Lie Derivatives yet again

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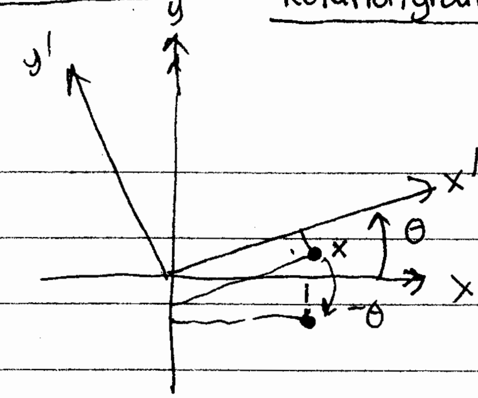
4 short pages on the Lie derivative starting with rotations and angular momentum operators..

- lieder1991.pdf 5 pages

BACKGROUND | Rotation group and Angular momentum operator on wavefunction in Quantum Mechanics



$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = f_\theta \begin{pmatrix}x\\ y\end{pmatrix} \quad \text{active}$$

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}\cos\theta & +\sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = f_{-\theta} \begin{pmatrix}x\\ y\end{pmatrix} \quad \text{passive}$$

$$\frac{d}{d\theta}\Big|_{\theta=0}\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{pmatrix}\Big|_{\theta=0}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}y\\ -x\end{pmatrix}$$

drag along coordinate functions by active transformation

any function:

$$\varphi'(\text{new point}) = \varphi(\text{old point})$$

$$\varphi'(\bar{x},\bar{y}) = \varphi(x,y) \quad \text{compose with } f_{-\theta}$$

$$\varphi'(x,y) = \varphi(f_{-\theta}(x,y))$$

new field $\longrightarrow \varphi' = \varphi \circ f_{-\theta}$

$$= \varphi(x',y') \quad \text{same function of new coordinates as of old.}$$

$$\frac{d}{d\theta}\Big|_{\theta=0}\varphi' = \Big[\frac{\partial\varphi}{\partial x}\underbrace{\frac{\partial x'}{\partial\theta}}_{y} + \frac{\partial\varphi}{\partial y}\underbrace{\frac{\partial y'}{\partial\theta}}_{-x}\Big]\Big|_{\theta=0} = -\underbrace{\Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big)}_{L_3}\varphi(x,y)$$

L_3 angular momentum operator

"generator of rotations"

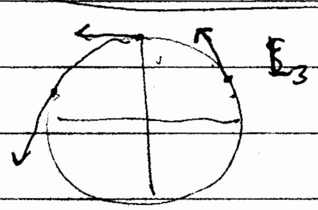
$$\Big(\frac{d}{d\theta}\Big)^n\Big|_{\theta=0}\varphi' = (-L_3)^n\varphi$$

$$\varphi' = \sum_{n=0}^{\infty}\frac{\theta^n}{n!}\Big(\frac{d}{d\theta}\Big)^n\Big|_{\theta=0}\varphi = \underbrace{e^{-\theta L_3}}_{\text{rotation operator.}}\varphi$$

interpretation of vector field generator: representation on the space of functions of the group of rotations.

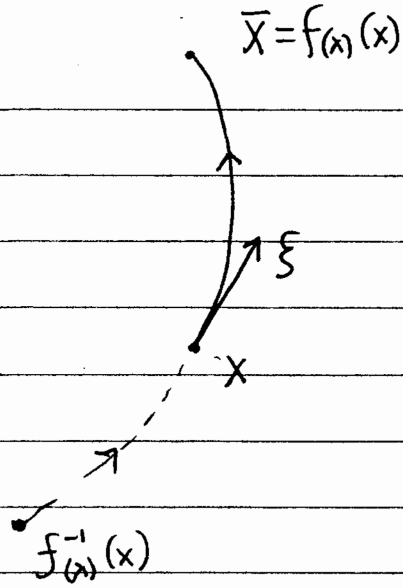
$$(x^\mu) = (x,y)$$

$$L_3 = \xi^\mu\frac{\partial}{\partial x^\mu} \qquad \xi^\mu = \frac{d}{d\theta}\Big|_{\theta=0} f^\mu(x^\nu,\theta)$$



tangent to orbits

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} \approx \begin{pmatrix}x\\ y\end{pmatrix} + \theta\begin{pmatrix}L_3^1\\ L_3^2\end{pmatrix}$$



one parameter family of point transformations:

$$x^\mu \to f^\mu_{(\lambda)}(x) \,, \quad f^\mu_{(0)}(x) = x^\mu$$

$$\xi^\mu(x) = \frac{d f^\mu_{(0)}}{d\lambda}(x) \,, \quad \xi = \xi^\mu(x) \frac{\partial}{\partial x^\mu}$$

$$\bar{x}^\mu = f^\mu_{(0)}(x) + \lambda \frac{d f^\mu_{(0)}(x)}{d\lambda} + \frac{1}{2}\lambda^2 \frac{d^2 f^\mu_{(0)}}{d\lambda^2} + \cdots$$

$$\approx x^\mu + \lambda \xi^\mu(x) \qquad \lambda \ll 1$$

$$f^{-1\,\mu}_{(\lambda)}(x) = \cdots \approx x^\mu - \lambda \xi^\mu(x)$$

$$\phi_{(\lambda)}(x) \equiv \phi(f^{-1}_{(\lambda)}(x)) \qquad \text{dragged along field (for each } \lambda)$$

$$\frac{d}{d\lambda}\Big|_{\lambda=0} \phi_{(\lambda)}(x) = \frac{\partial \phi}{\partial x^\mu}(\underbrace{f^{-1}_{(0)}(x)}_{x}) \underbrace{\frac{d f^{-1\,\mu}_{(0)}(x)}{d\lambda}}_{-\xi^\mu(x)} = -\xi^\mu(x) \frac{\partial}{\partial x^\mu}\phi(x)$$

$$= -\xi \phi(x)$$

$\boxed{\text{Lie derivative}}$ along vector field ξ^μ

$$\pounds_\xi \phi \equiv -\frac{d}{d\lambda}\Big|_{\lambda=0} \phi_{(\lambda)} = \xi \phi = \xi^\mu \frac{\partial \phi}{\partial x^\mu}$$

derivative of ϕ along vector field ξ.

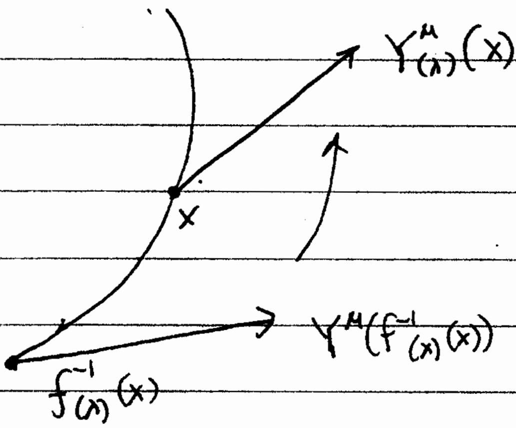
$$\phi_{(\lambda)} \approx \phi - \lambda \pounds_\xi \phi \qquad \text{describes how } \phi \text{ begins to change under dragging.}$$

$$= \underbrace{e^{-\lambda \pounds_\xi}}_{\text{"dragging operator"}} \phi \qquad \text{(power series)}$$

dragged along coordinates $\quad x'^\mu = (f^{-1})^\mu_{(\lambda)}(x) \quad$ coordinates of point $f^{-1}_{(\lambda)}(x)$ from which x came

or $\quad x^\mu = f^\mu_{(\lambda)}(x')$

How to drag along a vector field?



$$\underbrace{Y'^{\mu}_{(\lambda)}(x)}_{\text{new components of new field at new point}} = \underbrace{Y^\mu(f^{-1}_{(\lambda)}(x))}_{\text{old components of old field at old point}}$$

old components of new field at new point:

$$Y^\mu_{(\lambda)}(x) = \underbrace{\underbrace{\frac{\partial x^\mu}{\partial x'^\nu}(x')}_{\frac{\partial f^\mu_{(\lambda)}}{\partial x^\nu}(\underbrace{x'}_{f^{-1}_{(\lambda)}(x)})}}_{\substack{\text{differential of } f_{(\lambda)} \\ \text{maps tangent space at} \\ f^{-1}_{(\lambda)}(x) \text{ to } x.}} \; \underbrace{Y'^\nu_{(\lambda)}(x)}_{Y^\mu(f^{-1}_{(\lambda)}(x))}$$

$$(\mathcal{L}_\xi Y^\mu)(x) \equiv -\frac{d}{d\lambda}\Big|_{\lambda=0} Y^\mu_{(\lambda)}(x) = -\frac{d}{d\lambda}\Big|_{\lambda=0}\left[\frac{\partial f^\mu_{(\lambda)}}{\partial x^\nu}(f^{-1}_{(\lambda)}(x))\, Y^\mu(f^{-1}_{(\lambda)}(x))\right]$$

$$= -\underbrace{\frac{\partial f^\mu_{(0)}}{\partial x^\nu}}_{\delta^\mu_\nu}(\underbrace{f^{-1}_{(0)}(x)}_{x})\, \frac{\partial Y^\mu}{\partial x^\rho}(\underbrace{f^{-1}_{(0)}(x)}_{x})\, \underbrace{\frac{d f^{-1}_{(0)}{}^\rho(x)}{d\lambda}}_{-\xi^\rho(x)} - \frac{\partial}{\partial x^\nu}\left[\underbrace{\frac{d f^\mu_{(0)}}{d\lambda}}_{\xi^\mu}(\underbrace{f^{-1}_{(0)}(x)}_{x}) + \underbrace{\frac{d\,\partial f^\mu_{(0)}}{d\lambda\,\partial x}}_{\delta^\mu_\nu}\underbrace{(f^{-1}_{(0)}(x))}_{0}\right] Y^\nu(\underbrace{f^{-1}_{(0)}(x)}_{x})$$

$$= \underbrace{\xi^\rho(x)\frac{\partial}{\partial x^\rho}}\, Y^\mu(x) - Y^\nu(x)\frac{\partial}{\partial x^\nu}\,\xi^\mu(x)$$

$$= \xi Y^\mu - Y\xi \;=\; [\underbrace{\xi, Y}_{\xi Y - Y\xi}]^\mu$$

exercise.

commutator of operators ≡ Lie bracket of vector fields covector field?

$$Z_{(\lambda)\mu}(x) = \underbrace{\frac{\partial f^{-1\nu}_{(\lambda)}(x)}{\partial x^\mu}}_{\substack{\text{inverse}\\ \text{differential:}\ \frac{\partial x'^\nu}{\partial x^\mu}}} Z_\nu\left(f^{-1}_\lambda(x)\right)$$

other contraction.

$$\mathcal{L}_\xi \phi = \xi^\rho \frac{\partial\phi}{\partial x^\rho}$$

$$\mathcal{L}_\xi Y^\mu = \xi^\rho \frac{\partial Y^\mu}{\partial x^\rho} - \frac{\partial \xi^\mu}{\partial x^\rho} Y^\rho$$

$$\mathcal{L}_\xi Z_\mu = \underbrace{\xi^\rho \frac{\partial Z_\mu}{\partial x^\rho}}_{\substack{\text{dragging of}\\ \text{component functions}}} + \underbrace{Z_\rho \frac{\partial \xi^\rho}{\partial x^\mu}}_{\substack{\text{dragging}\\ \text{of coordinate axes.}}}$$

opposite sign

(For rotation group: orbital angular momentum spin orbital momentum.)

$$\mathcal{L}_\xi T^{\mu\cdots}{}_{\nu\cdots} = \xi^\rho \frac{\partial}{\partial x^\rho} T^{\mu\cdots}{}_{\nu\cdots} - \frac{\partial \xi^\mu}{\partial x^\rho} T^{\rho\cdots}{}_{\nu\cdots} - \cdots + \frac{\partial \xi^\rho}{\partial x^\nu} T^{\mu\cdots}{}_{\rho\cdots} + \cdots$$

$$= T^{\mu\cdots}{}_{\nu\cdots,\rho}\,\xi^\rho - \xi^\mu{}_{,\rho} T^{\rho\cdots}{}_{\nu\cdots} - \cdots + \xi^\rho{}_{,\nu} T^{\mu\cdots}{}_{\rho\cdots} + \cdots$$

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu,\rho}\,\xi^\rho + \xi^\rho{}_{,\mu} g_{\rho\nu} + \xi^\rho{}_{,\nu} g_{\mu\rho}$$

$$= \cdots\ g_{\mu\nu;\rho}\,\xi^\rho + \xi^\rho{}_{;\mu} g_{\rho\nu} + \xi^\rho{}_{;\nu} g_{\mu\rho}$$

$$= \xi_{\nu;\mu} + \xi_{\mu;\nu}$$

$$T_{(\lambda)}{}^{\mu\cdots}{}_{\nu\cdots}(x) = e^{-\lambda\mathcal{L}_\xi}\, T^{\mu\cdots}{}_{\nu\cdots}(x) \approx T^{\mu\cdots}{}_{\nu\cdots} - \lambda \mathcal{L}_\xi T^{\mu\cdots}{}_{\nu\cdots}$$

$$T_{(\lambda)}{}^{\mu\cdots}{}_{\nu\cdots}(x) = T^{\mu\cdots}{}_{\nu\cdots}(x) \quad \text{invariance for all } \lambda \iff \mathcal{L}_\xi T^{\mu\cdots}{}_{\nu\cdots} = 0$$

these notes March 1991
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$\boxed{\text{useful text: B. Schutz Geometrical Methods of Mathematical Physics Cambridge U. Press (paperback)}}$