H^3. On S^3 one has two different translation subgroups of its full group of motions. Using the correspondence of S^3 with $SU(2)$, these two translation groups are just the left and right translations respectively, leading to left and right harmonics on S^3 defined in JMP 19, 1163 (1978), while the harmonics adapted to the spherical symmetry are called adjoint harmonics.

On H^3 one has a choice of a continuous family of translation subgroups of the full symmetry group and one can develop harmonics with respect to each of these, leading to eigenfunctions which are also not normalizable and generalize the plane wave expansions of E^3 to H^3. An example is discussed by Lukash and Starobinsky (?) in the proceedings of GR8 (Waterloo, 1978).

The Lifshitz discussion of the linearized EE's goes as follows:

$\alpha, \beta = 0123$
my conventions are MTW and are the opposite of LK
$i,j = 123$

- Consider the linearized Einstein equations for $h_{\alpha\beta}$.

1) Impose linearized synchronous gauge: $h_{0\alpha} = 0$, leaving only h_{ij} nonzero.

2) Use $h^i{}_j = a^{-2} h_{ij}$ instead of h_{ij} so that contraction of this symmetric tensor commutes with $d/d\eta$.

3) For a similar reason consider the following form of the linearized EE's:

$$\delta(G^\alpha{}_\beta - T^\alpha{}_\beta) = 0.$$

Since $\delta T^i{}_j = -\delta^i{}_j \frac{dp}{d\rho}\delta T^0{}_0$ is diagonal, the offdiagonal space-space equations dont involve the fluid variables. The same is true for the differences of the diagonal components. The background field equations can be used to eliminate the fluid variables from the trace of the space-space equations, leading to 6 evolution equations for h_{ij} independent of the perturbed fluid variables which can therefore be defined as functions of the h_{ij} using the $\delta(G^0{}_0 - T^0{}_0) = 0 = \delta(G^i{}_0 - T^i{}_0)$ equations.

Thus one can solve the evolution equations for h_{ij} and then define the fluid perturbations which correspond to this metric perturbation using the remaining EE's.

4) The $h^i{}_j$ may be expanded in the complete set of tensor harmonics:

(i) built from the scalar harmonics:	$Q^i{}_j, P^i{}_j$	$\delta\epsilon \neq 0$, $\delta v^i \propto \partial_i \delta\epsilon$	"density perturbations"
(ii) built from the vector harmonics:	$S^i{}_j$	$\delta\epsilon = 0$, $\delta v^i \neq 0$ and $\vec{\nabla} \times \vec{v} \neq 0$	"velocity" or "rotational perturbations"
(iii) the transverse (zero divergence) tracefree harmonics:	$G^i{}_j$	$\delta\epsilon = 0 = \delta v$	"gravitational wave perturbations"

The latter two types of harmonics lead to uncoupled ODE's for their expansion coefficients, but the expansion coefficients of Q^i_j and $P^i{}_j$ for each (n)–eigenspace are coupled, leading to ODE's in two functions, λ and μ in their notation.
In discussing these equations they make a change in variables $(\lambda, \mu) \to (\xi, \zeta)$ to simplify the analysis.