INTRODUCTION TO COSMOLOGICAL MODELS : PART V

May 1988

Tying things together

Anisotropic cosmological models

Gravitational dynamics, DeWitt metric

Generalization of Kepler and rigid body dynamics

Bye.

TYING THINGS TOGETHER

Obviously parts I–IV need to be brought together in a final application, but since my time was cut short, I was only able to sketch this in an accidental lecture for a few students. Now I really don't have time to expand on that lecture. All I can do is ~~outline~~ outline such a discussion.

We started by studying spaces and subspaces of maximal symmetry as an introduction to the cosmological models of highest symmetry. To get more interesting models one must break the symmetry, which can be done in different ways, but each needs the machinery of Lie group theory. So we studied the geometry of transformation groups and the groups themselves, including dynamics involving Lie groups, in the rigid body problem. Then we looked at fiber bundles in preparation for higher dimensional cosmological models where groups again play a fundamental role.

The two main directions one can take in breaking the symmetry are

1) breaking the homogeneity by considering linear perturbation theory which involves harmonic analysis
2) breaking the isotropy by considering anisotropic but still (spatially) homogeneous models, requiring the geometry of homogeneous spaces, the simplest of which are Lie groups themselves. — here one can treat the full nonlinear field equations since they reduce to ordinary differential equations.

The first case I have treated in other years and I had hoped to get to the second this year but there was not enough time.

Anisotropy is already demanded at the spacetime level differentiating between the time and space directions to ~~explain~~ yield expanding cosmological models. In higher dimensional models, one has an obvious asymmetry between "inner" and "outer" space dimensions. ~~Anisotropic spatial~~ Spatially anisotropic models which are spatially homogeneous are quite useful in studying some aspects of the nonlinear dynamics of the field equations, ie, how spacetimes evolve.

GRAVITATIONAL DYNAMICS

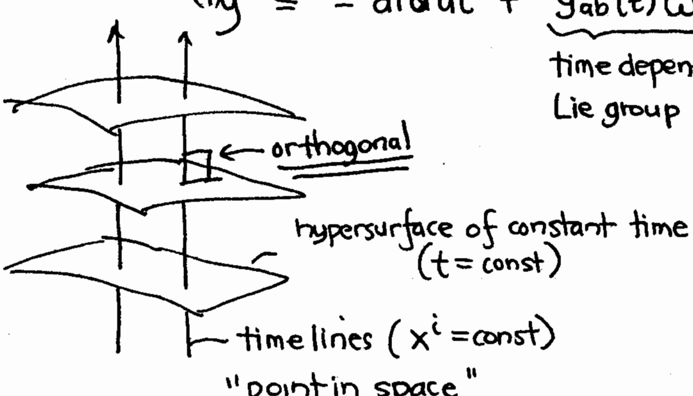
To consider gravitational dynamics, one needs to extend the classical mechanics of finite dimensional systems to fields. One has configuration spaces, phase spaces, Lagrangians, Hamiltonians, metrics on the configuration spaces, canonical variables, symmetry and gauge groups, etc. These concepts, which can be developed in general by a space-plus-time split of spacetime, return to familiar finite dimensional case for spatially homogeneous models.

In this finite dimensional problem, the configuration space for gravity is the space of matrices of positive definite inner products, representing the components of the spatial metric in an appropriate symmetry adapted frame. Example:

$$ {}^{(4)}g = -dt \otimes dt + \underbrace{g_{ab}(t)\, \omega^a \otimes \omega^b} $$

time dependent left invariant metric on 3-dimensional Lie group, like $SO(3,\mathbb{R})$ or the group of translations.

- spatial coordinates x^i
- Lie algebra $\{e_a\}$ basis
- dual basis $\{\omega^a\}$



For this 4-dim spacetime, t is the proper time measured orthogonally to the space sections, which are homogeneous, but in general anisotropic.
In vacuum, Einstein's equations determine the evolution of the variables g_{ab} which live on a 6-dimensional configuration space. The velocities $\dot{g}_{ab} = \frac{d}{dt} g_{ab} = -2K_{ab}$ are closely related to the extrinsic curvature discussed in part I.
One can discuss the velocity and momentum phase spaces. $\{g_{ab}, \dot{g}_{ab}\}$ are coord's on velocity phase space, $\{g_{ab}, \pi^{ab}\}$ on momentum phase space.

To discuss dynamics one needs a Lagrangian on velocity phase space from which one obtains a Hamiltonian on momentum phase space. This has a kinetic part and a potential part. The kinetic part is associated with a ~~Lorentz~~ metric on the configuration space called the DeWitt metric. (Let $\underline{K}=(K^a{}_b)$)

$$\mathcal{T} = g^{1/2}(\mathrm{Tr}\,\underline{K}^2 - \mathrm{Tr}^2\underline{K}) = \underbrace{g^{1/2}(g^{a(c}g^{d)b} - g^{ab}g^{cd})}_{\mathcal{G}^{abcd}} K_{ab}K_{cd}$$

(factor of 2 off usual convention) $= \frac{1}{4}\mathcal{G}^{abcd}\dot{g}_{ab}\dot{g}_{cd}$

$\mathcal{G}^{abcd} \sim$ "covariant" components of DeWitt metric in coord's "g_{ab}"

The potential part comes from the spatial curvature

$\mathcal{U} = -g^{1/2}R$, where R is the scalar curvature of the spatial metric.

so $\mathbb{L} = \mathcal{T} - \mathcal{U}$,

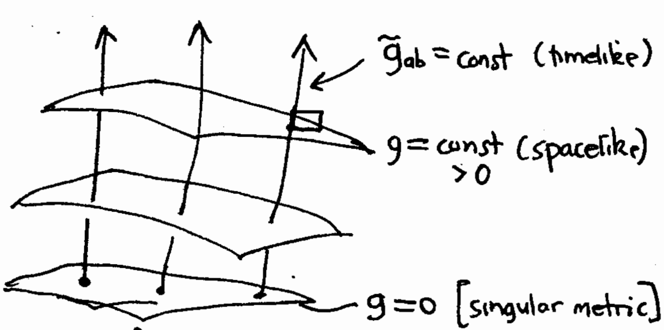
$H = \mathcal{T} + \mathcal{U}$, $\mathcal{T} = \frac{1}{4}\mathcal{G}^{-1}_{ab\,cd}\,\pi^{ab}\pi^{cd}$,

where π^{ab} are canonically conjugate to g_{ab} :

$\pi^{ab} = \partial L/\partial\dot{g}_{ab} = \frac{1}{2}\mathcal{G}^{abcd}\dot{g}_{ab}$ (index "lowering" with DeWitt metric)

For a homogeneous spatial metric, the spatial curvature becomes a quadratic function of the structure constants depending on both g_{ab} and its inverse. This gives rise to a force field $F = -d(\mathcal{U}) = -\partial\mathcal{U}/\partial g_{ab}\, dg_{ab} = -g^{1/2}G^{ab}dg_{ab}$ which is the negative of the spatial Einstein tensor density.

The spatial metric can be conformally split : $g_{ab} = g^{+1/3}\tilde{g}_{ab}$, $\det(\tilde{g}_{ab}) = 1$
It is easy to see at the point δ_{ab} of the configuration space, that g corresponds to the single timelike direction, with the tracefree matrices being spacelike an associated with the 5 directions tangent to the unit determinant metric matrices.



space of metric matrices

The general linear group acts on this space by change of frame formula:

$g_{ab} \to A^{-1c}{}_a A^{-1d}{}_b\, g_{cd}$.

The special linear group is an isometry group (8-dim) acting on the spacelike surfaces g = const — in fact these are the orbits making each a homogeneous space. Uniform scale transformations change $g^{1/2}$ and so are only <u>homothetic</u> transformations of the DeWitt metric.

The same is true of the metric dynamics. Very roughly, one can introduce a parametrization of the metric matrix corresponding to new variables adapted to the symmetry group:

$$g_{ab} = (e^{\beta} \underline{S})^{-1c}{}_{d}\, \delta_{cd}\, (e^{\beta} \underline{S})^{-1d}{}_{b} \qquad = g(e_a, e_b)$$

This makes an orthonormal frame:

$$e_{\hat a} = e_b (e^{\beta} \underline{S})^{-1b}{}_{a} \quad = \underbrace{e_c\, S^{-1c}{}_{b}}_{\substack{\text{orthogonalize} \\ (=\text{diagonalize } g_{ab})}} \underbrace{(e^{-\beta})^{b}{}_{a}}_{\substack{\text{normalize orthogonal} \\ \text{frame vectors}}}$$

$\underline{S}$ diagonalizing variables (like angular variables in Kepler problem)
↓
generalization of rigid body dynamics involving parameters $\underline{\beta}$
↓
conserved momenta (like angular momentum) →

$\beta^0\, \beta^+ \beta^-$ scaling variables (like radial variable in Kepler problem)
↓
flat Lorentz dynamics
↓
eliminate symmetry variables to get reduced system (like radial problem in Kepler problem)

The result is a mix of translationlike variables describing the rescaling of length scales along orthogonal directions and angular like variables describing the changing orientation of the orthogonal directions. A generalization of rigid body dynamics applies to the angularlike variables associated with spatial gauge transformations which dont change the 3 geometry. Using their conserved momenta like angular momenta one can reduce the problem to one for the translationlike variables involving effective potentials just like in the Kepler problem. Thus one has an interesting mix of the Kepler problem and rigid body dynamics and all of the tools developed in the classical mechanics–Lie group detour we took apply.

The diagonal submanifold $(g_{ab}) = \mathrm{diag}(g_{11}, g_{22}, g_{33})$ is a conformally flat subspace; let $\underline{\beta} = \frac{1}{2}\ln(g_{ab}) = \frac{1}{2}\mathrm{diag}(\ln g_{11}, \ln g_{22}, \ln g_{33}) = \mathrm{diag}(\beta^1, \beta^2, \beta^3)$.

By taking linear combinations of these logarithmic variables, one can get conformally flat Lorentz orthonormal coordinates

$$\underline{\beta} = \underbrace{\beta^0 \underline{1}}_{\text{pure trace}} + \underbrace{\beta^+ \mathrm{diag}(1,1,-2) + \beta^- \mathrm{diag}(\sqrt{3},-\sqrt{3},0)}_{\text{tracefree}}$$

$$\mathcal{G} = g^{1/2}\cdot(g^{a(c}g^{d)b} - g^{ab}g^{cd})\, dg_{ab}\otimes dg_{cd}\,\Big|_{diag}$$

$$= \underbrace{24\, e^{3\beta^0}}_{\substack{\text{conformal}\\ \text{factor.}}}(-d\beta^0\otimes d\beta^0 + d\beta^+\otimes d\beta^+ + d\beta^-\otimes d\beta^-)$$

Translations in β^0 lead to a uniform rescaling of the spatial metric, while translations in β^+, β^- lead to anisotropic rescalings of the framevectors $\{e_a\}$ which dont change the volume element factor.

The variables $\underline{\beta}$ are adapted to the action of the group of scale transformations on the configuration space.

For a given Lie group G specifying the homogeneity, there are special diffeomorphisms of G into itself which leave the group structure invariant. These are called automorphisms. Left invariant metrics which are related to each other by an automorphism are isometric and represent the same abstract 3-geometry. The orbits of the automorphism group correspond to the equivalence classes of spatial metrics under ~~isomorphism~~. the isometry equivalence relation. These are "spatial gauge transformations". When an automorphism acts on the Lie algebra, it leaves the structure constants invariant. The potential U will then be invariant as long as the linear transformation of the Lie algebra is unimodular (doesnt change the determinant function). Thus we obtain a symmetry group of both the kinetic and potential energy.

When a symmetry group exists, one can adapt the coordinates on the configuration and phasespaces to it. The Kepler problem is a good example. A spherically symmetric potential has the rotation group as a symmetry group. Angular momentum is conserved and one can simplify the dynamics by introducing spherical coordinates.

If the symmetry group is abelian, all "angular momenta" must vanish leading to a purely radial problem with no potential. The resulting solutions of the equations of motion are related to geodesics of the DeWitt metric on the diagonal subspace of metric matrices. Nonzero potentials (nonabelian symmetry groups) excite both the angular variables and deflect the solution curves from geodesics.

Matter leads to additional potentials.

In higher dimensions additional complications arise and even in 4 spacetime dimensions there are some complications.

Perhaps the only impression you can walk away with is that classical mechanics, Lie group theory and differential geometry all become important when considering all but the simplest cosmological models, both at the spacetime level and on the space of dynamical variables.

I am sorry that only 3 weeks of lectures have been insufficient from your point of view, but at least you have been exposed to these ideas. If you are inspired you can learn more about the mathematical techniques on your own. If you are not, at least you know that it is possible to understand many things you've seen in physics in a much more geometric and unified way.